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Locally minimal topological groups and their embeddings into products of o-bounded groups

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Abstract. It is proven that an infinite-dimensional Banach space (considered as an Abelian topological group) is not topologically isomorphic to a subgroup of a product of σ-compact (or more generally, o-bounded) topological groups. This answers a question of M. Tkachenko.

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In this paper we answer in the negative the following question of M. Tkachenko posed in [Tk, Problem 3.1]: *Does every second countable topological group embed into a product of σ-compact groups?* Namely, we show that an infinite-dimensional Banach space (considered as an Abelian topological group) admits no such an embedding. In fact, we prove a bit more: no infinite-dimensional Banach space admits an embedding into a product of o-bounded groups.

Let us recall some definitions, see [Tk]. All topological groups considered in this note are Hausdorff. A subset B of a topological group G is defined to be *totally bounded* if for every neighborhood U of the origin in G there exists a finite set $F \subset G$ such that $B \subset (F \cdot U) \cap (U \cdot F)$. A topological group G is defined to be *σ-bounded* if G is a countable union $G = \bigcup_{n=1}^{\infty} B_n$ of totally bounded subsets.

A topological group G is defined to be *$\aleph_0$-bounded* if for every neighborhood U of the origin in G there exists a subset $F \subset G$ with $|F| \leq \aleph_0$ and $G = F \cdot U$, see [Gu]. It is known that each second countable group is $\aleph_0$-bounded and each $\aleph_0$-bounded group embeds into a product of second countable groups ([Gu]).

A topological group G is called *o-bounded* if for every sequence $(U_n)_{n\in\omega}$ of neighborhoods of the origin in G there exists a sequence $(F_n)_{n\in\omega}$ of finite subsets in G such that $G = \bigcup_{n\in\omega} F_n \cdot U_n$, see [Tk, 3.9], [He].

According to [Tk] for a topological group G we have the implications

(σ-bounded)$\Rightarrow$(o-bounded)$\Rightarrow$($\aleph_0$-bounded),

no of which can be reversed. The considered three classes of groups are closed with respect to the operations of taking subgroups and continuous homomorphic images. M. Tkachenko asked in [Tk, Problem 3.1] if every $\aleph_0$-bounded group embeds isomorphically into a product of σ-bounded groups.

The following theorem answers this question in the negative.

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Main Theorem. *An infinite-dimensional Banach space* (*considered as an Abelian topological group*) *admits no isomorphic embedding into a product of o-bounded groups.*

As a by-product of the proof we get a characterization of Lie groups in terms of embeddings into products of o-bounded groups as well as a theorem on equivalence of o-boundedness and σ-boundedness for groups which are continuous homomorphic images of second countable Weil complete groups. A topological group is called *Weil complete* if it is complete in its left (equivalently, right) uniformity.

Interplay between o-boundedness and σ-boundedness

The main result of this section is

Equivalence Theorem. *Suppose that a topological group G is a continuous homomorphic image of a second countable Weil complete group. The group G is o-bounded if and only if it is σ-bounded.*

Proof: By hypothesis there exists a surjective continuous group homomorphism $h : H \to G$, where H is a second countable Weil complete group. Let d be any left-invariant complete metric on H and $B(\varepsilon) = \{x \in H : d(x,e) \le \varepsilon\}$, $\varepsilon > 0$, denote the closed ε-ball around the neutral element e of the group H.

The "if" part of the theorem is trivial. To prove the "only if" part, suppose G is an o-bounded group. We claim that the image $h(U)$ of some neighborhood U of the identity in H is left-bounded in H, i.e., for every neighborhood W of the identity in G there is a finite subset $F \subset G$ with $h(U) \subset F \cdot W$.

Assume that it is not so. To get a contradiction, we shall show that the group G is not o-bounded. For this we shall construct by induction a sequence $(\varepsilon_n)_{n=1}^\infty \subset (0,1]$ of real numbers and a sequence $(U_n)_{n=1}^\infty$ of neighborhoods of the origin in G such that

(1) $h(B(\varepsilon_n/2)) \not\subset F \cdot U_n \cdot U_n^{-1}$ for any finite set $F \subset G$;
(2) $h(B(\varepsilon_{n+1})) \subset U_n$;
(3) $\varepsilon_n \le \varepsilon_{n-1}/2$.

Let $\varepsilon_1 = 1$ and assume that for some n numbers $\varepsilon_1, \dots, \varepsilon_n$ and neighborhoods $U_1, \dots, U_{n-1}$ satisfying the conditions (1)–(3) have been constructed. By our assumption, the set $h(B(\varepsilon_n/2))$ is not left bounded in G. Hence, there exists a neighborhood $W \subset G$ of the origin such that $h(B(\varepsilon_n/2)) \not\subset F \cdot W$ for any finite set $F \subset G$. Let U_n be a neighborhood of the origin in G such that $U_n \cdot U_n^{-1} \subset W$. Clearly, the condition (1) is satisfied. Finally, using the continuity of h, choose any ε_{n+1} to satisfy $0 < \varepsilon_{n+1} \le \varepsilon_n/2$ and $h(B(\varepsilon_{n+1})) \subset U_n$. This finishes the inductive construction of the sequences $(\varepsilon_n)_{n=1}^\infty$ and $(U_n)_{n=1}^\infty$.

It rests to verify that $\bigcup_{n=1}^\infty F_n \cdot U_n \ne G$ for any sequence $(F_n)_{n=1}^\infty$ of finite subsets of G. For this, given such a sequence (F_n), we shall construct inductively a sequence $(x_n)_{n=1}^\infty$ of points of the group H such that the following conditions are satisfied for every $n \ge 1$:

(4) $h(x_n \cdot B(\varepsilon_n)) \cap F_{n-1} \cdot U_{n-1} = \emptyset$;
(5) $x_{n+1} \in x_n \cdot B(\varepsilon_n/2)$.

Let $x_0 = e$ and assume that for some $n \geq 0$ the points $x_0, \dots, x_n$ satisfying (4) and (5) have been defined. It follows from (1) that $h(x_n \cdot B(\varepsilon_n/2)) \not\subset F_n \cdot U_n \cdot U_n^{-1}$ and hence there exists a point $x_{n+1} \in x_n \cdot B(\varepsilon_n/2)$ with $h(x_{n+1}) \notin F_n \cdot U_n \cdot U_n^{-1}$. Multiplying this by U_n, we get $h(x_{n+1}) \cdot U_n \cap F_n \cdot U_n = \emptyset$. Then by (2), $h(x_{n+1} \cdot B(\varepsilon_{n+1})) \cap F_n \cdot U_n = \emptyset$. This finishes the construction of the sequence $(x_n)_{n=1}^{\infty}$.

It follows from (3) and (5) that the sequence $(x_n)_{n=1}^{\infty}$ is Cauchy with respect to the metric d and thus converges to some point $x_\infty \in H$. We claim that $x_\infty \in x_n \cdot B(\varepsilon_n)$ for every $n \geq 1$. Indeed, using (5) and (3), we get

$$d(x_\infty, x_n) \leq \sum_{i=n}^{\infty} d(x_i, x_{i+1}) \leq \sum_{i=n}^{\infty} \frac{\varepsilon_i}{2} \leq \sum_{i=n}^{\infty} \frac{\varepsilon_n}{2 \cdot 2^{i-n}} = \varepsilon_n.$$

Then by (4), $h(x_\infty) \notin F_n \cdot U_n$ for every $n \geq 1$ which implies $h(x_\infty) \notin \bigcup_{n=1}^{\infty} F_n \cdot U_n$ and $G \neq \bigcup_{n=1} F_n \cdot U_n$.

This contradiction shows that the image $h(U)$ of some symmetric neighborhood $U = U^{-1}$ of the identity in H is left bounded in G. Then for every points $x, y \in G$ the set $x \cdot h(U)$ is left bounded while the set $x \cdot h(U) \cap h(U) \cdot y$ is totally bounded. Now fix a dense countable subset $(d_n)_{n \in \omega}$ in H. Then $H = \bigcup_{i,j \in \omega} (d_i \cdot U) \cap (U \cdot d_j)$ and consequently, $G = h(H) = \bigcup_{i,j \in \omega} (h(d_i) \cdot h(U)) \cap (h(U) \cdot h(d_j))$ is a countable union of totally bounded subsets. □

Locally minimal groups

Recall that a topological group G is called *minimal* if G admits no strictly weaker Hausdorff group topology.

We define a topological group G to be *locally minimal* if there exists a neighborhood U of the origin in G such that G admits no strictly weaker Hausdorff group topology for which U is a neighborhood of the origin.

Clearly, each minimal group is locally minimal. It can be easily shown that each locally compact group is locally minimal. There are also non-locally compact locally minimal groups:

Proposition 1. *A normed linear space* (*considered as an Abelian topological group*) *is locally minimal.*

Proof: Let X be a normed linear space and let B denote the unit open ball in X with the center at the origin. Suppose τ is a weaker Hausdorff group topology on X such that B is a neighborhood of the origin in (X, τ). To prove our proposition it suffices to verify that for every $n \in \mathbb{N}$ the set $\frac{1}{n}B = \{x \in X : \|x\| < \frac{1}{n}\}$ is a neighborhood of the origin in (X, τ). Let $U \subset B$ be an open neighborhood of the origin in (X, τ). By the continuity of the group operation on (X, τ), the set $V = \{x \in X : nx \in U\}$ is open (X, τ). We claim that $V \subset \frac{1}{n}B$. Indeed, assuming the converse, we would find $x \in V$ with $\|x\| \geq \frac{1}{n}$. Then $\|nx\| \geq 1$, a contradiction with $nx \in U \subset B$. □

We call a topological group G a *group without small subgroups* if there exists a neighborhood of the origin in G containing no non-trivial subgroup. It is easy to see that each normed space is a group without small subgroups. Locally minimal groups without small subgroups have the following remarkable property.

Proposition 2. *Let $G \subset \prod_{i\in\mathcal{I}} G_i$ be a subgroup of a product of topological groups. If G is a locally minimal group without small subgroup, then there exists a finite subset $F \subset \mathcal{I}$ such that the projection $\mathrm{pr}_F : G \to \prod_{i\in F} G_i$ is an isomorphic embedding.*

Proof: Let U be a neighborhood of the origin in G containing no non-trivial subgroup and V be a neighborhood of the origin in G such that G admits no strictly weaker Hausdorff group topology for which V remains a neighborhood of the origin. By definition of the product topology on $\prod_{i\in\mathcal{I}} G_i$, there exists a finite subset $F \subset \mathcal{I}$ and a neighborhood W of the origin e of the group $\prod_{i\in F} G_i$ such that $\mathrm{pr}_F^{-1}(W) \subset U \cap V$. We claim that the projection $\mathrm{pr}_F : G \to \prod_{i\in F} G_i$ is an isomorphic embedding. Observe that $\mathrm{pr}_F^{-1}(e) \subset U$ is a trivial subgroup of G (by the choice of U) and thus the map $pr_F : G \to \prod_{i\in F} G_i$ is injective. Then $\tau = \{\mathrm{pr}_F^{-1}(O) : O$ is an open subset in $\prod_{i\in F} G_i\}$ is a weaker Hausdorff group topology on G. Since V is a neighborhood of the origin in (G, τ), the topology τ coincides with the original topology of the group G and thus the map $\mathrm{pr}_F : G \to \prod_{i\in F} G_i$ is an isomorphic embedding. □

Problem. *Investigate the class of locally minimal groups.*

A characterization of Lie groups

Characterization Theorem. *A second countable group G is a Lie group if and only if the following conditions are satisfied:*

1. *G is a locally minimal Weil complete group without small subgroups;*
2. *G embeds isomorphically into a product of o-bounded groups.*

Proof: The "only if" part of the theorem is trivial. To prove the "if" part, suppose that a second countable group G satisfies the conditions (1)–(2). By Proposition 2, the group G embeds isomorphically into a finite product $G_1 \times \cdots \times G_n$ of o-bounded groups. Since subgroups of o-bounded groups are o-bounded, we may assume that the projection of G on each G_i coincides with G_i. Then according to Equivalence Theorem, each G_i, being a continuous homomorphic image of a Weil complete group G, is σ-bounded. Consequently, the product $G_1 \times \cdots \times G_n$ as well as its subgroup G is σ-bounded. Now Weil completeness of G implies that G is σ-compact and hence, being second countable, must be locally compact. Since G has no small subgroups, G is a Lie group according to the well known Gleason-Montgomery-Zippin Theorem ([Gl], [MZ]). □

Question. *Is Characterization Theorem valid for Raĭkov complete groups, i.e., groups complete with respect to the two-sided uniformity?*

Proof of Main Theorem

Suppose that an infinite-dimensional Banach space X embeds into a product $\prod_{i\in\mathcal{I}} G_i$ of o-bounded groups. The groups G_i, being o-bounded, are $\aleph_0$-bounded. Then the subgroup X of their product $\prod_{i\in\mathcal{I}} G_i$ is $\aleph_0$-bounded ([Gu]). Next, the group X, being metrizable and $\aleph_0$-bounded, is second countable. Thus X is a second countable Weil complete abelian group without small subgroups (see Proposition 1) which embeds into a product of o-bounded groups. By Characterization Theorem, X must be a Lie group, a contradiction with the infinite-dimensionality of X. □

References

[Gl] Gleason A.M., *Groups without small subgroups*, Ann. Math. **56** (1952), 193–212.

[Gu] Guran I., *On topological groups close to being Lindelöf*, Soviet Math. Dokl. **23** (1981), 173–175.

[He] Hernández C., *Topological groups close to being σ-compact*, Topology Appl. **102** (2000), 101–111.

[MZ] Montgomery D., Zippin L., *Topological transformation groups*, Interscience, N.Y., 1955.

[Tk] Tkachenko M., *Introduction to topological groups*, Topology Appl. **86** (1998), 179–231.

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