Smooth invariants and ω-graded modules over $k[X]$

Fred Richman

Abstract. It is shown that every ω-graded module over $k[X]$ is a direct sum of cyclics. The invariants for such modules are exactly the smooth invariants of valuated abelian p-groups.

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1. Quivers and smooth invariants

Let R be a discrete valuation domain with prime p and residue class field k. An ω**-filtered module** is an R-module A together with a chain of submodules

$$A = A(0) \supset A(1) \supset A(2) \supset \cdots$$

such that $p(A(n)) \subset A(n+1)$. Such a structure arises when A is a submodule of a module B, in which case $A(n) = A \cap p^n B$. In this case the filtration can be extended to any ordinal α. The structure is also known as a **valuated module**, with $vx = \alpha$ if x is in $A(\alpha)$ but not in $A(\alpha+1)$, and $vx = \infty$ if no such α exists.

If B is a module of finite length, then a submodule A of B is determined, up to an automorphism of B, by its ω-filtered module structure [4, Theorem 32]. However, even if A has finite length and $A(6) = 0$, these structures defy classification (see [3] for $A(7) = 0$). Contrast this to finite-length R-modules whose structure theory is the same as that of finite abelian p-groups.

Each ω-filtered module A induces a representation of the quiver

$$0 \to 1 \to 2 \to \cdots$$

by vector spaces over the residue class field k, namely

$$\frac{A(0)}{A(1)} \xrightarrow{p} \frac{A(1)}{A(2)} \xrightarrow{p} \frac{A(2)}{A(3)} \xrightarrow{p} \cdots$$

where p stands for the linear transformation induced by multiplication by p in A. The **Ulm invariants** of the ω-filtered module A are (the dimensions of) the kernels of these maps. The **derived Ulm invariants** of A, introduced in [2], are the cokernels of these maps.

If $A(n) = 0$, these are representations of the quiver

$$0 \to 1 \to 2 \to \cdots \to n-1$$

and are well understood: they can be thought of as modules over the ring of triangular n-by-n matrices with entries in k. There are n simple representations of this quiver, obtained by putting the field k in position i and 0's everywhere else. There are $\binom{n+1}{2}$ nonzero uniserial representations, one for each nonempty string of consecutive integers in $\{0, 1, \ldots, n-1\}$. For example, if $n > 5$, then 345 is the representation that has k in positions 3, 4 and 5, with the two nonzero maps being the identity. The socle of 345 is 5, and when you divide 345 by its socle you get 34. Every representation is a direct sum of uniserials, essentially uniquely, as in the case of finite abelian p-groups.

So, if A is a finite-length ω-filtered module, then the number of uniserials of each type in the decomposition of the associated representation are isomorphism invariants of A. These turn out to be exactly the **smooth invariants** of A, introduced in [1, p. 484] as

$$s(n,m) = \frac{A(m+1) + \{x \in p^{m-n}A(n) : px \in A(m+2)\}}{A(m+1) + \{x \in p^{m-n+1}A(n-1) : px \in A(m+2)\}}$$

for $n \leq m$.

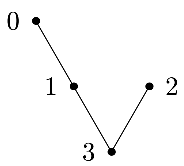
The Ulm invariants live in the socle of the representation — 345 gives an Ulm invariant at 5 and a derived Ulm invariant at 3. Decomposing the representation into a direct sum of uniserials matches the Ulm invariants with the derived Ulm invariants via the smooth invariants, which was the point of the smooth invariants in the first place.

The representation associated with the finite-length cyclic ω-filtered module 0245 is

$$k \to 0 \to k \to 0 \to k \to k$$

with the last map being the identity. It's easy to read off the smooth invariants: 0, 2, and 45. So it's obvious how to read off the invariants of the representation associated with any finite-length cyclic, or direct sum of finite-length cyclics.

The simplest indecomposable noncyclic ω-filtered modules are given by trees. The nodes on the tree are generators of the module, and if the node x sits above the node y, then $px = y$. There are numbers on the nodes indicating the values of the generators. These ω-filtered modules are said to be **simply presented**. The smallest one comes from the tree



which gives the representation

$$k \xrightarrow{1} k \xrightarrow{0} k \xrightarrow{1} k$$

and hence has smooth invariants 01 and 23, the same as for the direct sum cyclics corresponding to 01 and 23. Thus the smooth invariants are not a complete set of invariants for finite-length ω-filtered R-modules A, even when $A(4) = 0$. An even simpler example is provided by the cyclic module 02 which has the same smooth invariants as the direct sum of the cyclic modules 0 and 2. The ω-filtered R-modules A with $A(4) = 0$ are classifiable: they are direct sums of cyclics and copies of the simply presented module corresponding to the displayed tree (see [3]).

2. ω-graded modules over $k[X]$

We will show that the structure theory for representations of the finite quiver

$$0 \to 1 \to 2 \to \cdots \to n-1$$

holds for representations of the infinite quiver

$$0 \to 1 \to 2 \to \cdots$$

Such a representation

$$A_0 \to A_1 \to A_2 \to \cdots$$

can be thought of as an ω-graded module $A_0 \oplus A_1 \oplus A_2 \oplus \cdots$ over the polynomial ring $k[X]$, the action of X being given by the linear transformations $A_i \to A_{i+1}$. Such modules, unlike $\mathbf{Z}$-graded modules over $k[X]$ (which can be as complicated as abelian p-groups with no elements of infinite height), are direct sums of cyclics.

Theorem 1. *An ω-graded module over $k[X]$ is a direct sum of cyclics.*

Proof: Let

$$A = A_0 \oplus A_1 \oplus A_2 \oplus \cdots$$

be an ω-graded module over $k[X]$. Decompose

$$A_0 = (A_{01} \oplus A_{02} \oplus A_{03} \oplus \cdots) \oplus A_{0\infty}$$

where $A_{01} = \{a \in A_0 : Xa = 0\}$, $A_{01} \oplus A_{02} = \{a \in A_0 : X^2 a = 0\}$, and so on. Now decompose

$$A_1 = (A_{11} \oplus A_{12} \oplus A_{13} \oplus \cdots) \oplus A_{1\infty}$$

with $A_{11} \supset XA_{02}$, $A_{12} \supset XA_{03}$, $A_{13} \supset XA_{04}$, and so on including $A_{1\infty} \supset XA_{0\infty}$. We end up with

$$\begin{array}{l}
A_{01} \\
A_{02} \to A_{11} \\
A_{03} \to A_{12} \to A_{21} \\
A_{04} \to A_{13} \to A_{22} \to A_{31} \\
\quad\vdots \\
A_{0\infty} \to A_{1\infty} \to A_{2\infty} \to \cdots
\end{array}$$

where the maps are all multiplication by X and all monic. Take bases of all these spaces that extend bases of the previous space. The union will be a set of independent generators. □

Note that representations of the finite quivers can be thought of as bounded representations of the infinite quiver, so this theorem includes the finite case also.

Contrast this with the situation for ω^*-graded $k[X]$-modules, that is, representations of the infinite quiver

$$0 \leftarrow 1 \leftarrow 2 \leftarrow \cdots$$

Many X-primary $k[X]$-modules admit an ω^* grading. These modules include the analogues of simply presented p-groups, and $p^{\omega+1}$-projectives, and are closed under direct sums and torsion products.

The smooth invariants of an ω-filtered R-module A are given by passing to the representation

$$\frac{A(0)}{A(1)} \xrightarrow{p} \frac{A(1)}{A(2)} \xrightarrow{p} \frac{A(2)}{A(3)} \xrightarrow{p} \cdots$$

considered as an ω-graded module M over $k[X]$. They are the cardinal numbers $S(\alpha, n)$, where $\alpha \in \omega$ and $n \in \omega \cup \{\infty\}$, that count the number of cyclics of order X^{n+1} and grade α in a decomposition of M.

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Florida Atlantic University, Boca Raton, FL 33431, USA

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