Abstract initiality

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Abstract. We study morphisms that are initial w.r.t. all functors in a given conglomerate. Several results and counterexamples are obtained concerning the relation of such properties to different notions of subobject. E.g., strong monomorphisms are initial w.r.t. all faithful adjoint functors, but not necessarily w.r.t. all faithful monomorphism-preserving functors; morphisms that are initial w.r.t. all faithful monomorphism-preserving functors are monomorphisms, but need not be extremal; and (under weak additional conditions) a morphism is initial w.r.t. all faithful functors that map extremal monomorphisms to monomorphisms iff it is an extremal monomorphism.

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Initiality of morphisms or sources w.r.t. a functor is one of the most important notions of category theory and plays a central role e.g. in categorical topology or in the theory of fibrations (cf. [1], [2], [3], [4]). It is usually studied from the point of view of concrete categories, i.e. categories equipped with a fixed forgetful functor into a base category. In this paper, we adopt a more 'abstract' point of view, in the sense that we study morphisms that are initial with respect to *all* functors that satisfy a given property (e.g. preservation of monomorphisms); seen this way, initiality becomes a property of morphisms in abstract categories.

Morphisms with such initiality properties have a tendency to be monomorphisms. The usual additional properties such as extremality or regularity interrelate with initiality properties in various ways; e.g., strong monomorphisms are initial w.r.t. all faithful adjoint functors, and extremal monomorphisms are initial w.r.t. all solid functors. Several statements of this kind, along with counterexamples taken from categories of non-connected spaces showing, e.g., that extremal monomorphisms need not be initial w.r.t. faithful adjoint functors, are collected in Section 2.

Our main result, proved in Section 3, states that morphisms that are initial w.r.t. all faithful functors that preserve a given class of monomorphisms must belong to the closure of that class under composition and left cancellation; this generalizes a statement proved in [10]. The proof makes use of the semicategory method introduced in [10], which is briefly summarized at the end of Section 1. As a corollary we obtain, under weak completeness conditions, a characterization of initiality for the case of preservation of extremal monomorphisms. Moreover, we present an example which shows that monomorphisms can be initial w.r.t. all faithful monomorphism-preserving functors without being extremal; thus, initiality in this sense defines an interesting new class of monomorphisms.

For the sake of readability, we have restricted the exposition to initiality of morphisms; however, all results presented below (possibly with the exception of Corollary 3.7) are easily carried over to initiality of sources. Unexplained categorical terminology is referred to [1].

1. Basic concepts

The notion of initiality can be defined w.r.t. arbitrary functors; we recall the definition for morphisms as given e.g. in [1]:

Definition 1.1. Let $F : \mathbf{A} \to \mathbf{B}$ be a functor. A morphism $f : A \to B$ in $\mathbf{A}$ is called *initial* w.r.t. F if, whenever $g : C \to B$ is a morphism in $\mathbf{A}$ and $h : FC \to FA$ is a morphism in $\mathbf{B}$ such that $Ffh = Fg$, then there exists a unique morphism $\bar{h}$ in $\mathbf{A}$ such that $F\bar{h} = h$ and $f\bar{h} = g$.

If F is faithful, then this definition agrees with the usual one. In the theory of fibrations, the term 'cartesian' is usually used instead. We extend this notion to conglomerates of functors:

Definition 1.2. Let $\mathfrak{G}$ be a conglomerate of functors. A morphism f in a category $\mathbf{A}$ is called $\mathfrak{G}$*-initial* if it is initial with respect to all functors in $\mathfrak{G}$ with domain $\mathbf{A}$.

We will liberally use obvious terms such as 'faithful-initial' or 'adjoint-initial'. E.g., it has been shown in [10] that a morphism is faithful-initial iff it is a section, and functor-initial iff it is an isomorphism. The former statement will turn out to be a special case of a more general result presented below.

We will pay special attention to initiality with respect to the following conglomerates of functors:

Definition 1.3. Let $\mathcal{M}$ be a class of monomorphisms in a category $\mathbf{A}$. A functor is called $\mathcal{M}$*-preserving* if it maps all elements of $\mathcal{M}$ to monomorphisms. The conglomerate of (faithful) $\mathcal{M}$-preserving functors with domain $\mathbf{A}$ will be denoted by $\mathfrak{G}_{\mathcal{M}}$ ($\mathfrak{F}_{\mathcal{M}}$).

For technical purposes, we will need the following notions introduced in [11]:

Definition 1.4. A class $\mathcal{M}$ of morphisms in a category $\mathbf{A}$ is called *coclosed* if it contains all identities, is left cancellable in the sense that $fg \in \mathcal{M}$ implies $g \in \mathcal{M}$, and is closed under composition. A further class $\mathcal{A}$ of morphisms in $\mathbf{A}$ is called $\mathcal{M}$*-coclosed* if $\mathcal{A}$ is closed under composition with arbitrary morphisms from the left and under left cancellation of $\mathcal{M}$-morphisms (i.e. $g \in \mathcal{A}$ implies $fg \in \mathcal{A}$ for all f, and $mg \in \mathcal{A}$ implies $g \in \mathcal{A}$ for all $m \in \mathcal{M}$). The coclosure and the $\mathcal{M}$-coclosure in the obvious sense of a class $\mathcal{S}$ are denoted by $\mathrm{cl}\,\mathcal{S}$ respectively $\mathrm{cl}_{\mathcal{M}}\mathcal{S}$.

E.g., the class of all monomorphisms in a category is coclosed, and given a morphism f, the class of all morphisms g such that $fx = fy$ implies $gx = gy$ is mono-coclosed. The smallest coclosed class in a category is always the class of all sections. Moreover,

Proposition 1.5. *Let $\mathfrak{G}$ be a conglomerate of functors; then in any category, the class of $\mathfrak{G}$-initial morphisms is coclosed.* □

Since the complement of a coclosed class $\mathcal{M}$ is obviously $\mathcal{M}$-coclosed, we have

Lemma 1.6. *Let $\mathcal{M} \subset \mathrm{Mor}\mathbf{A}$ be coclosed, and let $\mathcal{S} \subset \mathrm{Mor}\mathbf{A}$. Then*

$$\mathrm{cl}_{\mathcal{M}}\mathcal{S} \cap \mathrm{Ident}(\mathbf{A}) = \emptyset \iff \mathrm{cl}_{\mathcal{M}}\mathcal{S} \cap \mathcal{M} = \emptyset \iff \mathcal{S} \cap \mathcal{M} = \emptyset.$$

In the proof of the main result, we will need the semicategory method introduced in [10], which allows us to construct extensions of categories by adding artificial morphisms without having to define all of the newly arising composites. We briefly review the involved concepts; further details and full proofs (not needed for the understanding of the present paper) can be found in [8,10].

A *semicategory* is a structure consisting of objects, morphisms and a composition operation in the usual sense which may fall short of being a category inasmuch as the composite fg, where the domain of f coincides with the codomain of g, need not always be defined (even when f or g is an identity). The identity and associativity laws are required to hold in the following form: For a morphism $f : A \to B$, the composites fid_A and $id_B f$ are equal to f whenever they are defined; moreover, if composites fg and gh are defined, then $(fg)h$ and $f(gh)$ are defined and equal.

The morphisms of a semicategory and their composition can be regarded as generators and relations; in this sense, every semicategory $\mathbf{A}$ freely generates a category $\mathbf{A}^*$ (the hom-set condition being ignored for the moment) which is constructed by first taking the category of paths over $\mathbf{A}$ in the obvious sense (where identities are admitted as components of paths) and then factoring out the smallest congruence that makes the map that sends an $\mathbf{A}$-morphism to the corresponding path of length 1 a functor. As an application of the Church-Rosser technique (cf. [7]), it can be shown that each morphism $A \to B$ in $\mathbf{A}^*$ has a unique normal form of the type $f_n \dots f_1 : A \to B$, $n \geq 0$, where the f_i are $\mathbf{A}$-morphisms, none of the f_i is an identity, and none of the composites $f_{i+1} f_i$ is defined in $\mathbf{A}$. In particular, $\mathbf{A}$ injects into $\mathbf{A}^*$.

2. Initiality vs. algebraicity

We begin with a number of observations that illustrate the rule of thumb that, the more 'algebraic' functors get, the more monomorphisms are initial with respect to them.

Proposition 2.1. *Strict monomorphisms are $\mathfrak{F}_{\mathrm{Mono}}$-initial.*

Proof: Let $m : A \to B$ be a strict monomorphism in a category $\mathbf{A}$, and let $U : \mathbf{A} \to \mathbf{B}$, $U \in \mathfrak{F}_{\mathrm{Mono}}$. To show that m is initial w.r.t. U, let $g : C \to B$ be a morphism in $\mathbf{A}$, and let $h : UC \to UA$ be a morphism in $\mathbf{B}$ such that $Umh = Ug$. Then $xm = ym$ in $\mathbf{A}$ implies $xg = yg$, since U is faithful; hence there exists $\bar{h}$ such that $m\bar{h} = g$, and since Um is a monomorphism, we have $U\bar{h} = h$ as required. $\square$

Remark 2.2. As the above proof shows, every strict monomorphism m is even $\mathfrak{F}_{\{m\}}$-initial.

Proposition 2.3. *Strong monomorphisms are faithful-adjoint-initial.*

Proof: Let $m : A \to B$ be a strong monomorphism in a category $\mathbf{A}$, let $(\eta, \varepsilon) : F \dashv U : \mathbf{A} \to \mathbf{B}$ be an adjoint situation, where U is faithful, and let $g : C \to B$ and $h : UC \to UA$ such that $Umh = Ug$. Then

$$m\varepsilon_A Fh = \varepsilon_B FUmFh = \varepsilon_B FUg = g\varepsilon_C.$$

Now ε_C is an epimorphism because U is faithful; thus the square $m(\varepsilon_A Fh) = g\varepsilon_C$ admits a diagonal d:

$$\begin{tikzcd} FUC \arrow[r, "\varepsilon_C"] \arrow[d, "\varepsilon_A Fh"'] & C \arrow[d, "g"] \arrow[dl, "d"'] \\ A \arrow[r, "m"'] & B \end{tikzcd} .$$

In particular, we have $md = g$, and since Um is monic, $Ud = h$ as required. $\square$

Remark 2.4. Of course, we have not fully used the fact that U is faithful and adjoint in the above proof; in fact, it suffices that U preserves monomorphisms (or even just m) and that there exist a functor $F : \mathbf{B} \to \mathbf{A}$ and a pointwise epimorphic natural transformation $\varepsilon : FU \to id_{\mathbf{A}}$ (which implies that U is faithful).

Proposition 2.5. *Extremal monomorphisms are solid-initial.*

Proof: Let $m : A \to B$ be an extremal monomorphism in $\mathbf{A}$, and let $U : \mathbf{A} \to \mathbf{B}$ be a solid functor (cf. [1]). Define a U-structured sink $\mathcal{T}$ with codomain A by

$$\mathcal{T} = \{(C, h) \mid \exists g : C \to B : Ug = Umh\}.$$

By solidity of U, there exists a semifinal arrow (e, D) for $\mathcal{T}$, and by semifinality of (e, D), there exists $f : D \to B$ such that $Ufe = Um$. Since $(A, id_{UA}) \in \mathcal{T}$, there exists $\bar{e} : A \to D$ in $\mathbf{A}$ such that $U\bar{e} = e$. $\bar{e}$ is an epimorphism, because semifinal arrows are generating; thus $\bar{e}$ is an isomorphism, since $f\bar{e} = m$ by faithfulness of U. This implies that for each $(C, h) \in \mathcal{T}$, there exists $\bar{h} : C \to A$ such that $U\bar{h} = h$, i.e. m is initial w.r.t. U. $\square$

This series of statements is completed by the remark that, given any reasonable definition of algebraic functor, monomorphisms are algebraic-initial (cf. [1,5,6]).

While we will see below that $\mathfrak{F}_{\mathrm{Mono}}$-initial morphisms must be monic, and that this statement extends to the situation of Remark 2.4, no such partial converses hold for Propositions 2.3 and 2.5. (Note, however, Remark 3.2.) For instance, if **A** is a category such that for each comonad on **A**, the counit consists of isomorphisms, then every adjoint functor with domain **A** is full and faithful, and hence all **A**-morphisms are adjoint-initial. An example of this kind is the category with precisely two morphisms id and f, where $ff = f$ (in particular, f is not monic).

Moreover, the obvious attempts to weaken the conditions in the above propositions fail: It is well known that not all monomorphisms are solid-initial (or even topological-initial); the corresponding generalizations of Propositions 2.1 and 2.3 are dealt with by the following counterexamples:

Example 2.6. Let **A** be the full subcategory of the category of topological spaces spanned by the spaces of cardinality at most 1 and the non-connected spaces; let $U : \mathbf{A} \to \mathbf{Set}$ denote the usual forgetful functor. U is adjoint, since **A** contains all discrete spaces. Furthermore, it is easily checked that a morphism in **A** is epic iff it is surjective.

Now let X be the discrete space with carrier set $\{0, 1\}$, and let Y be the space $\{0, 1, 2\}$ with open sets $\emptyset$, $\{2\}$, $\{0, 1\}$, and Y; let $m : X \hookrightarrow Y$ denote the inclusion. m is an extremal monomorphism: If $m = ge$, where e is an epimorphism, then e is bijective and hence an isomorphism, since any space in **A** of cardinality 2 is discrete. However, m is not initial w.r.t. U: Let $g : Y \to X$ be the map given by $g(0) = 0$ and $g(1) = g(2) = 1$; then mg is continuous, since the subspace $\{0, 1\}$ of Y is indiscrete, but g is not, since $g^{-1}[\{1\}] = \{1, 2\}$ is not open.

Example 2.7. Let **B** be the full subcategory of the category of topological spaces spanned by the spaces with precisely two connected components. It is easily verified that a morphism in **B** is monic iff it is injective (i.e. the forgetful functor $V : \mathbf{B} \to \mathbf{Set}$ preserves monomorphisms) and epic iff it is surjective.

Now take $m : X \hookrightarrow Y$ as in the previous example. It is seen as above that m is not initial w.r.t. V; however, m is a strong monomorphism in **B**: Let W and Z be spaces in **B**, and let $e : W \to Z$, $f : W \to X$, and $h : Z \to Y$ be continuous maps, where e is surjective, such that $he = mf$. We have to show that this commutative square admits a diagonal d:

$$\begin{tikzcd} W \arrow[r, "e"] \arrow[d, "f"'] & Z \arrow[d, "h"] \arrow[dl, "d"'] \\ X \arrow[r, "m"'] & Y \end{tikzcd} .$$

Of course, d exists as a map; we can assume w.l.o.g. that d (and hence f) is surjective. To see that d is continuous, we have to show that $d^{-1}[\{0\}]$ and $d^{-1}[\{1\}]$ are open, i.e. that these sets form the unique decomposition of Z into disjoint nonempty open sets. But this is clear, since $e^{-1}[d^{-1}[\{0\}]] = f^{-1}[\{0\}]$ and $e^{-1}[d^{-1}[\{1\}]] = f^{-1}[\{1\}]$ form the unique decomposition of W (and the map $A \mapsto e^{-1}[A]$ is injective).

3. A new class of monomorphisms

As indicated above, Proposition 2.1 has a partial converse, which is part of the following more general statement:

Theorem 3.1. *Let $\mathcal{M}$ be a class of monomorphisms in a category* $\mathbf{A}$*. Then every $\mathfrak{F}_{\mathcal{M}}$-initial morphism belongs to* $\mathrm{cl}\,\mathcal{M}$*.*

Proof: Since $\mathfrak{F}_{\mathcal{M}} = \mathfrak{F}_{\mathrm{cl}\,\mathcal{M}}$, we can assume that $\mathcal{M}$ is coclosed. Let $f : A \to B$ be a morphism in $\mathbf{A}$ such that $f \notin \mathcal{M}$. Extend $\mathbf{A}$ by a new morphism $x : A \to A$ and define composition incompletely by

$$gx = g \quad \text{for each} \quad g \in \mathrm{cl}_{\mathcal{M}}\{f\}.$$

This defines a semicategory $\mathbf{B}$ in the sense explained in Section 1: The identity and associativity laws hold, because $id_A \notin \mathrm{cl}_{\mathcal{M}}\{f\}$ by Lemma 1.6, respectively because $\mathrm{cl}_{\mathcal{M}}\{f\}$ is closed under composition from the left. Thus, the morphisms in the freely generated category $\mathbf{B}^*$ have a unique normal form of the type

$$g_r x g_{r-1} \dots x g_1, \; r \geq 1,$$

where the g_i are $\mathbf{A}$-morphisms such that $g_i \notin \mathrm{cl}_{\mathcal{M}}\{f\}$, $i = 2, \dots, r$ (this normal form is obtained from the normal form discussed in Section 1 by just filling in identities).

In particular, the functor $E : \mathbf{A} \to \mathbf{B}^*$ is indeed an embedding, x is really a new morphism, and $\mathbf{B}^*$ satisfies the hom-set condition. Moreover, E preserves $\mathcal{M}$: Let $m \in \mathcal{M}$, and let g and h be morphisms in $\mathbf{B}^*$ with normal forms $g_r x \dots x g_1$ respectively $h_s x \dots x h_1$ such that $mg = mh$. Then, since $\mathrm{cl}_{\mathcal{M}}\{f\}$ is stable under left cancellation of $\mathcal{M}$-morphisms, $m g_r x \dots x g_1$ and $m h_s x \dots x h_1$ are normal forms of the same morphism; this implies $g = h$ as required.

Thus $E \in \mathfrak{F}_{\mathcal{M}}$; however, f is not E-initial, since $fx = f$, but x does not belong to $\mathbf{A}$. □

(The construction applied in the above proof has been introduced in [9].)

Remark 3.2. It is easily seen that the extension E constructed in the above proof has a left inverse (namely, the functor that identifies x and id_A); thus, the improved version of Proposition 2.3 indicated in Remark 2.4 does have a partial converse in the sense that every morphism that is initial w.r.t. all functors of the mentioned type is a monomorphism.

Remark 3.3. Similarly as in [10], Theorem 3.1 is easily generalized to sources: Every $\mathfrak{F}_{\mathcal{M}}$-initial source meets $\mathrm{cl}\,\mathcal{M}$ (cf. [1] for the definition of initial source). Noticing furthermore that a source is initial w.r.t. the unique functor into the terminal category iff it is a product, one obtains a characterization of $\mathfrak{G}_{\mathcal{M}}$-initiality: a source is $\mathfrak{G}_{\mathcal{M}}$-initial iff it is a product and meets $\mathrm{cl}\,\mathcal{M}$ (the point being that, given a source $\mathcal{S}$ with the latter property, $F\mathcal{S}$ is a monosource for each $F \in \mathfrak{G}_{\mathcal{M}}$).

This statement is, of course, entirely uninteresting in the special case of 1-sources (i.e. morphisms); but even the general case mostly yields examples of limited interest, since product projections tend to be retractions.

The picture changes if the scope is extended to include initiality of cones, where a cone μ with codomain D is called *initial* w.r.t. a functor F if, whenever h is a morphism and ν is a cone with codomain D such that $F\mu h = F\nu$, then there exists a unique morphism $\bar{h}$ such that $\mu\bar{h} = \nu$ and $F\bar{h} = h$. Indeed, a cone is $\mathfrak{G}_{\mathcal{M}}$-initial iff it is a limit and meets $\mathrm{cl}\,\mathcal{M}$; e.g., the cones associated to equalizers and intersections are $\mathfrak{G}_{\mathrm{RegMono}}$-initial respectively $\mathfrak{G}_{\mathrm{Mono}}$-initial.

The converse of the above theorem is, of course, false in the general case; however, in conjunction with Propositions 2.1 and 1.5, we obtain

Corollary 3.4. *Let* $\mathbf{A}$ *be a category, and let* $\mathcal{M} \subset \mathrm{StrictMono}(\mathbf{A})$. *Then a morphism is* $\mathfrak{F}_{\mathcal{M}}$*-initial iff it belongs to* $\mathrm{cl}\,\mathcal{M}$. □

The special case $\mathcal{M} = \mathrm{Sect}(\mathbf{A})$ has been treated in [10]. As a further application, we have

Example 3.5. In **Cat**, every extremal epimorphism can be factored into two regular epimorphisms: Let $F : \mathbf{A} \to \mathbf{B}$ be an extremal epimorphism in **Cat** (i.e. $F[\mathbf{A}]$ generates $\mathbf{B}$), and let G be the coequalizer of the congruence relation of F. Then there exists a functor H such that $HG = F$. It is easily seen that H is bijective on objects and full; hence H is a regular epimorphism.

As a consequence,

$$\mathrm{ExtrEpi}(\mathbf{Cat}) = \mathrm{cl}\,\mathrm{RegEpi}(\mathbf{Cat}),$$

where cl denotes the *closure* of a class of epimorphisms, defined dually to Definition 1.4 (recall that the class of extremal epimorphisms is always closed under right cancellation, and closed under composition in categories with pullbacks). Thus, by the dual of the corollary above, a morphism in **Cat** is final w.r.t. to all faithful functors that preserve regular epimorphisms iff it is an extremal epimorphism.

A similar argument produces a characterization of $\mathfrak{F}_{\mathrm{ExtrMono}}$-initiality in sufficiently well-behaved categories. The proof needs the following

Lemma 3.6. *Let* $\mathbf{A}$ *be a category, and let* $\mathcal{M} \subset \mathrm{Mono}\mathbf{A}$ *be coclosed and closed under intersections. Then the class of* $\mathfrak{F}_{\mathcal{M}}$*-initial morphisms is closed under intersections.*

Proof: By assumption on $\mathcal{M}$ and by the above theorem, the intersection m of a family of $\mathfrak{F}_{\mathcal{M}}$-initial subobjects belongs to $\mathcal{M}$; hence Um is a monomorphism for each $U \in \mathfrak{F}_{\mathcal{M}}$. Using this fact, initiality of m w.r.t. U is easily verified along the same lines as in Proposition 2.1. □

Corollary 3.7. *Let* $\mathbf{A}$ *be a category with equalizers and intersections. Then the* $\mathfrak{F}_{\mathrm{ExtrMono}}$*-initial morphisms in* $\mathbf{A}$ *are precisely the extremal monomorphisms.*

(Under the given conditions, extremal monomorphisms and strong monomorphisms coincide; cf. [1].)

PROOF: By [1], Corollary 14.20, ExtrMono($\mathbf{A}$) is the closure of RegMono($\mathbf{A}$) under composition and intersections; in particular, ExtrMono($\mathbf{A}$) is coclosed, since extremal monomorphisms are always stable under left cancellation. Moreover, RegMono($\mathbf{A}$) $\subset \mathcal{I}$ by Proposition 2.1, where $\mathcal{I}$ denotes the class of $\mathfrak{F}_{\mathrm{ExtrMono}}$-initial morphisms; by Proposition 1.5 and the above Lemma, this implies ExtrMono($\mathbf{A}$) $\subset \mathcal{I}$. Conversely, $\mathcal{I} \subset$ ExtrMono($\mathbf{A}$) by Theorem 3.1. $\square$

(Note that Theorem 3.1 is invoked for both inclusions!)

To justify the title of the section, we conclude with a counterexample which shows that $\mathfrak{F}_{\mathrm{Mono}}$-initial morphisms, which are monomorphisms by Theorem 3.1, need not be extremal (the question whether every $\mathfrak{F}_{\{m\}}$-initial monomorphism m is extremal remains open; cf. Remark 2.2). As seen in Example 2.7, even strong monomorphisms need not be $\mathfrak{F}_{\mathrm{Mono}}$-initial; thus the $\mathfrak{F}_{\mathrm{Mono}}$-initial morphisms form a class of monomorphisms that is contained in the class of strict monomorphisms and incomparable to the usual broader notions.

Example 3.8. Let $\mathbf{B}$ be the free category over the graph

$$\begin{tikzcd}
\bullet \arrow[drr, "c"] \arrow[ddr, "d"'] & & & \\
& A \arrow[r, "n"'] \arrow[d, "f"] & B \arrow[d, "g"'] \arrow[ddr, "b"] & \\
& \bullet \arrow[r, "m"] \arrow[drr, "a"'] & \bullet & \\
& & & \bullet
\end{tikzcd},$$

and let $\sim$ denote the equivalence relation generated by

$$gn \sim mf, \quad bn \sim af, \quad \text{and} \quad gc \sim md.$$

Then $\sim$ is already a congruence on $\mathbf{B}$; let $\mathbf{A}$ be the associated quotient category of $\mathbf{B}$. Note that $ad \neq bc$ in $\mathbf{A}$.

Now m and n are monomorphisms in $\mathbf{A}$; n is not extremal, since it is also an epimorphism in $\mathbf{A}$. However, n is $\mathfrak{F}_{\mathrm{Mono}}$-initial: Let $U : \mathbf{A} \to \mathbf{C}$ be a faithful functor that preserves monomorphisms, and let $Unh = Ux$ for morphisms $x : X \to B$ and $h : UX \to UA$, where X is an $\mathbf{A}$-object. Then $x = n$ implies $h = id_{UA} = Uid_A$. If $x = id_B$, then Un is an isomorphism with inverse h; thus,

$\delta = Ufh$ is a diagonal for the squares $UaUf = UbUn$ and $UmUf = UgUn$ and hence for the square $UgUc = UmUd$, because Um is a monomorphism. This implies $U(ad) = Ua\delta Uc = U(bc)$, in contradiction to faithfulness of U. The only remaining case is $x = c$. In this case, $UmUfh = UgUnh = UgUc = UmUd$ and hence $Ufh = Ud$; thus $U(ad) = UaUfh = UbUnh = U(bc)$, again a contradiction.

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