Surjective factorization of holomorphic mappings

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Abstract. We characterize the holomorphic mappings f between complex Banach spaces that may be written in the form $f = T \circ g$, where g is another holomorphic mapping and T belongs to a closed surjective operator ideal.

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1. Introduction and preliminary results

In recent years many authors [1], [2], [7], [9], [10], [15], [19], [20] have studied conditions on a holomorphic mapping f between complex Banach spaces so that it may be written in the form either $f = g \circ T$ or $f = T \circ g$, where g is another holomorphic mapping and T a (linear bounded) operator belonging to certain classes of operators.

A rather thorough study of the factorization of the form $f = g \circ T$, where T is in a closed injective operator ideal, was carried out by the authors in [10]. In the present paper we analyze the case $f = T \circ g$.

If $f = T \circ g$, with T in the ideal of compact operators, and g is holomorphic on a Banach space E then, since g is locally bounded, f will be "locally compact" in the sense that every $x \in E$ has a neighborhood V_x such that $f(V_x)$ is relatively compact. It is proved in [2] that the converse also holds: every locally compact holomorphic mapping f can be written in the form $f = T \circ g$, with T a compact operator. Similar results were given in [20] for the ideal of weakly compact operators, in [15] for the Rosenthal operators, and in [19] for the Asplund operators. We extend this type of factorization to every closed surjective operator ideal.

Throughout, E, F and G will denote complex Banach spaces, and $\mathbb{N}$ will be the set of natural numbers. We use B_E for the closed unit ball of E, and $B(x, r)$ for the open ball of radius r centered at x. If $A \subset E$, then $\bar{\Gamma}(A)$ denotes the absolutely convex, closed hull of A, and if f is a mapping on E, then

$$\|f\|_A := \sup\{|f(x)| : x \in A\}.$$

We denote by $\mathcal{L}(E, F)$ the space of all operators from E into F, endowed with the usual operator norm. A mapping $P : E \to F$ is a k-homogeneous (continuous)

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polynomial if there is a k-linear continuous mapping $A : E \times \overset{(k)}{\cdots} \times E \to F$ such that $P(x) = A(x, \dots , x)$ for all $x \in E$. The space of all such polynomials is denoted by $\mathcal{P}(^kE, F)$. A mapping $f : E \to F$ is *holomorphic* if, for each $x \in E$, there are $r > 0$ and a sequence (P_k) with $P_k \in \mathcal{P}(^kE, F)$ such that

$$f(y) = \sum_{k=0}^{\infty} P_k(y - x)$$

uniformly for $\|y - x\| < r$. We use the notation

$$P_k = \frac{1}{k!}\, d^k f(x),$$

while $\mathcal{H}(E, F)$ stands for the space of all holomorphic mappings from E into F.

We say that a subset $A \subset E$ is *circled* if for every $x \in A$ and complex λ with $|\lambda| = 1$, we have $\lambda x \in A$.

For a general introduction to polynomials and holomorphic mappings, the reader is referred to [5], [16], [17]. The definition and general properties of operator ideals may be seen in [18].

An operator ideal $\mathcal{U}$ is said to be *injective* ([18, 4.6.9]) if, given an operator $T \in \mathcal{L}(E, F)$ and an injective isomorphism $i : F \to G$, we have that $T \in \mathcal{U}$ whenever $iT \in \mathcal{U}$. The ideal $\mathcal{U}$ is *surjective* ([18, 4.7.9]) if, given $T \in \mathcal{L}(E, F)$ and a surjective operator $q : G \to E$, we have that $T \in \mathcal{U}$ whenever $Tq \in \mathcal{U}$. We say that $\mathcal{U}$ is *closed* ([18, 4.2.4]) if for all E and F, the space $\mathcal{U}(E, F) := \{T \in \mathcal{L}(E, F) : T \in \mathcal{U}\}$ is closed in $\mathcal{L}(E, F)$.

Given an operator $T \in \mathcal{L}(E, F)$, a procedure is described in [4] to construct a Banach space Y and operators $k \in \mathcal{L}(E, Y)$ and $j \in \mathcal{L}(Y, F)$ so that $T = jk$. We shall refer to this construction as the DFJP *factorization.* It is shown in [12, Propositions 1.6 and 1.7] (see also [8, Proposition 2.2] for simple statement and proof) that given an operator $T \in \mathcal{L}(E, F)$ and a closed operator ideal $\mathcal{U}$,

(a) if $\mathcal{U}$ is injective and $T \in \mathcal{U}$, then $k \in \mathcal{U}$;

(b) if $\mathcal{U}$ is surjective and $T \in \mathcal{U}$, then $j \in \mathcal{U}$.

We say that $\mathcal{U}$ is *factorizable* if, for every $T \in \mathcal{U}(E, F)$, there are a Banach space Y and operators $k \in \mathcal{L}(E, Y)$ and $j \in \mathcal{L}(Y, F)$ so that $T = jk$ and the identity I_Y of the space Y belongs to $\mathcal{U}$.

We now give a list of closed operator ideals which are injective, surjective or factorizable. We recall the definition of the most commonly used, and give a reference for the others.

An operator $T \in \mathcal{L}(E, F)$ is (*weakly*) *compact* if $T(B_E)$ is a relatively (weakly) compact subset of F; T is (*weakly*) *completely continuous* if it takes weak Cauchy sequences in E into (weakly) convergent sequences in F; T is *Rosenthal* if every sequence in $T(B_E)$ has a weak Cauchy subsequence; T is *unconditionally converging* if it takes weakly unconditionally Cauchy series in E into unconditionally convergent series in F.

Closed operator ideals	*Injective*	*Surjective*	*Factorizable*
compact operators	**Yes**	**Yes**	No
weakly compact	**Yes**	**Yes**	**Yes**
Rosenthal	**Yes**	**Yes**	**Yes**
completely continuous	**Yes**	No	No
weakly completely continuous	**Yes**	No	No
unconditionally converging	**Yes**	No	No
Banach-Saks [13,§3]	**Yes**	**Yes**	**Yes**
weakly Banach-Saks [13, §3]	**Yes**	No	No
strictly singular [18, 1.9]	**Yes**	No	No
separable range	**Yes**	**Yes**	**Yes**
strictly cosingular [18, 1.10]	No	**Yes**	No
limited [3]	No	**Yes**	No
Grothendieck [6]	No	**Yes**	No
decomposing (Asplund) [18, 24.4]	**Yes**	**Yes**	**Yes**
Radon-Nikodým [18, 24.2]	**Yes**	No	No
absolutely continuous [14, §3]	**Yes**	No	No

The results on this list may be found in [18] and the other references given, for the injective and surjective case. The factorizable case may be seen in [12].

If $\mathcal{U}$ is an operator ideal, the *dual ideal* $\mathcal{U}^d$ is the ideal of all operators T such that the adjoint T^* belongs to $\mathcal{U}$. Easily, we have:

$$\mathcal{U} \text{ is closed injective } \implies \mathcal{U}^d \text{ is closed surjective}$$

$$\mathcal{U} \text{ is closed surjective } \implies \mathcal{U}^d \text{ is closed injective}$$

The list above might therefore be completed with some more dual ideals.

Moreover, to each $T \in \mathcal{L}(E,F)$ we can associate an operator $T^q : E^{**}/E \to F^{**}/F$ given by $T^q(x^{**}+E) = T^{**}(x^{**})+F$. Let $\mathcal{U}^q := \{T \in \mathcal{L}(E,F) : T^q \in \mathcal{U}\}$. Then, if $\mathcal{U}$ is injective (resp. surjective, closed), so is $\mathcal{U}^q$ ([8, Theorem 1.6]).

Remark 1. There is another notion of factorizable operator ideal which may be used. We say that $\mathcal{U}$ is DFJP *factorizable* ([8, Definition 2.3]) if, for every $T \in \mathcal{U}$, the identity of the intermediate space in the DFJP factorization of T belongs to $\mathcal{U}$. Clearly, every DFJP factorizable operator ideal is factorizable. The following example shows that the converse is not true. Let $\mathcal{A}$ be the ideal of all the operators that factor through a subspace of c_0. Clearly, $\mathcal{A}$ is factorizable. Consider the operator $T : \ell_2 \to \ell_2$ given by $T((x_n)) := (x_n/n)$. We have $T \in \mathcal{A}$. The intermediate space in the DFJP factorization is an infinite dimensional reflexive space. Clearly, the identity map on it does not belong to $\mathcal{A}$.

All the factorizable ideals on the table above are DFJP factorizable ([8]). Note also that, if $\mathcal{U}$ is DFJP factorizable, then so are $\mathcal{U}^d$ and $\mathcal{U}^q$ ([8]).

2. Surjective factorization

In this section, we study the factorizations in the form $T \circ g$, with $T \in \mathcal{U}$, where $\mathcal{U}$ is a closed surjective operator ideal.

Lemma 2 ([13, Proposition 2.9]). *Given a closed surjective operator ideal $\mathcal{U}$, let $S \in \mathcal{L}(E, F)$ and suppose that for every $\epsilon > 0$ there are a Banach space D_ϵ and an operator $T_\epsilon \in \mathcal{U}(D_\epsilon, F)$ such that*

$$S(B_E) \subseteq T_\epsilon(B_{D_\epsilon}) + \epsilon B_F.$$

Then, $S \in \mathcal{U}$.

We denote by $\mathcal{C}_{\mathcal{U}}(E)$ the collection of all $A \subset E$ so that $A \subseteq T(B_Z)$ for some Banach space Z and some operator $T \in \mathcal{U}(Z, E)$ (see [21]).

The following probably well-known properties of $\mathcal{C}_{\mathcal{U}}$ will be needed:

Proposition 3. *Let $\mathcal{U}$ be a closed surjective operator ideal. Then:*

(a) *if $A \in \mathcal{C}_{\mathcal{U}}(E)$ and $B \subset A$, then $B \in \mathcal{C}_{\mathcal{U}}(E)$;*
(b) *if $A_1, \ldots, A_n \in \mathcal{C}_{\mathcal{U}}(E)$, then $\cup_{i=1}^n A_i \in \mathcal{C}_{\mathcal{U}}(E)$ and $\sum_{i=1}^n A_i \in \mathcal{C}_{\mathcal{U}}(E)$;*
(c) *if $A \subset E$ is bounded and, for every $\epsilon > 0$, there is a set $A_\epsilon \in \mathcal{C}_{\mathcal{U}}(E)$ such that $A \subseteq A_\epsilon + \epsilon B_E$, then $A \in \mathcal{C}_{\mathcal{U}}(E)$.*
(d) *if $A \in \mathcal{C}_{\mathcal{U}}(E)$, then $\bar{\Gamma}(A) \in \mathcal{C}_{\mathcal{U}}(E)$;*

Proof: (a) is trivial and (b) is easy. Both are true without any assumption on the operator ideal $\mathcal{U}$.

(c) For $A \subset E$ bounded, consider the operator

$$T : \ell_1(A) \longrightarrow E \quad \text{given by} \quad T\left((\lambda_x)_{x\in A}\right) = \sum_{x\in A} \lambda_x x.$$

Given $\epsilon > 0$, there is $A_\epsilon \in \mathcal{C}_{\mathcal{U}}(E)$ such that $A \subseteq A_\epsilon + \epsilon B_E$. Therefore,

$$A \subseteq T\big(B_{\ell_1(A)}\big) \subseteq \bar{\Gamma}(A) \subseteq \Gamma(A) + \epsilon B_E \subseteq \Gamma(A_\epsilon) + 2\epsilon B_E.$$

Clearly, $\Gamma(A_\epsilon) \in \mathcal{C}_{\mathcal{U}}(E)$. Hence, $T \in \mathcal{U}$ (by Lemma 2), and $A \in \mathcal{C}_{\mathcal{U}}(E)$.

(d) If $A \in \mathcal{C}_{\mathcal{U}}(E)$, there is a space Z and $T \in \mathcal{U}(Z, E)$ such that $A \subseteq T(B_Z)$. Therefore, for all $\epsilon > 0$,

$$\bar{\Gamma}(A) \subseteq \overline{T(B_Z)} \subseteq T(B_Z) + \epsilon B_E.$$

Now, it is enough to apply part (c). □

We shall denote by $\mathcal{H}_{\mathcal{U}}(E, F)$ the space of all $f \in \mathcal{H}(E, F)$ such that each $x \in E$ has a neighborhood V_x with $f(V_x) \in \mathcal{C}_{\mathcal{U}}(F)$. Easily, a polynomial $P \in \mathcal{P}({}^kE, F)$ belongs to $\mathcal{H}_{\mathcal{U}}(E, F)$ if and only if $P(B_E) \in \mathcal{C}_{\mathcal{U}}(F)$. The set of all such polynomials will be denoted by $\mathcal{P}_{\mathcal{U}}({}^kE, F)$.

The following result is an easy consequence of the Hahn-Banach theorem and the Cauchy inequality

Lemma 4 ([20, Lemma 3.1]). *Given $f \in \mathcal{H}(E,F)$, a circled subset $U \subset E$, and $x \in E$, we have*

$$\frac{1}{k!}\, d^k f(x)(U) \subseteq \bar{\Gamma}(f(x+U))$$

for every $k \in \mathbb{N}$.

Proposition 5. *Let $\mathcal{U}$ be a closed surjective operator ideal, and $f \in \mathcal{H}(E,F)$. The following assertions are equivalent:*

(a) $f \in \mathcal{H}_{\mathcal{U}}(E,F)$;
(b) *there is a zero neighborhood $V \subset E$ such that $f(V) \in \mathcal{C}_{\mathcal{U}}(F)$;*
(c) *for every $k \in \mathbb{N}$ and every $x \in E$, we have that $d^k f(x) \in \mathcal{P}_{\mathcal{U}}({}^kE,F)$;*
(d) *for every $k \in \mathbb{N}$, we have that $d^k f(0) \in \mathcal{P}_{\mathcal{U}}({}^kE,F)$.*

Proof: (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (d) follow from Lemma 4.

(d) $\Rightarrow$ (a). Let $x \in E$. There is $\epsilon > 0$ such that

$$f(y) = \sum_{k=0}^{\infty} \frac{1}{k!}\, d^k f(0)(y)$$

uniformly for $y \in B(x,\epsilon)$ ([17, §7, Proposition 1]). By Proposition 3 (b), for each $m \in \mathbb{N}$, we have

$$\left\{ \sum_{k=0}^{m} \frac{1}{k!}\, d^k f(0)(y) : y \in B(x,\epsilon) \right\} \in \mathcal{C}_{\mathcal{U}}(F).$$

Using the uniform convergence on $B(x,\epsilon)$, and Proposition 3 (c), we conclude that $f(B(x,\epsilon)) \in \mathcal{C}_{\mathcal{U}}(F)$.

(a) $\Rightarrow$ (b) and (c) $\Rightarrow$ (d) are trivial. □

If A is a closed convex balanced, bounded subset of F, F_A will denote the Banach space obtained by taking the linear span of A with the norm given by its Minkowski functional.

Theorem 6. *Let $\mathcal{U}$ be a closed surjective operator ideal, and $f \in \mathcal{H}(E,F)$. The following assertions are equivalent:*

(a) $f \in \mathcal{H}_{\mathcal{U}}(E,F)$;
(b) *there is a closed convex, balanced subset $K \in \mathcal{C}_{\mathcal{U}}(F)$ such that f is a holomorphic mapping from E into F_K;*
(c) *there is a Banach space G, a mapping $g \in \mathcal{H}(E,G)$ and an operator $T \in \mathcal{U}(G,F)$ such that $f = T \circ g$.*

Proof: (a) $\Rightarrow$ (b) follows the ideas in the proof of [2, Proposition 3.5] and [20, Theorem 3.7].

For each $m \in \mathbb{N}$ and $x \in E$, define

$$A_m(x) := \left\{ \lambda y : y \in B\Big(x, \frac{1}{m}\Big) \text{ and } |\lambda| \leq 1 \right\}$$

and

$$U_m := \bigcup \left\{ B\Big(x, \frac{1}{m}\Big) : \|x\| \leq m \text{ and } \|f\|_{A_m(x)} \leq m \right\}.$$

For each $x \in E$ there is a neighborhood of the compact set $\{\lambda x : |\lambda| \leq 1\}$ on which f is bounded. Hence, there is $m \in \mathbb{N}$ so that $\|f\|_{A_m(x)} \leq m$, which shows that $E = \cup_{m=1}^{\infty} U_m$.

Let W_m be the balanced hull of U_m. Since the sets $A_m(x)$ are balanced, we have $|f(x)| \leq m$ for all $x \in W_m$. Let $V_m := 2^{-1} W_m$. We have $E = \cup_{m=1}^{\infty} V_m$ and hence

$$f(E) = \bigcup_{m=1}^{\infty} f(V_m). \tag{1}$$

For each $k, m \in \mathbb{N}$, define

$$K_{mk} := \bar{\Gamma}\Big(\frac{1}{k!}\, d^k f(0)(W_m) \Big) \in \mathcal{C}_{\mathcal{U}}(F).$$

By Proposition 3, we obtain that the set

$$K_m := \left\{ \sum_{k=0}^{\infty} 2^{-k} z_k : z_k \in K_{mk} \right\}$$

belongs to $\mathcal{C}_{\mathcal{U}}(F)$. Easily, $f(V_m) \subseteq K_m$. Hence $f(V_m) \in \mathcal{C}_{\mathcal{U}}(F)$ for all $m \in \mathbb{N}$. By Proposition 3, we can select numbers $\beta_m > 0$ with $\sum \beta_m < \infty$ so that

$$K := \bar{\Gamma}\Big(\bigcup_{m=1}^{\infty} \beta_m f(V_m) \Big) \in \mathcal{C}_{\mathcal{U}}(F).$$

It follows from (1) that f maps E into F_K.

It remains to show that $f \in \mathcal{H}(E, F_K)$. Let $x \in E$. Easily, there are $\epsilon > 0$ and $r \in \mathbb{N}$ such that $f(B(x, 2\epsilon)) \subseteq rK$. By Lemma 4,

$$\frac{1}{k!}\, d^k f(x)\,(B(0, 2\epsilon)) \subseteq rK \tag{2}$$

for all $k \in \mathbb{N}$. Now, for each $n \in \mathbb{N}$ and $a \in B(0, \epsilon)$, we have

$$f(x+a) - \sum_{k=0}^{n} \frac{1}{k!}\, d^k f(x)(a) = 2^{-n} \sum_{k=n+1}^{\infty} 2^{n-k} \frac{1}{k!}\, d^k f(x)(2a).$$

Since K is convex and closed, we get from (2) that

$$\sum_{k=n+1}^{\infty} 2^{n-k} \frac{1}{k!} d^k f(x)(2a) \in rK.$$

Hence,

$$f(x+a) - \sum_{k=0}^{n} \frac{1}{k!} d^k f(x)(a) \in 2^{-n} rK,$$

and so, the F_K-norm of the left hand side is less than or equal to $2^{-n}r$, for all $a \in B(0,\epsilon)$. Thus, f is holomorphic.

(b) $\Rightarrow$ (c). It is enough to note that, by Lemma 2, the natural inclusion $F_K \to F$ belongs to $\mathcal{U}$.

(c) $\Rightarrow$ (a). Each $x \in E$ has a neighborhood V_x such that $g(V_x)$ is bounded in G. Hence, $f(V_x) = T(g(V_x)) \in \mathcal{C}_{\mathcal{U}}(F)$. $\square$

Theorem 7. *Let $\mathcal{U}$ be a closed surjective, factorizable operator ideal and take a mapping $f \in \mathcal{H}(E,F)$. Then $f \in \mathcal{H}_{\mathcal{U}}(E,F)$ if and only if there are a Banach space G, a mapping $g \in \mathcal{H}(E,G)$ and $T \in \mathcal{U}(G,F)$ such that $I_G \in \mathcal{U}$ and $f = T \circ g$.*

Remark 8. Theorem 7 implies that, if $\mathcal{U}$ is the ideal of weakly compact (resp. Rosenthal, Banach-Saks or Asplund) operators and $f \in \mathcal{H}_{\mathcal{U}}(E,F)$, then f factors through a Banach space G which is reflexive (resp. contains a copy of ℓ_1, has the Banach-Saks property or is Asplund).

Moreover, if $\mathcal{U} = \{T : T^q \text{ has separable range}\}$, then G is isomorphic to $G_1 \times G_2$, with G_1^{**} separable and G_2 reflexive ([22]). If $\mathcal{U} = \{T : T^* \text{ is Rosenthal}\}$, then G contains no copy of ℓ_1 and no quotient isomorphic to c_0 ([11]).

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