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Cauchy-Neumann problem for a class of nondiagonal parabolic systems with quadratic growth nonlinearities I. On the continuability of smooth solutions

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Abstract. A class of nonlinear parabolic systems with quadratic nonlinearities in the gradient (the case of two spatial variables) is considered. It is assumed that the elliptic operator of the system has a variational structure. The behavior of a smooth on a time interval $[0,T)$ solution to the Cauchy-Neumann problem is studied. For the situation when the "local energies" of the solution are uniformly bounded on $[0,T)$, smooth extendibility of the solution up to $t=T$ is proved. In the case when $[0,T)$ defines the maximal interval of the existence of a smooth solution, the singular set at the moment $t=T$ is described.

Keywords: boundary value problem, nonlinear parabolic systems, solvability

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Global in time weak solvability of the Cauchy-Dirichlet problem for a class of nondiagonal parabolic systems with quadratic growth nonlinearities in the gradient was proved by the author in [1], [2]. In these papers, we analyzed the parabolic systems provided that the number of spatial variables equals two and that the corresponding elliptic operator has a variational structure. More exactly,we constructed a solution $u : Q \to \mathbb{R}^N$, $N > 1$, where $Q = \Omega \times (0,T)$, Ω is a bounded domain in $\mathbb{R}^2$, and T is any positive number. This solution is smooth in $\overline{\Omega} \times (0,T)$ with the exception of at most a finite number of points. This result was proved for quasilinear systems in [1] and it was generalized to the nonlinear case in [2].

To construct the global solution we attract two important facts: 1) local in time classical solvability theorem, 2) the result on the extension of smooth solutions from an interval $[0,T_0)$ to the closed interval $[0,T_0]$. Such an idea of proof was originally used by M. Struwe in [3], where the author constructed heat flows of harmonic maps in the case of two spatial variables.

In this paper we study the Cauchy-Neumann problem for the same type of parabolic systems. We prove the existence of weak global in time solution possessing the same properties as in the case of the Dirichlet boundary condition. The work consists of two parts.

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In the presented below Part I, we prove the main analytic result. It concerns the extension of a smooth solution given on an interval $[0, T_0)$ up to $t = T_0$ provided that "local energies" of the system remain small for $t \in [0, T_0]$ (Theorem 1).

Next paper (Part II) will be devoted to the solvability results. First,we shall prove local in time classical solvability to nonlinear nondiagonal parabolic systems under nonlinear boundary conditions. This result has a general meaning. By this we mean that we do not assume any structural restriction and growth conditions on the nonlinearities. In the Part II we also prove weak global solvability of the Cauchy-Neumann problem for the class of the parabolic systems considered in the Part I.

Let Ω be a bounded domain in $\mathbb{R}^2$ with sufficiently smooth boundary $\partial\Omega$, $Q = \Omega \times (0, T)$, $u: \overline{Q} \to \mathbb{R}^N$, $N > 1$, $u = (u^1, \dots, u^N)$.

We consider the functional

$$\mathcal{E}[u] = \int_{\Omega} f(x, u, u_x)\, dx + \int_{\partial\Omega} G(x, u)\, ds, \tag{1}$$

where f and G are scalar-valued functions, $x = (x_1, x_2)$.

Here we study the initial boundary value problem

$$\begin{aligned} &u_t^k - \frac{d}{dx_\alpha} f_{p_\alpha^k}(x, u, u_x) + f_{u^k}(x, u, u_x) = 0 \quad \text{in} \quad Q, \\ &f_{p_\alpha^k}(x, u, u_x) \cos(\mathbf{n}, x_\alpha) + g^k(x, u)\big|_\Gamma = 0, \quad \Gamma = \partial\Omega \times (0, T),\ k \le N, \\ &u\big|_{t=0} = \varphi, \end{aligned} \tag{2}$$

where $g(x, u) = \nabla_u G(x, u)$, $\varphi: \Omega \to \mathbb{R}^N$ is a given function, $\mathbf{n} = \mathbf{n}(x)$ is the outward to Ω normal vector at a point $x \in \partial\Omega$, $\alpha = 1, 2$.

It is obvious that the elliptic operator in (2) is the Euler operator of functional (1) and the natural boundary condition is defined at the lateral surface Γ of the cylinder Q.

Now we fix a number $\alpha_0 \in (0, 1)$ and suppose that Ω, f, G, g and φ satisfy the following conditions.

CONDITION A_1. Ω is a bounded domain in $\mathbb{R}^2$, $\partial\Omega \in C^{2+\alpha_0}$.

CONDITION A_2. $\varphi \in C^{2+\alpha_0}(\overline{\Omega})$, the compatibility condition holds:

$$f_{p_\alpha^k}(x, \varphi, \varphi_x) \cos(\mathbf{n}, x_\alpha) + g^k(x, \varphi)\big|_{x \in \partial\Omega} = 0, \quad k \le N.$$

CONDITION B_1. The function f is defined and has continuous derivatives f_u, f_{ux}, f_{uu}, f_p, f_{px}, f_{up}, f_{pp} on the set $\mathcal{M} = \bar{\Omega} \times \mathbb{R}^N \times \mathbb{R}^{2N}$.

The following growth conditions hold on $\mathcal{M}$:

$$\nu_0 |p|^2 \le f \le \mu_0 |p|^2 + \mu_1, \tag{3}$$

$$|f_u| + |f_{ux}| + |f_{uu}| \le \mu_2(1+|p|^2),\\ |f_p| + |f_{px}| + |f_{pu}| \le \mu_2(1+|p|), \tag{4}$$

$$|f_{pp}| \le \mu_2, \quad \frac{\partial^2 f}{\partial p_\alpha^k \partial p_\beta^l}\, \xi_\beta^l \xi_\alpha^k \ge \nu |\xi|^2, \tag{5}$$

where $\nu_0,\ \nu,\ \mu_0,\ \mu_2 = \mathrm{const} > 0,\ \ \mu_1 = \mathrm{const} \ge 0$.

CONDITION B_2. On any compact subset of $\mathcal{M}$, the functions $f_{px},\ f_{pu},\ f_{pp}$ satisfy Hölder condition with respect to x, u, p with the exponent α_0.

CONDITION C_1. The function G is defined and has continuous derivatives $G_x,\ G_u,\ G_{xu},\ G_{uu},\ G_{uxx},\ G_{uux},\ G_{uuu}$ on the set $\mathcal{M}_0 = \overline{\Omega} \times \mathbb{R}^N$. The following inequalities hold on $\mathcal{M}_0$:

$$G \ge h_0|u|^2 - h_1, \quad h_0 = \mathrm{const} \ge 0, \quad |G| + |G_x| \le h_2(1+|u|^2), \tag{6}$$

and for $g = \nabla_u G$ we suppose that

$$|g| + |g_x| + |g_{xx}| \le h_3(1+|u|), \quad |g_u| + |g_{ux}| + |g_{uu}| \le h_3, \tag{7}$$

where $h_1, h_2, h_3 = \mathrm{const} > 0$.

CONDITION C_2. On any compact subset of $\mathcal{M}_0$, the functions g_u and g_{xx} satisfy Hölder condition in x and u with the exponent α_0.

As an example of f we introduce

$$f(x,u,p) = \frac{1}{2} \sum_{\substack{k,l \le N \\ \alpha,\beta \le 2}} \mathcal{A}_{kl}^{\alpha\beta}(x,u) p_\beta^l p_\alpha^k, \tag{8}$$

where the matrix $A = \{A_{kl}^{\alpha\beta}\}$ is smooth enough, $\mathcal{A}_{kl}^{\alpha\beta} = \mathcal{A}_{lk}^{\beta\alpha}$, and

$$\langle A(x,u)\xi,\ \xi\rangle \ge \nu|\xi|^2, \quad \forall\, \xi \in \mathbb{R}^{2N}, \quad \nu = \mathrm{const} > 0.$$

We may also put $G \equiv 0$ or $G = \frac{1}{2}|u|^2$.

Remark 1. System (1) with f represented by quadratic form (8) is the quasilinear system of parabolic equations with nondiagonal principal matrix $A(x,u)$ and quadratic growth nonlinearity ($f_u(x,u,p) \sim |p|^2,\ \ |p| \to \infty$).

Remark 2. The continuation theorem we shall prove (Theorem 1) is valid under more general assumptions on f and G. For example, we may suppose that $f = f(x,t,u,p)$ and $G = G(x,t,u)$,

$$|f_t| + |f_{tu}| \le \mu_2(1+|p|^2), \quad |f_{pt}| \le \mu_2(1+|p|), \quad \text{on} \quad \overline{Q} \times \mathbb{R}^N \times \mathbb{R}^{2N},\\ |G_t| \le h_2(1+|u|^2), \quad |g_t| \le h_3(1+|u|), \quad \text{on} \quad \overline{\Gamma} \times \mathbb{R}^N, \tag{9}$$

and conditions $A_1, \dots, C_2$ hold.

Nevertheless, to save place we study the situation when $f_t = G_t = 0$ because in Part II of the paper a weak global solution will be constructed under such restriction.

We shall use the following notation:

$$
\begin{aligned}
&B_R(x^0) = \{x \in \mathbb{R}^2 : \ |x - x^0| < R\}, \quad S_R(x^0) = \{x \in \mathbb{R}^2 : \ |x - x^0| = R\},\\
&B_R^+(x^0) = B_R(x^0) \cap \{x_2 > x_2^0\}, \quad \Omega_R(x^0) = B_R(x^0) \cap \Omega,\\
&\gamma_R(x^0) = B_R(x^0) \cap \partial\Omega,\\
&Q^{t_1,t_2} = \Omega \times (t_1, t_2), \quad Q = Q^T = Q^{0,T}, \quad \Omega^t = \Omega \times \{t\}.
\end{aligned}
$$

For $u : \overline{Q} \to \mathbb{R}^N$ we write

$$
\begin{aligned}
&u_x = \{u^k_{x_\alpha}\}^{k\le N}_{\alpha\le 2}, \quad |u_x|^2 = \sum_{\substack{k\le N\\ \alpha\le 2}} (u^k_{x_\alpha})^2, \quad u_{xt} = \{u^k_{x_\alpha t}\}^{k\le N}_{\alpha\le 2},\\
&|u_{xt}|^2 = \sum_{\substack{k\le N\\ \alpha\le 2}} (u^k_{x_\alpha t})^2, \quad u_{xx} = \{u^k_{x_\alpha x_\beta}\}^{k\le N}_{\alpha,\beta\le 2}, \quad |u_{xx}|^2 = \sum_{\substack{k\le N\\ \alpha,\beta\le 2}} (u^k_{x_\alpha x_\beta})^2.
\end{aligned}
$$

For a set $A \subset \mathbb{R}^k$ we write $|A|_k = \text{meas}_k\, A$.

We simply write B_R, S_R, B_R^+, ... , instead of $B_R(0)$, $S_R(0)$, $B_R^+(0)$, ... , for brevity. We write $\|\cdot\|_{p,\Omega}$ instead of $\|\cdot\|_{L^p(\Omega)}$.

The definition of the spaces can be found in [4].

For $\beta, \gamma \in (0,1)$ and a continuous in $\overline{Q}$ function v we put

$$
\begin{aligned}
\langle v\rangle^{(\beta)}_{x,Q} &= \sup_{\substack{(x,t),(x',t)\in\overline{Q}\\ x\ne x'}} \frac{|v(x,t) - v(x',t)|}{|x - x'|^\beta},\\
\langle v\rangle^{(\gamma)}_{t,Q} &= \sup_{\substack{(x,t),(x,t')\in\overline{Q}\\ t\ne t'}} \frac{|v(x,t) - v(x,t')|}{|t - t'|^\gamma},\\
[v]^{(\beta)}_Q &= \langle v\rangle^{(\beta)}_{x,Q} + \langle v\rangle^{(\beta/2)}_{t,Q}.
\end{aligned}
$$

$C^{\beta,\gamma}(\overline{Q})$ is the space of continuous in $\overline{Q}$ functions with the finite norm

$$
\|v\|_{C^{\beta,\gamma}(\bar{Q})} = \sup_{\bar{Q}} |v| + \langle v\rangle^{(\beta)}_{x,Q} + \langle v\rangle^{(\gamma)}_{t,Q}.
$$

Let

$$
\delta(z^1, z^2) = \max\{|x^1 - x^2|, |t^1 - t^2|^{1/2}\}, \quad \forall\, z^1, z^2 \in \mathbb{R}^{n+1},
$$

be the parabolic metric.

We denote by $L^{2,\lambda}(\Omega)$, $\mathcal{L}^{2,\lambda}(\Omega)$ and $L^{2,\lambda}(Q;\delta)$, $\mathcal{L}^{2,\lambda}(Q;\delta)$ the Morrey spaces and the Campanato spaces in euclidean and parabolic metrics, respectively.

$\mathcal{H}^{2+\gamma,1+\gamma/2}(\overline{Q})$ is the space of continuous in $\overline{Q}$ functions $u = u(x,t)$ possessing continuous in $\overline{Q}$ derivatives u_t, u_x, u_{xx}, with the norm

$$\begin{aligned}\|u\|_{\mathcal{H}^{2+\gamma,1+\gamma/2}(\bar{Q})} &= \sup_{\bar{Q}}|u| + \sup_{\bar{Q}}|u_x| + \sup_{\bar{Q}}|u_{xx}| \\ &+ \sup_{\bar{Q}}|u_t| + [u_t]_Q^{(\gamma)} + [u_{xx}]_Q^{(\gamma)} + \langle u_x\rangle_{t,Q}^{(1+\gamma)/2},\end{aligned}$$

(see [4, Chapter I, §1]).

For fixed $\alpha_0 \in (0,1)$ and $(t_1,t_2) \in [0,T]$, we define the class

$$\mathcal{K}\{[t_1,t_2]\} = \{u:\ \overline{Q}' \to \mathbb{R}^N \mid u \in \mathcal{H}^{2+\alpha_0,1+\alpha_0/2}(\overline{Q}')\}.$$

where $Q' = Q^{t_1,t_2}$.

We write $u \in \mathcal{K}\{[t_1,t_2)\}$ if $u \in \mathcal{K}\{[t_1,\tau]\}\ \ \forall\, \tau < t_2$.

We denote by $V(Q)$ the space $L^\infty((0,T);L^2(\Omega)) \cap L^2((0,T);W_2^1(\Omega))$ of functions v with the norm

$$|\!|\!|v|\!|\!|_Q = \big(\operatorname*{esssup}_{(0,T)}\|v(\cdot,t)\|_{2,\Omega}^2 + \|v_x\|_{2,Q}^2\big)^{1/2} < +\infty.$$

If $v \in V(Q)$ and $\dim\Omega = 2$, then $v \in L^4(Q)$ and

$$\|v\|_{4,Q} \le q_0\left(1 + \left(\frac{T}{|\Omega|_2}\right)^{1/4}\right)|\!|\!|v|\!|\!|_Q, \tag{10}$$

where $q_0 = \mathrm{const} > 0$ depends only on C^1 characteristic of $\partial\Omega$ (see [4, Chapter 2, §3]).

We denote by c, c_i positive constants which may depend on the parameters $\nu_0,\ldots,h_3$ from conditions (3)–(7) and on the $C^{2+\alpha_0}$ characteristic of $\partial\Omega$, $\|\varphi\|_{C^{2+\alpha_0}(\bar{\Omega})}$. The dependence on T is stressed by writing $c(T)$.

Now we formulate the theorem on the extendibility of smooth solutions.

Theorem 1. *Let conditions $A_1,\ldots,C_2$ hold and u be a solution of the class $\mathcal{K}\{[0,T)\}$ to problem* (2). *Then there exist $\varepsilon_0 > 0$ and $R_0(\varepsilon_0) > 0$ such that the inequality*

$$\sup_{[0,T)}\sup_{x^0\in\bar{\Omega}}\|u_x(\cdot,t)\|_{2,\Omega_{R_0}(x^0)}^2 < \varepsilon_0 \tag{11}$$

implies the inclusions $u \in \mathcal{K}\{[0,T]\}$ and $u_{xt} \in L^{2,2+2\alpha_0}(Q;\delta)$. The number ε_0 is determined by parameters $\nu_0,\nu,\mu_0,\ldots,\mu_2,h_0,\ldots,h_3$ and by C^{1+1} characteristics of $\partial\Omega$.

The proof of the theorem is contained in Lemmas 1–7 and Propositions 1, 2. The proofs of Lemmas 2–4 are similar to the corresponding proofs in [1] and we omit them here.

Now we put

$$E[u(t)] = \|u_x(\cdot,t)\|^2_{2,\Omega} + \|u(\cdot,t)\|^2_{2,\partial\Omega},$$
$$E[u(t),\Omega_r(x^0)] = \|u_x(\cdot,t)\|^2_{2,\Omega_r(x^0)} + \|u(\cdot,t)\|^2_{2,\gamma_r(x^0)},$$

and we write $\mathcal{E}[u(t)]$ for $\mathcal{E}[u]$, $u = u(x,t)$ (see (1)).

Lemma 1. *If $u \in \mathcal{K}\{[0,T)\}$ is a solution to problem (2) then the following inequalities hold:*

$$\int_{t_1}^{t_2}\int_{\Omega} |u_t|^2 dx\,dt + \mathcal{E}[u(t_2)] \le \mathcal{E}[u(t_1)], \quad \forall\, t_1 \le t_2 < T, \tag{12}$$

$$\|u_t\|^2_{2,Q} + \sup_{[0,T]} E[u(t)] \le c_1 E[\varphi] + c_2 \equiv E_0, \tag{13}$$

where $c_1, c_2 = const > 0$ depend on the parameters $\nu_0, \mu_0, \mu_1, h_0 - h_2$. If $h_0 = 0$ in condition (6) then c_1, c_2 also depend on T and on C^{1+1} characteristics of $\partial\Omega$.

Moreover, the following local energy-type estimate holds:

$$\begin{aligned}\int_{t_1}^{t_2}\int_{\Omega_R(x^0)} |u_t|^2 dx\,dt + \sup_{[t_1,t_2]} E[u(t),\Omega_R(x^0)] &\le c_3(R+(t_2-t_1))\\ + c_4 E[u(t_1),\Omega_{2R}(x^0)] + \frac{c_5(t_2-t_1)E_0}{R^2}&,\\ \forall\, t_1,t_2 \in [0,T],\ \forall\, x^0 \in \overline{\Omega},\ R \le \min\{1, \operatorname{diam}\Omega\}&.\end{aligned} \tag{14}$$

The constants $c_3,\dots,c_5$ in (14) depend on the same parameters as c_1, c_2 in inequality (13).

Proof: The function u satisfies the integral identity

$$\int_{t_1}^{t_2}\int_{\Omega} \left(u^k_t \eta^k + f_{p^k_\alpha}(x,u,u_x)\eta^k_{x_\alpha} + f_{u^k}(x,u,u_x)\eta^k\right) dx\,dt + \int_{t_1}^{t_2}\int_{\partial\Omega} g^k(x,u)\eta^k\, ds\,dt = 0, \tag{15}$$

where $t_1 \le t_2 < T$ and η is a smooth function on the set $\overline{\Omega} \times [t_1,t_2]$.

From (15) with $\eta = u_t$, estimate (12) follows. To derive (13) we consider two cases.

First, we suppose that $h_0 \neq 0$ in (6). From (12) it follows that

$$\int_{t_1}^{t}\int_{\Omega} |u_t|^2 dx\,dt + \min\{h_0,\nu_0\}\,E[u(t)] \le (h_1+h_2)|\partial\Omega| + \mu_1|\Omega| + \max\{\mu_0,h_2\}\,E[u(t_1)],$$

and the inequality (13) holds.

If $h_0 = 0$ in (6) then

$$(16)\quad \int_{t_1}^{t_2}\int_{\Omega} |u_t|^2 dx\,dt + \nu_0 \sup_{[t_1,t_2]} \|u_x(\cdot,t)\|^2_{2,\Omega} \le c_1 E[u(t_1)] + c_2, \quad \forall\, t_1 \le t_2 < T,$$

where $c_1, c_2 = \mathrm{const} > 0$ do not depend on T.

Let λ_α, $\alpha = 1,2$, be Lipschitz in $\overline{\Omega}$ functions such that $\lambda_\alpha\big|_{\partial\Omega} = \cos(\mathbf{n}, x_\alpha)$. The following inequalities are valid:

$$\begin{aligned}\|u(\cdot,t)\|^2_{2,\partial\Omega} &\le 2\int_{\partial\Omega} |u(x,t)-u(x,t_1)|^2(\lambda_1^2+\lambda_2^2)\,dx + 2\|u(\cdot,t_1)\|^2_{2,\partial\Omega}\\ &\le 2\int_{\Omega}\Big(|u(x,t)-u(x,t_1)|^2\lambda_\alpha(x)\Big)_{x_\alpha} dx + 2\|u(\cdot,t_1)\|^2_{2,\partial\Omega}\\ &\le \|u_x(\cdot,t)\|^2_{2,\Omega} + c\|u(\cdot,t)-u(\cdot,t_1)\|^2_{2,\Omega} + 2E[u(t_1)].\end{aligned}$$

Moreover,

$$\|u(\cdot,t)-u(\cdot,t_1)\|^2_{2,\Omega} \le (t_2-t_1)\|u_t\|^2_{2,\Omega\times(t_1,t_2)} \underset{(16)}{\le} (t_2-t_1)c_1E[u(t_1)] + (t_2-t_1)c_2.$$

This implies the estimate

$$(17)\quad \frac{\nu_0}{2}\,\|u(\cdot,t)\|^2_{2,\partial\Omega} \le \frac{\nu_0}{2}\,\|u_x(\cdot,t)\|^2_{2,\Omega} + \nu_0 E[u(t_1)] + c(t_2-t_1)(E[u(t_1)]+1).$$

Now we sum (16) and (17) to obtain the inequality

$$\int_{t_1}^{t_2}\int_{\Omega} |u_t|^2 dx\,dt + \frac{\nu_0}{2}\sup_{[t_1,t_2]} E[u(t)] \le c_1(T)\,E[u(t_1)] + c_2(T), \quad \forall\, t \le t_2,$$

and estimate (13) follows.

To derive (14), fix a point $x^0 \in \partial\Omega$, $R \le \min\{1, \operatorname{diam}\Omega\}$ and set $\eta = u_t\xi^2$ in (15), where $\xi = \xi(x)$ is a cut-off function on $B_{2R}(x^0)$, $\xi = 1$ in $B_R(x^0)$. If $h_0 \neq 0$ then (14) follows immediately.

If $h_0 = 0$ in (6) then we get the inequality

$$\int_{t_1}^{t_2}\int_{\Omega_{2R}} |u_t|^2\xi^2 dx\, dt + \nu_0 \sup_{[t_1,t_2]} \| |u_x(t)|\xi\|^2_{2,\Omega_{2R}(x^0)} \\ \leq c_0(R+R^2+(t_2-t_1)) + c_1 E[u(t_1),\Omega_{2R}(x^0)] + \frac{c_2(t_2-t_1)}{R^2} E_0. \tag{18}$$

Furthermore, as above, we derive the inequality

$$\frac{\nu_0}{2}\|u(\cdot,t)\xi\|^2_{2,\gamma_{2R}} \leq \frac{\nu_0}{2}\|u_x(\cdot,t)\xi\|^2_{2,\Omega_{2R}} + \nu_0 E[u(t_1),\Omega_{2R}] + \frac{c(t_2-t_1)E_0}{R^2}. \tag{19}$$

From (18) and (19), inequality (14) follows. □

Remark 1. Taking into account the estimate $\|u_t\|^2_{2,Q} \leq E_0$ we derive that

$$\sup_{[0,T]} \|u(\cdot,t)\|^2_{2,\Omega} \leq 2TE_0 + 2\|\varphi\|^2_{2,\Omega} \equiv E_1. \tag{20}$$

Estimate (10) with $v = u$ guarantees that

$$\|u\|_{4,Q} \leq c(E_0,T). \tag{21}$$

Remark 2. The variational structure of the elliptic operator of system (2) was only assumed in order to prove Lemma 1. Later on we do not use this fact and consider our problem in the form

$$\begin{aligned} &u_t^k - \frac{d}{dx_\alpha} a_\alpha^k(x,u,u_x) + b^k(x,u,u_x) = 0, \quad (x,t)\in Q,\\ &a_\alpha^k(x,u,u_x)\cos(\mathbf{n},x_\alpha) + g^k(x,u)\big|_\Gamma = 0,\\ &u\big|_{t=0} = \varphi, \end{aligned} \tag{22}$$

where $a_\alpha^k(x,u,u_x) = f_{p_\alpha^k}(x,u,u_x)$ and $b^k(x,u,p) = f_{u^k}(x,u,p)$. From assumptions (4), (5) it follows that the functions $a = \{a_\alpha^k\}_{\alpha\leq 2}^{k\leq N}$ satisfy the natural growth conditions:

$$\begin{aligned} &|a| + |a_x| + |a_u| \leq \mu_2(1+|p|), \quad |a_p| \leq \mu_2,\\ &\frac{\partial a_\alpha^k}{\partial p_\beta^l}\xi_\alpha^k\xi_\beta^l \geq \nu|\xi|^2, \quad \forall\,\xi\in\mathbb{R}^{2n},\\ &|b| + |b_x| + |b_u| \leq \mu_2(1+|p|^2), \quad |b_p| \leq \mu_2(1+|p|). \end{aligned} \tag{23}$$

To estimate $\|u_x\|_{4,Q}$ and $\|u_{xx}\|_{2,Q}$ we study problem (2) in the local setting.

Let V be a neighborhood of a fixed point of $\partial\Omega$ such that under a C^{1+1} diffeomorphism $y = y(x)$, the set $V \cap \Omega$ is mapped to $B_2^+ = B_2 \cap \{y_2 > 0\}$ and the set $V \cap \partial\Omega$ to $\gamma_2 = B_2(0) \cap \{y_2 = 0\}$. We denote by $x = x(y)$, $y \in \overline{B_2^+}$, the inverse transformation to $y = y(x)$ and by $v(y,t) = u(x(y),t)$ a solution to the following problem:

$$
\begin{aligned}
& v_t^k - (A_\alpha^k(y,v,v_y))_{y_\alpha} + \mathbb{B}^k(y,v,v_y) = 0, \quad y \in B_2^+, \quad t \in (0,T), \\
& - A_2^k(y,v,v_y) + \hat{g}^k(y_1,v)\big|_{\gamma_2\times(0,T)} = 0, \quad k = 1,\dots,N, \\
& v\big|_{t=0} = \psi(y), \quad y \in B_2^+.
\end{aligned} \tag{24}
$$

Here

$$
\begin{aligned}
& A_\alpha^k(y,v,q) = a_\beta^k\left(x(y), u, q\,\frac{\partial y}{\partial x}\right)\frac{\partial y_\alpha}{\partial x_\beta}, \\
& \mathbb{B}^k(y,v,q) = b^k\left(x(y), u, q\,\frac{\partial y}{\partial x}\right) - A_\alpha^k(y,v,q)\,\frac{J_{y_\alpha}(y)}{J(y)}, \\
& J(y) = \left|\det \frac{\partial x(y)}{\partial y}\right| > 0 \quad \text{in} \quad \overline{B_2^+}, \\
& \psi(y) = \varphi(x(y)), \quad \hat{g}(y_1,v) = \frac{g(x(y_1,0),u)H(y_1)}{J(y_1,0)}, \\
& H(y_1) = \left(\left(\frac{\partial x_1(y_1,0)}{\partial y_1}\right)^2 + \left(\frac{\partial x_2(y_1,0)}{\partial y_1}\right)^2\right)^{1/2}, \quad |y_1| \le 2.
\end{aligned}
$$

On the set $\mathcal{M}^+ = \overline{B_2^+} \times \mathbb{R}^N \times \mathbb{R}^{2N}$, the following conditions hold (see (23)):

$$
\begin{aligned}
& |A| + |A_y| + |A_v| \le l_1(1 + |q|), \\
& |A_q| \le l_2, \quad \frac{\partial A_\alpha^k}{\partial q_\gamma^m}\,\theta_\alpha^k\theta_\gamma^m \ge \nu_*|\theta|^2, \quad \forall\,\theta \in \mathbb{R}^{2N},
\end{aligned} \tag{25}
$$

$$
|\mathbb{B}| + |\mathbb{B}_y| + |\mathbb{B}_v| \le l_3(1 + |q|^2), \quad |\mathbb{B}_q| \le l_3(1 + |q|), \tag{26}
$$

where the positive constants ν_*, $l_1,\dots,l_3$ depend on the parameters ν, μ_2 and C^{1+1} characteristics of functions $x(y)$ and $y(x)$.

Furthermore,

$$
\begin{aligned}
|\hat{g}| + |\hat{g}_{y_1}| &\le l_4(1 + |v|), \\
|\hat{g}_v| + |\hat{g}_{vy_1}| + |\hat{g}_{vv}| &\le l_4,
\end{aligned} \tag{27}
$$

with $l_4 = \text{const} > 0$ depending on h_3 (see (7)) and C^{1+1} characteristics of $y(x)$ and $x(y)$.

Remark 3. The set $\overline{\Omega}$ can be covered by a finite number of neighborhoods V^1, ..., V^M such that a C^{1+1}-diffeomorphism $y^j = y^j(x)$ is defined on the set V^j and transforms $V^j \cap \Omega$ into a standard ball or a half-ball, $j = 1, \dots, M$. We may assume that parameters $l_1, \dots, l_4$ in the local problem (24)–(27) depend on the C^1 or C^{1+1} characteristics of $\partial\Omega$, but not on the fixed mapping y^j.

Remark 4. For a fixed neighborhood V and diffeomorphism $y : V \cap \Omega \to B_2^+$ there exists a number $\lambda > 0$ such that the image of $\omega_R(y^0) = B_2^+ \cap B_R(y^o)$ under the mapping $x = x(y)$ is contained in $\Omega_{\lambda R}(z^0)$ for all $y^0 \in \overline{B_2^+}$, $z^0 = x(y^0)$ and $R < 1/2$. Below we fix the same parameter $\lambda \geq 1$ for all neighborhoods $V^1, \dots, V^M$ covering $\partial\Omega$.

Lemma 2. *Let v be a smooth solution of* (24) *in $\overline{B_2^+} \times [0, T)$. There exists a number $\varepsilon_1 > 0$ depending on the parameters ν_*, $l_1, \dots, l_4$ from conditions* (25)–(27) *such that if*

$$\sup_{[0,T)} \sup_{y^0 \in \overline{B_2^+}} \|v_y(\cdot, t)\|^2_{2,\omega_{R_1}(y^0)} < \varepsilon_1 \tag{28}$$

with some $R_1 = R_1(\varepsilon_1) > 0$, then for any $y^0 \in \overline{B_{3/2}^+}$ the following estimate holds:

$$\begin{aligned} J = \int_0^T \int_{\omega_{R/4}(y^0)} \left(|v_y|^4 + |v_{yy}|^2 \right) dy\, dt \\ \leq c \left\{ T \left(E_1 + \frac{E_0}{R^2} + 1 \right) + \|\psi_y\|^2_{2,\omega_{2R}(y^0)} \right\}, \end{aligned} \tag{29}$$

where parameters E_0 and E_1 were defined in (13) *and* (20).

Now we only comment the idea of the proof of Lemma 2.

It is easy to see that v satisfies the identity

$$\begin{aligned} \int_0^t \int_{B_2^+} \left(v^k_{y_1 t} h^k + [A^k_\alpha]_{y_1} h^k_{y_\alpha} - \mathbb{B}^k h^k_{y_1} \right) dy\, d\tau \\ + \int_0^t \int_{\gamma_2} [\hat{g}^k]_{y_1} h^k ds\, d\tau = 0, \ \forall\, t < T_0, \end{aligned} \tag{30}$$

with any smooth function $h(y, \tau)$ which vanishes in the neighborhood of the set $S_2^+ = \{|y| = 2\} \cap \{y_2 > 0\}$ for any $\tau \in [0, t]$.

Here and below we denote by $[\dots]_{y_k}$ the total derivative with respect to y_k of the expression $[\dots]$. From (30) with $h = v_{y_1} \xi^2$, ξ is a cut-off function for B_2, transforming the boundary integral over γ_2 to the integral over B_2^+ we derive the inequality

$$
(31) \quad \frac{1}{2}\int\limits_{B_2^+} |v_{y_1}|^2\xi^2 dy\Big|_0^t + \frac{\nu_*}{2}\int\limits_0^t\int\limits_{B_2^+} |(v_{y_1})_y|^2\xi^2 dy\,dt
$$
$$
\leq c_1 \int\limits_0^t\int\limits_{B_2^+} \big[(|v_y|^2+|v|^2+1)\xi^2 + |v_y|^2|\xi_y|^2 + |v_y|^4\xi^2\big]dy\,dt \equiv P.
$$

To estimate the integral $\int_0^t\int_{B_2^+} |v_{y_2y_2}|^2\xi^2 dy\,dt$ we refer to system (24) and ellipticity condition (25). After that, by (31), we obtain the inequality

$$
(32) \qquad \int\limits_0^t\int\limits_{B_2^+} |v_{yy}|^2\xi^2 dy\,dt \leq c_2\bigg\{\int\limits_0^t\int\limits_{B_2^+} |v_t|^2\xi^2 dy\,dt + P\bigg\}.
$$

The integral $I = \int_0^t\int_{B_2^+} |v_y|^4\xi^2 dy\,dt$ in the expression P is estimated with the help of the inequality

$$
\|w\|_{4,\Omega}^4 \leq 2\|w\|_{2,\Omega}^2 \cdot \|w_x\|_{2,\Omega}^2
$$

for $w = |v_y|^2\xi$ and assumption (28) with some small ε_1 in the same way as it was done in the proof of Lemma 2.1 ([1]). Then estimate (29) follows from (32), (13), (14) and (20). As a consequence of (29), we have the estimate

$$
(33) \qquad \int\limits_0^t\int\limits_{B_{3/2}^+} \big(|v_y|^4 + |v_{yy}|^2\big)dy\,dt \leq c\left\{T\left(1+E_1+\frac{E_0}{R_1^2}\right) + \|\varphi\|_{W_2^1(\Omega)}^2\right\},
$$

where $R_1 > 0$ is the constant from Lemma 2.

Remark 5. Let $\varepsilon_1 > 0$ be the same number as in Lemma 2, let $c^* > 0$ be the constant from the inequality

$$
\int\limits_{\omega_r(y^0)} |v_y(y,t)|^2 dy \leq c^* \int\limits_{\Omega_{\lambda r}(x^0)} |u_x(x,t)|^2 dx,
$$

where $y^0 \in \overline{B_{3/2}^+}$, $x^0 = x(y^0)$, $r < 1/2$. The constant c^* depends on C^1 characteristics of $\partial\Omega$ only.

Now we suppose that for the solution $u \in \mathcal{K}\{[0,T)\}$ and $\varepsilon_0 = \varepsilon_1/c_*$ there exists $R_0 = R_0(\varepsilon_0)$ such that

$$\sup_{[0,T)} \sup_{x^0 \in \overline{\Omega}} \|u_x(\cdot,t)\|^2_{2,\Omega_{R_0}(x^0)} < \varepsilon_0. \tag{34}$$

Then $v(y,t) = u(x(y),t)$ satisfies (28) with $R_1 = R_0/\lambda$ and estimate (33) follows from Lemma 2. Inequality (34) coincides with condition (11) of Theorem 1. As a result, under the assumptions of Theorem 1, we obtain the estimate

$$J \equiv \int\limits_Q (|u_x|^4 + |u_{xx}|^2)dQ \le c(T,R_0). \tag{35}$$

Lemma 3. *Let u be a solution to problem (2). If the integral $J_0 = \int_Q (|u|^4 + |u_x|^4)dQ$ is finite then there exists $t_1 \in (0,T)$ such that for all $\gamma \in [0, \nu/(4\mu_2)]$ the following estimate holds:*

$$\sup_{[t_1,T)} \int\limits_\Omega |u_t(x,t)|^{2+2\gamma}dx + \int\limits_{t_1}^T \int\limits_\Omega \big(|u_{xt}|^2|u_t|^{2\gamma} + |u_t|^{3+2\gamma}\big)dx\,dt \le \varkappa_1(t_1), \tag{36}$$

where t_1 is determined by the parameters ν, μ_2, h_3, T, by C^{1+1} characteristics of $\partial\Omega$, and by the integral J_0; the constant $\varkappa_1(t_1)$ also depends on $\|u_t(\cdot,t_1)\|_{2+2\gamma,\Omega}$.

The proof of this lemma is similar to the proof of Lemma 1.3 in [1] and we omit it here.

Remark 6. The existence of $u_{xt} \in L^2(\Omega \times (0, T-\varepsilon))$ for any $\varepsilon > 0$, follows from the assumption that $u \in \mathcal{K}\{[0,T)\}$. Estimate (36) with $\gamma = 0$ guarantees that $\|u_{xt}\|_{2,\Omega\times(t_1,T)} < +\infty$. As a result, we have got the existence of the derivatives $u_{xt} \in L^2(Q)$.

We need also a local variant of Lemma 3.

Lemma 3°. *If the assumptions of Lemma 3 hold then for some $t_1 \in (0,T)$ and any $\gamma \in [0,\nu/(4\mu_2)]$ the following inequality is valid:*

$$\begin{aligned} \sup_{[t_1,T)} \int\limits_{\Omega_R(x^0)} |u_t(x,t)|^{2+2\gamma}dx + \int\limits_{t_1}^T \int\limits_{\Omega_R(x^0)} \big(|u_t|^{2\gamma}|u_{xt}|^2 + |u_t|^{3+2\gamma}\big)dx\,dt \\ \le \varkappa_1(t_1;2R), \forall\, x^0 \in \overline{\Omega}, \quad R \le \frac{1}{2}\operatorname{diam}\Omega, \end{aligned} \tag{37}$$

where t_1 depends on the same data as in Lemma 3, and $\varkappa_1(t_1;2R)$ is determined by parameters ν, μ_2, h_3, T, by C^{1+1} characteristics of $\partial\Omega$, R^{-1} and $\|u_t(\cdot,t_1)\|_{2+2\gamma,\Omega_{2R}(x^0)}$.

Now we introduce the function class

$$Y(Q) = W_2^{2,1}(Q) \cap L^\infty((0,T); W_2^1(\Omega)).$$

If $u \in Y(Q)$ then $u_x \in V(Q)$ and from (10) with $v = u_x$ it follows that

$$\|u_x\|_{4,Q} \le c(q_0, T) \left\{ \sup_{(0,T)} \|u_x(\cdot,t)\|_{2,\Omega} + \|u_{xx}\|_{2,Q} \right\}. \tag{38}$$

Remark 7. Let u be a weak solution of (2) from the class $Y(Q)$ and let the conditions $A_1, B_1, \dots, C_2$ be valid. Then for a fixed $\tau > 0$ the following estimate holds:

$$\sup_{[\tau,T)} \int_\Omega |u_t(x,t)|^{2+2\gamma} dx + \int_\tau^T\!\!\int_\Omega \big(|u_{xt}|^2|u_t|^{2\gamma} + |u_t|^{3+2\gamma}\big) dx\, dt \le c(T, \tau^{-1}) \int_Q (1 + |u_t|^2) dQ. \tag{39}$$

Furthermore, a local variant of (39) is valid:

$$\begin{aligned} &\sup_{[\tau,T)} \int_{\Omega_R(x^0)} |u_t|^{2+2\gamma} dx + \int_\tau^T \int_{\Omega_R(x^0)} \big(|u_{xt}|^2|u_t|^{2\gamma} + |u_t|^{3+2\gamma}\big) dx\, dt \\ &\le c_0(T, \tau^{-1}, R^{-1}) \int_0^T \int_{\Omega_{2R}(x^0)} (1 + |u_t|^2) dx\, dt, \quad \forall\, x^0 \in \overline{\Omega},\ R \le \frac{1}{2} \operatorname{diam} \Omega. \end{aligned} \tag{40}$$

(To derive (39) see Remark 1.3 in [1]. To prove (40), see the derivation of inequality (1.15) in [1].)

Lemma 4. *Let v be a smooth on $[0,T)$ solution of problem (24) and let the integral $J = \int_0^T \int_{B^+_{3/2}} |v_y|^4 dy\, dt$ be finite. Then there exist $t_2 \in (0,T)$ and $\gamma_0 \le \nu/(4\mu_2)$ such that for any $\gamma \le \gamma_0$*

$$\begin{aligned} &\sup_{[t_2,T)} \int_{\omega_{R/4}(y^0)} |v_y(y,t)|^{2+2\gamma} dy \\ &\quad \le c_1 \int_{t_2}^T \int_{\omega_{2R}(y^0)} (1 + |v|^4 + |v_t|^{2+\gamma} + R^{-2}|v_y|^{2+2\gamma}) dy\, dt \\ &\quad + c_2 \int_{\omega_{2R}(y^0)} |v_y(y,t_2)|^{2+2\gamma} dy \equiv \varkappa(t_2, R), \quad \forall\, y^0 \in \overline{B_1^+},\ R \le \frac{1}{4}. \end{aligned} \tag{41}$$

The proof of Lemma 4 is similar to the proof of Lemma 3.1 in [1]. We explain only some details. It is not difficult to derive the inequality

$$
\begin{aligned}
&\frac{1}{2(1+\gamma)} \int\limits_{\omega_{2R}(y^0)} |v_y|^{2+2\gamma}\xi^2 dy\Big|_{t_2}^{t} + \frac{\nu_*}{2}\int\limits_{t_2}^{t}\int\limits_{\omega_{2R}(y^0)} |v_y|^{2\gamma}|v_{yy}|^2\xi^2 dy\,dt \\
&\le c_3 \int\limits_{t_2}^{t}\int\limits_{\omega_{2r}(y^0)} |v_y|^{4+2\gamma}\xi^2 dy\,dt + c_4\int\limits_{t_2}^{T}\int\limits_{\omega_{2r}(y^0)} (1+|v|^4+|v_t|^{2+\gamma} \\
&+ |v_y|^{2+2\gamma}(1+R^{-2}))dy\,dt, \quad \forall\, y^0\in\overline{B_1^+},\ R\le\frac{1}{4}, \quad (t_2,t)\subset(0,T),
\end{aligned}
\tag{42}
$$

where ξ is a cut-off function for $B_{2R}(y^0)$, $\xi=1$ on $B_R(y^0)$ and $t_2\ge t_1$ (t_1 is fixed in Lemma 3). By (21) and(36), we estimate the integral with a constant c_4 in (42). We denote by I_R the integral with the coefficient c_3. To estimate I_R we apply inequality (10) for the function $|v_y|^{1+\gamma}\xi$ in $\Omega\times(t_2,t)$ and deduce:

$$
\begin{aligned}
I_R \le c(T,q_0)\Big(\int\limits_{t_2}^{T}\int\limits_{\omega_{2R}(y^0)} |v_y|^4 dy\,dt\Big)^{1/2}\Big\{ &\sup_{[t_2,t]}\int\limits_{\omega_{2R}(y^0)} |v_y|^{2+2\gamma}\xi^2 dy \\
&+ \int\limits_{t_2}^{t}\int\limits_{\omega_{2R}(y^0)} (|v_y|^{2\gamma}|v_{yy}|^2\xi^2 + |v_y|^{2+2\gamma}\xi_y^2)dy\,dt\Big\}.
\end{aligned}
\tag{43}
$$

As the integral $J_0=\displaystyle\int_0^T\int_{B_{3/2}^+}|v_y|^4 dy\,dt$ is absolutely continuous, for fixed $R>0$ and some $t_2\ge t_1$ the integral $\displaystyle\int_{t_2}^T\int_{\omega_{2R}(y^0)}|v_y|^4 dy\,dt$ will be small enough and (41) follows from (42) and (43).

Remark 8. By (41), we find that

$$
\sup_{[t_2,T)} \|v_y(\cdot,t)\|_{2+2\gamma_0,B_1^+} \le K_1,
\tag{44}
$$

for some $\gamma_0>0$.

Here and below we denote by K_i different constants that may depend on the parameters from conditions (3), …, (7), T, R_0^{-1}, $C^{2+\alpha_0}$ characteristics of $\partial\Omega$, $\|\varphi\|_{C^{2+\alpha_0}(\bar\Omega)}$, $\|u_t(\cdot,t_2)\|_{2+2\gamma_0,\Omega}$, $\|u_x(\cdot,t_2)\|_{2+2\gamma_0,\Omega}$ ($t_2\in(0,T)$ and $\gamma_0\in(0,1)$ are fixed in Lemma 4).

From now on we put $\Lambda_0 = (t_2, T)$. According to Lemmas 3 and 4, we have the estimates

$$\sup_{\Lambda_0} \|u_t(\cdot,t)\|_{p,\Omega} \le K_2, \tag{45}$$

$$\sup_{\Lambda_0} \|u_x(\cdot,t)\|_{p,\Omega} \le K_3, \quad p = 2 + 2\gamma_0. \tag{46}$$

As $W_p^1(\Omega) \hookrightarrow C^\beta(\overline{\Omega})$, $\beta = 1 - 2/p > 0$, we obtain that

$$\sup_{\Lambda_0} \|u(\cdot,t)\|_{C^\beta(\bar{\Omega})} \le K_4, \quad \beta = 1 - \frac{2}{p} > 0. \tag{47}$$

Remark 9. If $u \in Y(Q)$ is a weak solution of (2) then for any fixed $\tau \in (0,T)$ and $\gamma \le \gamma_0$ (γ_0 is defined in Lemma 4), $u_x(\cdot,t) \in L^{2+2\gamma}(\Omega)$, $\forall\, t \in (2\tau, T)$ and

$$\sup_{(2\tau,T)} \int_\Omega |u_x(x,t)|^{2+2\gamma} dx \le c(1+\tau^{-1}) \int_\tau^T \int_\Omega \left(1 + |u_t|^{2+\gamma} + |u|^4 + |u_x|^{2+2\gamma}\right) dx\, dt. \tag{48}$$

To derive (48) we consider the local variant (24) of problem (2). The proof is almost the same as the proofs of Lemma 3.1 and Remark 3.3 in [1]. The appearance of a nonlinear boundary condition does not essentially change the proof.

As a consequence of (39) and (48), we obtain estimates like (45)–(47).

The next step will be explained.

Lemma 5. *There exist constants K_5 and K_6 such that*

$$\sup_{\Lambda_0} \|u(\cdot,t)\|_{C^\delta(\bar{\Omega})} \le K_5, \quad \forall\, \delta \in (0,1), \tag{49}$$

$$\sup_{\Lambda_0} \|u_x(\cdot,t)\|_{C^{\delta_0}(\bar{\Omega})} \le K_6, \quad \textit{with some} \quad \delta_0 > 0. \tag{50}$$

Proof: As always, we denote by $v = v(y,t)$ a smooth on $[0,T)$ solution to (24). For a fixed number $t \in \Lambda_0$, v is the solution to the elliptic problem

$$\begin{aligned} &-\frac{d}{dy_\alpha} A_\alpha^k(y,v,v_y) + \mathbb{B}^k(y,v,v_y) = F^k(y,t), \quad y \in B_2^+, \\ &-A_2^k(y,v,v_y) + \hat{g}^k(y_1,v)\big|_{\gamma_2} = 0, \end{aligned} \tag{51}$$

where $\gamma_2 = B_2(0) \cap \{y_2 = 0\}$, $F(y,t) = -v_t(y,t)$.

From estimates (45)–(47) it follows that

$$\|F(\cdot,t)\|_{p,B_2^+}\le K_7,\quad \|v_y(\cdot,t)\|_{p,B_2^+}\le K_8,\quad \|v(\cdot,t)\|_{C^\beta(B_2^+)}\le K_9,\qquad \beta=1-2/p>0. \tag{52}$$

For a fixed $y^0\in B^+_{3/2}$, $R\le 1/4$, we study the model problem:

$$\begin{gathered}\frac{d}{dy_\alpha}\overset{\circ}{A}{}^k_\alpha(\theta_y)=0\quad\text{in}\quad \omega_R(y^0),\\ -\overset{\circ}{A}{}^k_2(\theta_y)+\overset{\circ}{g}{}^k\big|_{\gamma_R(y^0)}=0,\quad k\le N;\quad \theta\big|_{\partial\omega_R(y)\setminus\gamma_R(y^0)}=v,\end{gathered} \tag{53}$$

where $\overset{\circ}{A}{}^k_\alpha(\theta_y)=A^k_\alpha(y^0,v^0,\theta_y)$, $v^0=\frac{1}{|\omega_R|}\int_{\omega_R(y^0)}v(y,t)\,dy$, $\overset{\circ}{g}=\hat g(y^0_1,v^0_\Gamma)$, $v^0_\Gamma=\frac{1}{|\gamma_R|}\int_{\gamma_R(y^0)}v\,ds$, $\gamma_R(y^0)=B_R(y^0)\cap\{y_2=0\}$.

The Campanato-type integral estimates were derived in [8] for solutions to (53), $\dim\omega_R=2$:

$$\int_{\omega_\rho(y^0)}|\theta_y|^2dy\le c_0\left(\frac{\rho}{R}\right)^2\int_{\omega_R(y^0)}|\theta_y|^2dy, \tag{54}$$

$$\int_{\omega_\rho(y^0)}|\theta_y-(\theta_y)_{y^0,\rho}|^2dy\le c_0\left(\frac{\rho}{R}\right)^{2+2(1-2/q)}\int_{\omega_R(y^0)}|\theta_y-(\theta_y)_{y^0,R}|^2dy, \tag{55}$$

where $(\theta_y)_{y^0,r}=\frac{1}{|\omega_r|}\int_{\omega_r(y^0)}\theta_y(y,t)\,dy$ and the constants $c_0>0$, $q>2$ depend on the parameters l_2 and ν_* from conditions (25). The integral identities for v and θ provide the following equality:

$$\begin{aligned}&\int_{\omega_R}\big\{[A^k_\alpha(y^0,v^0,v_y)-A^k_\alpha(y^0,v^0,\theta_y)]\eta^k_{y_\alpha}+\Delta A^k_\alpha\eta^k_{y_\alpha}+\mathbb{B}^k(y,v,v_y)\eta^k\big\}dy\\ &+\int_{\gamma_R}\big[\hat g^k(y_1,v)-\hat g^k(y^0,v^0_\Gamma)\big]\eta^k ds=\int_{\omega_R}F^k\eta^k dy,\end{aligned} \tag{56}$$

where $\Delta A^k_\alpha=A^k_\alpha(y,v,v_y)-A^k_\alpha(y^0,v^0,v_y)$, η is a smooth function in $\bar\omega_R$, $\eta|_{\partial\omega_R\setminus\gamma_R}=0$, $\omega_R=\omega_R(y^0)$ and $\gamma_R=\gamma_R(y^0)$.

We denote $w=v-\theta$ and set $\eta=w$ in (56) in order to derive the inequality

$$\begin{aligned}\int_{\omega_R}|w_y|^2dy\le c\int_{\omega_R}&\big\{|v_y|^2|w|+R^2(1+|v_y|^2)\\ &+|v-v^0|^2(1+|v_y|^2)+R^2|F|^2\big\}\,dy+J_R,\end{aligned} \tag{57}$$

where the integral $J_R = \int_{\gamma_R} |\hat{g}(y_1, v) - \hat{g}(y_1^0, v_\Gamma^0)|\, |w|\, ds$ is estimated according to conditions (27) by:

$$|J_R| \le c \int\limits_{\gamma_R} (R(1+|v|) + |v - v^0|)|w|\, ds \le cR^\beta \int\limits_{\gamma} |w|\, dy$$
$$= CR^\beta \Big(- \int\limits_{\omega_R} (|w|)_{y_2} dy \Big) \le \frac{1}{2} \int\limits_{\omega_R} |w_y|^2 dy + \frac{c}{\varepsilon} R^{2+2\beta}.$$

Now by (57), we deduce the inequality

$$\int\limits_{\omega_R} |w_y|^2 dy \le c_1 \{\mathbb{P}_R(y^0) + R^{4\beta} + c_F R^{2+2\beta}\}, \tag{58}$$

where $\mathbb{P}_R(y^0) = \int_{\omega_R(y^0)} |v_y|^2 |w|\, dy$, $c_F = \|F\|^2_{p, B_2^+}$, and $c_1 > 0$ depends on the parameters from (25)–(27), K_8 and K_9.

To estimate $\mathbb{P}_R(y^0)$ in (58), we consider the identity for the solution v:

$$\int\limits_{B_2^+} \left[A_\alpha^k(y, v, v_y) h^k_{y_\alpha} + \mathbb{B}^k(y, v, v_y) h^k\right] dy + \int\limits_{\gamma_2} \hat{g}^k(y_1, v) h^k\, ds = \int\limits_{B_2^+} F^k h^k dy$$

with the function $h = (v - v^0)|w|$, $y \in \omega_R(y^0)$, $h = 0$ in $B_2^+ \setminus \omega_R(y^0)$. Using estimates (52), we obtain from the last equality:

$$\mathbb{P}_R(y^0) \le c_2 R^\beta \mathbb{P}_R(y^0) + c_3 \left[\varepsilon \int\limits_{\omega_R} |w_y|^2 dy + \frac{1}{\varepsilon}\big(R^{4\beta} + c_F R^{2+2\beta}\big)\right] \tag{59}$$

for any $\varepsilon > 0$, with constants c_2 and c_3 depending on the same parameters as c_1. We put $\varepsilon = 1/(4c_1c_3)$ and suppose that the radius $R \le 1/4$ satisfies the additional restriction $c_2 R^\beta \le 1/2$. Then by (58), (59), we find that

$$\begin{gathered} \int\limits_{\omega_R(y^0)} |w_y|^2 dy \le c_4 (R^{4\beta} + c_F R^{2+2\beta}), \\ c_4 = c_4(\nu_*, l_1, \dots, l_4, K_8, K_9), \quad c_F = \|F\|^2_{p, B_2^+}. \end{gathered} \tag{60}$$

For the function $H_\rho(y^0) = \int_{\omega_\rho(y^0)} |v_y|^2 dy$, (54) and (60) imply the inequality

$$H_\rho(y^0) \le c_5 \left[\Big(\frac{\rho}{R}\Big)^2 H_R(y^0) + R^{4\beta} + c_F R^{2+2\beta}\right], \qquad \forall\, p \le R,\ c_5 = c_5(c_0, c_4). \tag{61}$$

Note that $\beta = 1 - 2/p = \gamma_0/(1+\gamma_0) < 1/4$. By a well-known algebraic lemma (see, for example, [9, Chapter III, Lemma 2.1]), we derive from (61) that

$$H_\rho(y^0) \le c_6 \left\{ \left(\frac{\rho}{R}\right)^{4\beta} H_R(y^0) + (1+c_F)\rho^{4\beta} \right\}, \quad \forall \rho \le R. \tag{62}$$

Inequality (62) is valid for any $y^0 \in \overline{B^+_{3/2}}$ and the constant c_6 does not depend on y_0. It provides the estimates

$$\|v_y(\cdot,t)\|^2_{L^{2,4\beta}(B^+_{3/2})} \le K_{10}, \quad \|v(\cdot,t)\|^2_{\mathcal{L}^{2,2+4\beta}(B^+_{3/2})} \le K_{11}. \tag{63}$$

In the case of two spatial variables, the Campanato space $\mathcal{L}^{2,2+4\beta}(B^+_{3/2})$ is isomorphic to the Hölder space $C^{\beta_1}(\overline{B^+_{3/2}})$ and

$$\|v(\cdot,t)\|_{C^{\beta_1}(\overline{B^+_{3/2}})} \le K_{12}, \quad \beta_1 = 2\beta.$$

Now we can repeat our considerations interchanging β by β_1 and $B^+_{3/2}$ by $B^+_{1+(1/4)^2}$, $R \le 1/16$. As a result, we obtain estimate (62) with β_1 instead of β and

$$\|v(\cdot,t)\|_{C^{\beta_2}(\overline{B^+_{1+(1/4)^2}})} \le K_{13}, \quad \beta_2 = 2\beta_1 = 4\beta.$$

It is obvious that for a finite number M of steps, we get to the situation $2\beta_M = 2^{M+1}\beta \ge 1$.

Then by (61) with β_M instead of β, we obtain the estimate

$$H_\rho(y^0) \le c \left\{ \left(\frac{\rho}{R}\right)^{2-2\varepsilon} H_R(y^0) + (1+c_F)\rho^{2-2\varepsilon} \right\}, \tag{64}$$

valid for any $\varepsilon > 0$, $\rho \le R \le 1/4^M$ and $y^0 \in \overline{B^+_1}$.

It ensures us that for any fixed $\varepsilon > 0$

$$\|v_y\|_{L^{2,2(1-\varepsilon)}(B^+_1)} \le K_{14}, \quad \|v(\cdot,t)\|_{\mathcal{L}^{2,2+2(1-\varepsilon)}(B^+_1)} \le K_{15}, \tag{65}$$

and as a result we have the estimate

$$\|v(\cdot,t)\|_{C^{1-\varepsilon}(\overline{B^+_1})} \le K_{16}, \quad \forall \varepsilon > 0. \tag{66}$$

From (66) the global estimate (49) follows.

To derive (50) we note that for a fixed $t \in \Lambda_0$ and $\omega_R(y^0) \subset B^+_1$, the solution v satisfies the inequalities

$$\begin{aligned} &\max_{\omega_R(y^0)} |v(\cdot,t)| + R^{\varepsilon-1} \operatorname*{osc}_{\omega_R(y^0)} v(\cdot,t) \le K_{17}, \\ &\|v_y(\cdot,t)\|^2_{2,\omega_R(y^0)} \le K_{18} R^{2(1-\varepsilon)}, \quad \forall \varepsilon > 0. \end{aligned} \tag{67}$$

Instead of (60) we have the inequality

$$\int_{\omega_R(y^0)} |w_y|^2 dy \le c(R^{4(1-\varepsilon)} + c_F R^{2+2\beta}). \tag{68}$$

Using (55) and (68) for the function $M_\rho(y^0) = \int_{\omega_\rho(y^0)} |v_y - (v_y)_{y^0,p}|^2 dy,\ \rho \le R$, we derive:

$$M_\rho(y^0) \le c\left\{\left(\frac{\rho}{R}\right)^{2+2\beta_0} M_R(y^0) + R^{4(1-\varepsilon)} + c_F R^{2+2\beta}\right\}, \quad \forall\, \rho \le R, \tag{69}$$

where $\beta_0 = 1 - 2/q > 0$, $q > 2$ is the exponent from (55).

We put $\varepsilon = (1-\beta)/2,\ \hat\beta = \min(\beta_0, \beta)$. Due to the algebraic lemma mentioned above, it follows from (69) that

$$M_\rho(y^0) \le c\left\{\left(\frac{\rho}{R}\right)^{2+2\delta_0} M_R(y^0) + c_F \rho^{2+2\delta_0}\right\},\ \forall\, \rho \le R, \quad \text{if} \quad \delta_0 < \hat\beta. \tag{70}$$

Inequality (70) is valid for any $y^0 \in \overline{B^+_{1/2}}$ and $R \le 1/2$. It provides that

$$\|v_y(\cdot,t)\|^2_{\mathcal{L}^{2,2+2\delta_0}(B^+_{1/2})} \le c\{\|v_y(\cdot,t)\|^2_{2,B^+_1} + \|v_t(\cdot,t)\|^2_{p,B^+_1}\} \le K_{19}.$$

As a consequence, we get the estimate

$$\|v_y(\cdot,t)\|_{\mathcal{C}^{\delta_0}(\overline{B^+_{1/2}})} \le K_{20},$$

and now (50) follows. $\square$

Lemma 6. *The following estimates hold:*

$$\|u\|_{\mathcal{C}^{\delta,\delta_1}(\bar Q_0)} \le K_{21}, \quad \forall\, \delta \in (0,1), \quad \delta_1 = \frac{\delta}{2(1+\delta)}, \tag{71}$$

$$\|u_x\|_{\mathcal{C}^{\gamma,\gamma/2}(\bar Q_0)} \le K_{22} \quad \textit{for some} \quad \gamma \in (0,1), \quad Q_0 = \Omega \times \Lambda_0. \tag{72}$$

Proof: Inequality (71) is a consequence of the estimate $\|u_t\|_{2,Q} \le E_0$ and relation (49). (See, for example, [10, Lemma 4].) It is known that estimates (50) and (71) guarantee the validity of (72) ([4, Chapter 2, Lemma 3.1]). In (72) $\gamma = \frac{2\delta_0\delta_1}{1+\delta_0}$, where δ_0 and δ_1 are the exponents from (50) and (71), respectively. $\square$

Remark 10. In the case of the *quasilinear* system (2) (see Remark 1), the information we have got in Lemma 6 is sufficient to consider problem (2) as a linear one and to derive further regularity of $u(x,t)$. In the case of nonlinear Cauchy-Neumann problem (2) estimates (71) and (72) do not guarantee stronger regularity of the solution in the frame of the linear theory. In such situation some additional considerations are required.

We also recall here that estimates (71), (72) provided further regularity of a solution of the Cauchy-Dirichlet problem both for nonlinear and quasilinear operators ([1], [2]).

Now we shall describe some regularity results for weak solutions of the linear boundary-value problem in a local coordinate system.

Assume that $Q^+ = B_2^+ \times \Lambda$, $\Gamma^+ = \gamma_2 \times \Lambda$, $B_2^+ \subset \mathbb{R}^2$, $\Lambda = (t_0, T)$ with any $t_0 < T$.

Let $S: Q^+ \to \mathbb{R}^N$, be a weak solution of the problem

$$(73)\qquad \begin{aligned} &S_t^k - (\mathcal{A}_{kl}^{\alpha\beta}(\xi) S_{y_\beta}^l + r_{km}^\alpha(\xi) S^m + \lambda_\alpha^k(\xi))_{y_\alpha} + M_{kl}^\alpha(\xi) S_{y_\alpha}^l \\ &+ N_{kl}(\xi) S^l + \mathbb{P}^k(\xi) = 0, \quad \xi = (y,t) \in Q^+, \\ &\mathcal{A}_{kl}^{2\beta}(\xi) S_{y_\beta}^l + r_{km}^2(\xi) S^m + \lambda_2^k(\xi) + D_{kl}(\xi) S^l + d^k(\xi)\big|_{\Gamma^+} = 0, \\ &S\big|_{t=t_0} = \rho(\xi), \quad \xi \in B_2^+. \end{aligned}$$

We suppose that the following conditions hold:

I. $\mathcal{A} \in C^\gamma(\overline{Q^+}; \delta)$, $\rho \in C^\gamma(\overline{B_2^+})$, where $\gamma \in (0,1)$ is a fixed number; $r, M, N \in L^\infty(Q^+)$, $\lambda \in L^{2,2+2\gamma}(Q^+;\delta)$, $\mathbb{P} \in L^{2,2\gamma}(Q^+;\delta)$, $D \in L^\infty(\Gamma^+)$, $d \in L^{2,1+2\gamma}(\Gamma^+;\delta)$; $\langle \mathcal{A}(\xi)\eta, \eta\rangle \ge \nu|\eta|^2$ for any $\eta \in \mathbb{R}^{2N}$ and $\forall\, \xi \in Q^+$, $\nu = \text{const} > 0$.

II. In addition to **I** we suppose that

$$r, \lambda \in C^\gamma(\overline{Q^+}; \delta), \quad D, d \in C^\gamma(\Gamma^+; \delta), \quad \mathbb{P} \in \mathcal{L}^{2,2+2\gamma}(Q^+; \delta), \quad \rho \in C^{1+\gamma}(\overline{B_2^+})$$

and the compatibility condition holds:

$$(74)\qquad \mathcal{A}_{kl}^{2\beta}(\xi) \rho_{y_\beta}^l + r_{km}^2(\xi) \rho^m + \lambda_2^k(\xi) + D_{kl}(\xi) \rho^l + d^k(\xi)\big|_{\substack{\xi \in \gamma_2 \\ t = t_0}} = 0, \quad k \le N.$$

Proposition 1. *Let $S \in V(Q^+)$ be a solution to the linear problem* (73).

(1) *If conditions* **I** *hold then $S \in C^\gamma(\overline{Q'}; \delta)$, $S_y \in L^{2,2+2\gamma}(Q';\delta)$, $Q' = B_{3/2}^+ \times \Lambda$, and the following estimate is valid:*

$$(75)\qquad \begin{aligned} &\|S\|_{C^\gamma(\overline{Q'};\delta)} + \|S_y\|_{L^{2,2+2\gamma}(Q';\delta)} \le c\big\{\|S\|_{Q^+} + \|\lambda\|_{L^{2,2+2\gamma}(Q';\delta)} \\ &+ \|\mathbb{P}\|_{L^{2,2\gamma}(Q';\delta)} + \|d\|_{2,L^{2,1+2\gamma}(\Gamma^+;\delta)} + \|\rho\|_{C^\gamma(\overline{B_2^+})}\big\}, \end{aligned}$$

with the constant c depending on ν, $\|\mathcal{A}\|_{C^\gamma(\overline{Q'};\delta)}$, $\|r\|_{L^\infty(Q^+)}$, $\|M\|_{L^\infty(Q^+)}$, $\|N\|_{L^\infty(Q^+)}$ and $\|D\|_{L^\infty(\Gamma^+)}$.

(2) *If conditions* **II** *hold then $S, S_y \in C^\gamma(\overline{Q'}; \delta)$ and*

$$(76)\qquad \begin{aligned} &\|S\|_{C^\gamma(\overline{Q'};\delta)} + \|S_y\|_{C^\gamma(\overline{Q'};\delta)} \le c\big\{\|S\|_{Q^+} + \|\lambda\|_{C^\gamma(\overline{Q^+};\delta)} \\ &+ \|\mathbb{P}\|_{\mathcal{L}^{2,2+2\gamma}(Q';\delta)} + \|d\|_{C^\gamma(\Gamma^+;\delta)} + \|\rho\|_{C^{1+\gamma}(\overline{B_2^+})}\big\}, \end{aligned}$$

with the constant c depending on ν, $\|\mathcal{A}\|_{C^\gamma((\overline{Q^+};\delta)}$, $\|r\|_{C^\gamma(\overline{Q^+};\delta)}$, $\|M\|_{L^\infty(Q^+)}$, $\|N\|_{L^\infty(Q^+)}$ *and* $\|D\|_{C^\gamma(\Gamma^+;\delta)}$.

The second statement of Proposition 1 (the case $r = M = N = D = 0$) was proved in [5, Theorem 2.1]. Here we are interested in the result for the complete form (73) of linear operators. Analyzing the proof of the mentioned theorem it is not difficult to verify both of the statements of Proposition 1.

Now we continue the proof of Theorem 1.

Lemma 7. *Under assumptions of Theorem 1, $u_t, u_{xx} \in \mathcal{C}^{\alpha_0}(\overline{Q^0};\delta)$, and $u_{xt} \in L^{2,2+2\alpha_0}(Q^0;\delta)$, where $Q^0 = \Omega \times \Lambda_0$, $\Lambda_0 = (t_2, T)$.*

Proof: Suppose that $v(y,t) = u(x(y),t)$ is a solution of the local problem (see (24)):

$$\begin{aligned} &v_t^k - \frac{d}{dy_\alpha}\left(A_\alpha^k(y,v,v_y)\right) + \mathbb{B}^k(y,v,v_y) = 0, \quad (y,t) \in Q^+ = B_2^+ \times \Lambda_0, \\ &-A_2^k(y,v,v_y) + \hat{g}^k(y_1,v)\big|_{\Gamma^+} = 0, \quad k \le N. \end{aligned} \tag{77}$$

From Lemma 6 it follows that

$$\|v\|_{\mathcal{C}^\beta(\overline{Q^+};\delta)} + \|v_y\|_{\mathcal{C}^\beta(\overline{Q^+};\delta)} \le K_{23} \tag{78}$$

with some $\beta \in (0,1)$.

Moreover, due to estimates (13), (35) and Remark 6 we know that

$$|\!|\!|v_t|\!|\!|_{Q^+} \le K_{24}, \quad |\!|\!|v_y|\!|\!|_{Q^+} \le K_{25}, \tag{79}$$

where we denote by $|\!|\!|w|\!|\!|_{Q^+}$ the norm in the space $V(Q^+)$:

$$|\!|\!|w|\!|\!|_{Q^+}^2 = \sup_{\Lambda_0} \|w(\cdot,t)\|_{2,B_2^+}^2 + \|w_y\|_{2,Q^+}^2.$$

Now we differentiate in t system and boundary condition (77). Function $\theta = v_t$ is a solution from $V(Q^+)$ of the linear problem

$$\begin{aligned} &\theta_t^k - (\mathcal{A}_{kl}^{\alpha\beta}(\xi)\theta_{y_\beta}^l + r_{km}^\alpha(\xi)\theta^m)_{y_\alpha} + M_{kl}^\alpha(\xi)\theta_{y_\beta}^l + N_{kl}(\xi)\theta^l = 0, \ \xi \in Q^+, \\ &-(\mathcal{A}_{kl}^{2\beta}(\xi)\theta_{y_\beta}^l + r_{km}^2(\xi)\theta^m) + D_{kl}(\xi)\theta^l\big|_{\Gamma^+} = 0, \quad k \le N, \end{aligned} \tag{80}$$

where

$$\mathcal{A}_{kl}^{\alpha\beta}(\xi) = \frac{\partial A_\alpha^k}{\partial p_\beta^l}(\xi, v(\xi), v_y(\xi)), \quad r_{km}^\alpha(\xi) = \frac{\partial A_\alpha^k(\dots)}{\partial v^m}, \quad M_{kl}^\alpha(\xi) = \frac{\partial \mathbb{B}^k(\dots)}{\partial p_\beta^l},$$

$$N_{kl}^\alpha(\xi) = \frac{\partial \mathbb{B}^k(\dots)}{\partial v^l}, \quad \forall\, \xi \in Q^+, \quad D_{kl}(\xi) = -\frac{\partial \hat{g}^k}{\partial v^l}(y_1, t, v(\xi)), \quad \xi \in \Gamma^+.$$

We denote by $(\dots)$ the same arguments as functions $\frac{\partial a^k_\alpha}{\partial p^l_\beta}$ have.

Function θ satisfies the initial condition $\theta\big|_{t=t_2} = \rho$, where

$$\rho = \frac{d}{dy_\alpha} A^k_\alpha(y, v(y,t_2), v_y(y,t_2)) - \mathbb{B}^k(y, v(y,t_2), v_y(y,t_2)) \in C^{\alpha_0}(\overline{B^+_2});$$

α_0 is the exponent from condition B_2, $\|\rho\|_{C^{\alpha_0}(\overline{B^+_2})} \le c(1 + \|v(y,t_2)\|_{C^{2+\alpha_0}(\overline{B^+_2})})$.

Now we can assert that all coefficients of problem (80) satisfy conditions **I** of Proposition 1 with $\gamma = \beta \cdot \alpha_0$, β is defined in (78), $\lambda^k_\alpha = P^k = d^k = 0$, $\Lambda = \Lambda_0$.

In a result we obtain for $S = \theta = v_t$ the estimate (75). From it follows that

$$\|v_t\|_{C^\gamma(\overline{Q'};\delta)} + \|v_{ty}\|_{L^{2,2+2\gamma}(Q';\delta)} \le c\{K_{23} + K_{24} + \|v(\cdot,t_2)\|_{C^{2+\alpha_0}(\overline{B^+_2})} + 1\} \equiv K_{26}. \tag{81}$$

Differentiating (77) with respect to y_1, we derive that $\theta = v_{y_1}$ is a solution of the linear problem that is similar to (80). In this case, the coefficients of the linear system and of the boundary condition satisfy the conditions **II** of Proposition 1 with $\gamma = \beta\alpha_0$ and $\theta\big|_{t=t_2} = v_{y_1}(y,t_2) \in C^{1+\alpha_0}(\overline{B^+_2})$. Furthermore, for the linear system the compatibility condition holds on the set $\{y \in \gamma_2,\ t = t_2\}$. It provides estimate (76) for $S = \theta = v_{y_1}$ and

$$\|v_{y_1}\|_{C^\gamma(\overline{Q'};\delta)} + \|(v_{y_1})_y\|_{C^\gamma(\overline{Q'};\delta)} \le c\{K_{23} + K_{25} + \|v(\cdot,t_2)\|_{C^{2+\alpha_0}(\overline{B^+_2})} + 1\} \equiv K_{27}. \tag{82}$$

By (81) and (82), we derive from system (77) that

$$\|v_{y_2y_2}\|_{C^\gamma(\overline{Q'};\delta)} \le K_{28}. \tag{83}$$

Now we assert that estimate (78) is valid in Q' with $\beta = 1$. It provides that $v_t, v_{yy} \in C^{\alpha_0}(\overline{Q''};\delta)$ and $v_{ty} \in L^{2,2+2\alpha_0}(Q'';\delta)$, where $Q'' = B^+_1 \times \Lambda_0$. The result of Lemma 7 follows. □

Remark 11. It is assumed in Theorem 1 that $u \in \mathcal{K}\{[0,T']\}$ for any $T' < T$. Considering the local problem (77) in the cylinder $B^+_2 \times (0,T')$ and repeating the proof of Lemma 7, one can derive that $v_{yt} \in L^{2,2+2\alpha_0}(B^+_1 \times (0,T');\delta)$. It implies that $u_{xt} \in L^{2,2+2\alpha_0}(\Omega \times (0,T');\delta)$. Taking into account the result of Lemma 7 we obtain that $u_{xt} \in L^{2,2+2\alpha_0}(Q;\delta)$.

The last step to prove Theorem 1 is the estimation of $\langle u_x\rangle^{(1+\alpha_0)/2}_{t,Q_0}$.

Definition. A bounded domain Ω is said to be of type (A) if for a fixed number $A > 0$ and all $x \in \Omega$ and $r < \operatorname{diam}\Omega$,

$$|\Omega_r(x)| \ge Ar^n.$$

Now we prove the following statement.

Proposition 2. *Let Ω be a bounded domain of type (A) in $\mathbb{R}^n$, $n \geq 2$, $Q = \Omega \times (0,T)$. Suppose that the function $w\colon \overline{Q} \to \mathbb{R}^N$ is continuously differentiable in $\overline{Q}$ with respect to $x_1, \dots, x_n$ and that $w_t \in L^{2,n+2\alpha}(Q;\delta)$ for some $\alpha \in (0,1)$. If*

$$\langle w_x \rangle_{t,Q}^{\alpha/2} = l_1 < +\infty, \quad \|w_t\|_{L^{2,n+2\alpha}(Q;\delta)} = l_2,$$

then there exists a constant $c = c(l_1, l_2)$ such that

$$\langle w \rangle_{t,Q}^{(1+\alpha)/2} \leq c.$$

Proof: We fix $x \in \overline{\Omega}$, $t, t' \in [0,T]$, $t < t'$, and denote $\Delta = t' - t > 0$, $R = \Delta^{1/2}$. For $y \in \Omega_R(x)$ we have the inequalities

$$\begin{aligned}
|w(x,t) - w(x,t')| &\leq \left| \int_0^1 \frac{d[w(y+s(x-y),t) - w(y+s(x-y),t')]}{ds} \right| \\
&+ |w(y,t) - w(y,t')| \underset{(*)}{\leq} \left| \int_0^1 [w_{y_j}(\tilde{y},t) - w_{y_j}(\tilde{y},t')]\, ds\ (x_j - y_j) \right| \\
&+ \int_t^{t'} |w_\tau(y,\tau)|\, d\tau \leq l_1 \Delta^{\alpha/2} R + \int_t^{t'} |w_\tau(y,\tau)|\, d\tau.
\end{aligned}$$

Inequality $(*)$ holds for almost all $y \in \Omega_R(x)$, $\tilde{y} = y + s(x-y)$. Now we integrate the result with respect to y over $\Omega_R(x)$ and divide by $|\Omega_R|$:

$$\begin{aligned}
|w(x,t) - w(x,t')| &\leq l_1 \Delta^{(1+\alpha)/2} + \frac{|\Omega_R|^{1/2} |\Delta|^{1/2}}{AR^n} \left(\int_t^{t'} \int_{\Omega_R(x)} |w_\tau(y,\tau)|^2 dy\, d\tau \right)^{1/2} \\
&\leq (l_1 + c(A,n) l_2) \Delta^{(1+\alpha)/2}.
\end{aligned}$$

We apply Proposition 2 to the function $w = u_x$ on $Q_0 = \Omega \times \Lambda_0$, where u is the solution of (2) under investigation. Here $n = 2$, $\alpha = \alpha_0$ and the estimates of $\langle u_{xx} \rangle_{t,Q_0}^{\alpha_0/2}$ and $\|u_{xt}\|_{L^{2,2+2\alpha_0}(Q_0;\delta)}$ were derived in Lemma 7. It implies that

$$\langle u_x \rangle_{t,Q_0}^{(1+\alpha_0)/2} \leq K_{29}. \tag{84}$$

From Lemma 7, Remark 11 and estimate (84) it follows that $u \in \mathcal{K}\{[0,T]\}$. Theorem 1 is proved. $\square$

Remark 12. Suppose that for a domain $\Omega_1 \subset \Omega$ the inequality

$$\sup_{[0,T)} \sup_{x^0 \in \bar{\Omega}_1} \|u_x(\cdot,t)\|^2_{2,B_{R_0}(x^0)} < \varepsilon_0 \tag{11'}$$

holds instead of (11) with some R_0 and ε_0 as in Theorem 1. Then $u \in \mathcal{K}\{\overline{\Omega}_0 \times [0,T]\}$ for any $\Omega_0 \subset \Omega_1$ with $\operatorname{dist}(\Gamma_0, \Gamma_1) > 0$, $\Gamma_i = \partial\Omega_i \cap \Omega$, $i = 0,1$. To prove this assertion we analyze the proof of Theorem 1. In particular, we take into account the local estimates (14), (29), (37), (41) and the local method of the proof of Lemmas 5 and 7.

Further, analyzing the proof of Theorem 1, we can state the smoothness result for the solution $u \in Y(Q) = W_2^{2,1}(Q) \cap L^\infty((0,T); W_2^1(\Omega))$.

Theorem 2. *Suppose that for a fixed $\alpha_0 \in (0,1)$ conditions A_1, B_1, B_2, C_1 and C_2 hold. If $u \in Y(Q)$ is a solution to problem* (2) *then $u \in \mathcal{K}\{(0,T]\}$ and the derivatives $u_{xt} \in L^{2,2+2\alpha_0}(\Omega \times (\delta, T))$ for any $\delta > 0$.*

Proof: From Remarks 7 and 9 it follows that for any $\tau > 0$

$$\sup_{[\tau,T)} \|u_t(\cdot,t)\|_{p,\Omega} \leq M_1, \quad \sup_{[\tau,T)} \|u_x(\cdot,t)\|_{p,\Omega} \leq M_2 \tag{85}$$

with some $p > 2$, M_1 and M_2 being constants depending on the parameters from conditions (3)–(7), on $C^{2+\alpha_0}$-characteristics of $\partial\Omega$, T, $\|\varphi\|_{W_2^1(\Omega)}$, $\|u\|_{Y(Q)}$ and τ^{-1}. By estimate (85), we derive higher regularity of u in the same way as it was done in Theorem 1. As a result, $u \in \mathcal{K}\{[\tau,T]\}$ and $u_{xt} \in L^{2,2+2\alpha_0}(\Omega \times ((\tau,T);\delta)$. □

To construct a weak global in time solution of problem (2) we shall use the following uniqueness result.

Theorem 2′. *Problem* (2) *has not more than one solution in the class $Y(Q)$.*

The proof of Theorem 2′ is trivial when taking into account that $\operatorname{diam}\Omega = 2$ and, in particular, applying inequality (10) (see [2, Theorem 3] for the case of the Dirichlet boundary condition).

On the singular set of the solution.

To describe the singular set, we follow M. Struwe's idea [3]. Suppose that $u \in \mathcal{K}\{[0,T)\}$ is a solution of problem (2) and $T > 0$ defines the maximal interval of the existence of the smooth solution. It means that it is impossible to extend $u(x,t)$ as a smooth function up to $t = T$. According to Theorem 1, there exists a point $(\hat{x}, T)$, $\hat{x} \in \overline{\Omega}$, where condition (11) is not fulfilled, that is

$$\varlimsup_{t \nearrow T} \|u_x(\cdot,t)\|^2_{2,\Omega_R(\hat{x})} \geq \varepsilon_0, \tag{86}$$

where $\varepsilon_0 > 0$ is defined by the data of (2). Let σ denote the set of all such points $\hat{x}$ from $\overline{\Omega}$ and put $\Sigma_T = \sigma \times \{T\}$.

Let us fix points $x^1, \dots, x^M \in \sigma$ and choose a number $R \in (0,1)$ such that $B_{2R}(x^i) \cap B_{2R}(x^j) = \emptyset$ for any $i \neq j$, $i,j \leq M$ and $c_3 R < \varepsilon_0/16$. (Here and below, $c_3, \dots, c_5$ are the constants from inequality (14).)

We fix a positive number θ from the condition $\theta(c_3 + c_5 E_0) < \varepsilon_0/8$ and choose a $t^k \in (0,T)$ such that $t^k \geq T - \theta R^2 \equiv \hat{t}$ for any $k = 1, \dots, M$ and

$$\|u_x(\cdot, t^k)\|^2_{2,\Omega_R(x^k)} \geq \frac{\varepsilon_0}{2}\,. \tag{87}$$

By inequality (14), we obtain the estimate

$$\|u_x(\cdot, t^k)\|^2_{2,\Omega_R(x^k)} < \frac{\varepsilon_0}{4} + c_4 E\big[u(\hat{t}), \Omega_{2R}(x^k)\big]. \tag{88}$$

From (87)and (88) it follows that

$$c_4 E\big[u(\hat{t}), \Omega_{2R}(x^k)\big] > \frac{\varepsilon_0}{4}\,.$$

Taking into account that $\sup\limits_{[0,T]} E[u(t)] \leq E_0$ (see (13)), we have

$$E_0 \geq \sum_{k=1}^{M} E\big[u(\hat{t}), \Omega_{2R}(x^k)\big] > \frac{\varepsilon_0}{4c_4}\, M.$$

It means that $M < 4c_4 E_0/\varepsilon_0$, i.e., the singular set σ consists of at most a finite number of points.

Moreover, Remark 12 allows us to assert that the solution u can be extended smoothly to the set $\overline{Q} \setminus \Sigma_T$. We have proved the following result.

Theorem 3. *Suppose that conditions A_1–C_2 hold and $u \in \mathcal{K}\{[0,T)\}$ is a solution of problem* (2). *If $T > 0$ defines the maximal interval of the existence of a smooth solution u then there exist at most a finite number of points $\hat{x}^1, \dots, \hat{x}^M$ in $\overline{\Omega}$ such that the function u loses its smoothness in $(\hat{x}^j, T)$, $j \leq M$, more exactly*

$$\varlimsup_{t \nearrow T} \|u_x(\cdot, t)\|^2_{2,\Omega_R(\hat{x}^j)} > \varepsilon_0, \quad \forall\, R > 0,$$

where ε_0 is defined by parameters from conditions (3)–(7).

Remark 13. If we suppose that $h_0 \neq 0$ in (6) then the constants $c_1, \dots, c_5$ in (13), (14) are independent of T. In this case, analyzing the proof of Theorem 3, one can show that M (the number of singular points of the solution) is dominated by a constant that does not depend on T. The same fact is valid if we consider $G = 0$ on $\partial\Omega$ and put $E[u(t)] = \|u_x(t)\|^2_{2,\Omega}$ in (13) and (14).

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