On quasigroups with the left loop property

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Abstract. Some properties of quasigroups with the left loop property are investigated. In loops we point out that the left loop property is closely related to the left Bol identity and the particular case of homogeneous loops is considered.

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1. Introduction

In [1] V.D. Belousov raised the following problem: which identities guarantee that quasigroups possessing these identities are, in fact, loops? ([1, p. 217]. Note that Belousov attributes this problem to I.E. Burmistrovich.) The results of K. Kunen ([8], [9]) could be included in the context of solving this problem.

In this note we consider in quasigroups a special identity, the so-called left loop property. Our investigations stem from the works of A.A. Ungar [15], [16], where he showed the weight of the (left) loop property in Special Relativity and in the loop-theoretical interpretation of weakly associative groups and gyrogroups (see also [13], [14], [5]). In particular, he showed the usefulness of the left loop property in solving the equation $x.a = b$ in a given weakly associative group and thus such an algebraic structure is endowed with a quasigroup structure and, hence, is a loop.

Although the left loop property in an arbitrary quasigroup fails to imply that this quasigroup is a loop (see Example 4.1), for some classes of quasigroups the left loop property implies that these quasigroups are, in fact, loops (and by Theorem 3.2 they are left Bol loops). It turns out that, from the standpoint of Universal Algebra, the left loop property and its mirror (see Section 2) are certainly of interest. Indeed the left loop property in a left quasigroup implies that such a quasigroup is a left loop and, likewise, the mirror of the left loop property in a right quasigroup implies that such a quasigroup is a right loop (Theorem 2.3). Therefore any quasigroup satisfying both the left loop property and its mirror is a loop (Corollary 2.4). The situation of loops with the left loop property may be summarized as follows: any loop with the left loop property is a left Bol loop (Theorem 3.2).

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The present paper is organized as follows. In Section 2 we investigate the impact of the left loop property and its mirror in arbitrary quasigroups and also in some special types of quasigroups such as left or right inverse property quasigroups, totally symmetric or semisymmetric quasigroups, left or right alternative quasigroups. Section 3 deals with loops with the left loop property. These loops are left Bol loops and their connection with homogeneous loops is considered. Finally in Section 4, we suggest some constructions of nonloop quasigroups that have the left loop property but fail its mirror and conversely.

We resume the mirroring of identities (see [8], [9]) so that most of our statements or proofs are related primarily to the left loop property; the mirror statements ([8], [9]) then can be easily inferred.

2. The left loop property in quasigroups

We begin with some basic notions on quasigroups.

Let Q be a set. A system $(Q, ., \backslash, /)$ is called a *quasigroup* provided that each of $(Q, .)$, $(Q, \backslash)$, $(Q, /)$ is a groupoid and

$$a.(a\backslash b) = b, \quad a\backslash(a.b) = b \tag{1}$$

$$(a.b)/b = a, \quad (a/b).b = a \tag{2}$$

for every a, b in Q. A system $(Q, ., \backslash)$ such that each of $(Q, .)$, $(Q, \backslash)$ is a groupoid and the identities (1) hold, is called a *left quasigroup*. Likewise a *right quasigroup* is defined.

A quasigroup is also defined by means of left translations $L_a : Q \to Q, x \mapsto a.x$ and right translations $R_a : Q \to Q$, $y \mapsto y.a$, with a in Q: if L_a and R_a are permutations of Q for all a in Q, then $(Q, .)$ is a quasigroup. Note that $a\backslash b = L_a^{-1}b$ and $b/a = R_a^{-1}b$.

A *left loop* L is defined to be a left quasigroup with a right identity (i.e. an element $e \in L$ such that $x.e = x$, $\forall\, x \in L$), while a *right loop* R is a right quasigroup with a left identity (i.e. an element $e' \in R$ such that $e'.y = y$, $\forall\, y \in R$). A *loop* is a quasigroup with a two-sided identity element.

For further information on quasigroups and loops one may refer to [1], [3], [10]. In order to reduce the number of brackets, we shall use juxtaposition in place of dot whenever applicable that is, for example, $xy.z$ means $(x.y).z$.

For our purpose we make the following

Definition 2.1. A (left) quasigroup Q is called a *left loop property* (*left*) *quasigroup* (LLP (*left*) *quasigroup* for short) if the identity

$$(x.y)\backslash(x.yz) = (xy.y)\backslash(xy.yz) \quad \text{(left loop property (LLP))} \tag{3}$$

holds for any x, y, z in Q.

The identity (3) may be written as $l_{x,y}z = l_{x.y,y}z$, where $l_{a,b} = L_{a.b}^{-1}L_aL_b$ are the *left inner mappings* of Q. Beside (3), we shall consider its *mirror* in a (right) quasigroup Q, that is

$$(zy\,.\,yx)/(y\,.\,yx) = (zy\,.\,x)/(yx) \quad (MLLP) \tag{4}$$

for all x, y, z in Q.

Definition 2.2. A (right) quasigroup in which the identity (4) holds is called a MLLP (*right*) *quasigroup.*

The identity (4) may also be written as $r_{y,x}z = r_{y,y.x}z$, where $r_{a,b} = R_{a.b}^{-1}R_bR_a$. Instead of calling (4) the "right loop property" we prefer the term of "MLLP", where "M" stands for "mirror", in order to avoid confusion with Ungar's *right loop property* that reads $l_{x,y}z = l_{x,y.x}z$.

Lemma 2.1. *In a* (*left*) *quasigroup, the left loop property* (*LLP*) *is equivalent to*

$$(x\,.\,(z\backslash x))\,.\,y = x\,.\,(z\backslash(x\,.\,y)). \tag{5}$$

Likewise, in a (*right*) *quasigroup the mirror* (4)*, i.e. MLLP, is equivalent to*

$$((y\,.\,x)/z)\,.\,x = y\,.\,((x/z)\,.\,x). \tag{6}$$

Proof: We prove the first equivalence.
Let u be the unique element defined by $u = (x\,.\,y)\backslash(x\,.\,yz)$. Then, from LLP, we have $u = (xy\,.\,y)\backslash(xy\,.\,(x\backslash(xy\,.\,u)))$ that is $(xy\,.\,y)u = (xy)\,.\,(x\backslash(xy\,.\,u))$. Further, replacing y by $x\backslash v$, we get $(v\,.\,(x\backslash v))\,.\,u = v\,.\,(x\backslash(v\,.\,u))$ which is (5) since u ranges through the given quasigroup as x, y, z do (recall that $l_{x,y}$ are permutations).

Conversely, assume (5). Then replacing $z\backslash x$ by b, we have $(zb\,.\,b)\,.\,y = (zb)\,.\,(z\backslash(zb\,.\,y))$, i.e.

$$y = (zb\,.\,b)\backslash(zb\,.\,(z\backslash(zb\,.\,y))). \tag{7}$$

Now let w be the unique element defined by $w = b\backslash(z\backslash(zb\,.\,y))$, then from (7) we draw $(z\,.\,b)\backslash(z\,.\,bw) = (zb\,.\,b)\backslash(zb\,.\,bw)$ so that we get LLP. $\square$

Note that (5) and (6) are also mirrors of each other.

The identity $(a\,.\,ba)\,.\,c = a\,.\,(b\,.\,ac)$ in a quasigroup $(Q, .)$ is called the *left Bol identity*. A quasigroup $(Q, .)$ is said to have the *left inverse property* if there exists a permutation $J_l : a \mapsto a^l$ of Q such that $a^l\,.\,ax = x$ for every x in Q. Likewise $(Q, .)$ is said to have the *right inverse property* if there exists a permutation $J_r : a \mapsto a^r$ of Q such that $xa\,.\,a^r = x$ for every x in Q. If $(Q, .)$ has both the left and right inverse properties, then $(Q, .)$ is said to have the *inverse property*.

Corollary 2.2. *Any LLP quasigroup with the left inverse property is a left Bol quasigroup and any MLLP quasigroup with the right inverse property is a right Bol quasigroup.*

Proof: The left inverse property means that $L_a^{-1} = L_{a^\lambda}$, where a^λ is uniquely defined by $a^\lambda . ax = x$. Then the identity (5) can now be written as $(x . z^\lambda x) . y = x . (z^\lambda . xy)$ and one recognizes the left Bol identity. $\square$

Bol quasigroups are studied by D.A. Robinson in [12], where he constructed nonloop quasigroups that satisfy the right Bol identity but not the left one and he showed that neither of the Bol identities implies that a quasigroup is a loop. However, we recall that every quasigroup satisfying the right Bol identity has a left identity ([8]. We note that this result is attributed to A.C. Choudhury; see A.C. Choudhury, *Quasigroups and Nonassociative Systems*, I, Bull. Calcutta Math. Soc., **40** (1948), 183–194). One also observes that the right and left Bol identities are mirrors of each other. In this respect we point out the following

Theorem 2.3. *Any LLP quasigroup has a right identity and, therefore, any LLP left quasigroup is a left loop. A MLLP right quasigroup is a right loop.*

Proof: Let Q be a LLP quasigroup and consider some fixed elements x and e in Q such that $x . e = x$. Then, for any z in Q, $x . (z\backslash x) = x . (z\backslash (x . e)) = (x . (z\backslash x)) . e$ (by (5)). Next, let a be the unique element in Q defined by $a = x . (z\backslash x)$. Then we get $a = a . e$. And since a ranges through Q as z does, we conclude that e is a right identity in Q. If Q is a left quasigroup, then LLP implies that Q is a left loop. $\square$

The following corollary is obvious.

Corollary 2.4. *Any quasigroup satisfying both LLP and MLLP is a loop.* $\square$

As in the corollary above, LLP (or MLLP) together with some additional conditions in a given quasigroup turn this quasigroup into a loop. A quasigroup Q is said to be *semisymmetric* if the identity $y . xy = x$ (or $xy . x = y$) holds in Q. A *totally symmetric* (TS) quasigroup is a commutative semisymmetric quasigroup. The fundamental concepts on TS quasigroups could be found in [2]. A TS quasigroup which is a loop is called a *Steiner loop*. There is a close relationship between TS quasigroups and commutative Moufang loops ([6], [10]).

Theorem 2.5. *Let Q be a LLP quasigroup. If Q is semisymmetric, then Q is left alternative and hence, is a loop.*

Proof: Consider LLP in a quasigroup Q. Then the semisymmetric property in (5) implies $(x . xz) . y = x . (xy . z)$ (we observe that the semisymmetric property $y . xy = x$ means that $xy = y\backslash x$). Now set $z = e$ (where e is a right identity in Q) in this equality and we get the *left alternative law* $xx . y = x . xy$ in Q. But the left alternative law in a quasigroup implies there is a left identity ([9, Lemma 2.2]). Thus Q has a right and a left identity (they necessarily coincide) so that Q is a loop. $\square$

We observe that MLLP also turns a semisymmetric quasigroup into a loop.

Theorem 2.6. *Let Q be a LLP quasigroup. If Q is totally symmetric then Q is a commutative Moufang loop of exponent 2 and is a Steiner loop.*

Proof: Being totally symmetric, Q is commutative and has the inverse property (indeed, the commutativity in the semisymmetric property $y\,.\,xy = x$ implies $y\,.\,yx = x$ and $xy\,.\,y = x$ so that we get the left and right inverse properties). Therefore, by Corollary 2.2, Q turns out to be a commutative Moufang loop and is of exponent 2 by the total symmetry, and also Q is Steiner. □

Theorem 2.7. *Let Q be a LLP quasigroup. If Q is right alternative then the equation $(x\,.\,yy)\,.\,z = xy\,.\,yz$ in Q implies that Q is a group.*

Proof: The equation $(x\,.\,yy)\,.\,z = xy\,.\,yz$ implies that a quasigroup is a loop ([9, Theorem 3.2]). Then LLP (see (3)) together with the right alternative law imply the associativity $x\,.\,yz = xy\,.\,z$ in Q. Indeed, the right alternative law in (3) implies $(x\,.\,y)\backslash(x\,.\,yz) = (x\,.\,yy)\backslash(xy\,.\,yz)$ and thus $(x\,.\,y)\backslash(x\,.\,yz) = z$. □

One notes that $(x\,.\,yy)\,.\,z = xy\,.\,yz$ is one of the Fenyves' Bol-Moufang type identities ([4, identity (32)]). In Section 4 we mention a result by K. Kunen regarding the existence of a right alternative nonloop quasigroup and we point out that such a quasigroup is also a LLP quasigroup.

3. The left loop property in loops

We observe in this section that the occurrence of LLP (or MLLP) in loops leads to some well-known properties, one of the most important of which is the left (or right) Bol identity. The statements below are related to LLP.

Lemma 3.1. *Let Q be a LLP loop. Then*

(i) *Q has the left inverse property;*
(ii) *Q is left alternative.*

Proof: Let 1 denote the identity of Q and let $a^\lambda \in Q$ be the unique element defined by $a^\lambda a = 1$ for any $a \in Q$. In the left loop property (3), set $x = y^\lambda$ to obtain $y^\lambda\,.\,yz = z$, $\forall\, y, z \in Q$, which proves (i). Next, for (ii), one needs only to set $x = 1$ in (3). One also observes that (ii) is obtained by setting $z = 1$ in (5). □

A (*left*) *Bol loop* is a loop that satisfies the (left) Bol identity. Bol loops are investigated by D.A. Robinson [11] (see also [1]). As a straightforward consequence of the left inverse property in LLP loops we have the following

Theorem 3.2. *Any LLP loop is a left Bol loop.*

Proof: One needs only to apply Lemma 3.1 (i) and Corollary 2.2. □

The case of LLP loops that are homogeneous are particularly of interest in the context of gyrogroups (For the concept of a gyrogroup one may refer to [16]).

Homogeneous loops are defined to be loops with the left inverse property for which the left inner mappings $l_{a,b}$ are automorphisms for every a and b ([7]). In [5] we pointed out that nongyrocommutative gyrogroups could be seen as homogeneous loops with the left loop property. Therein ([5, Theorem 7]) we obtained that a *left inverse property loop* that has the left loop property is a left Bol loop. Thus the results of the present section (specifically Theorem 3.2) raise the redundancy of the assumption of the left inverse property that we made in [5].

4. Examples

Here we construct some nonloop quasigroups with either LLP or MLLP.

Example 4.1. *Let Z_3 be the field of integers modulo 3. Then $(Z_3, \circ)$ with $x \circ y = x + 2y$ is a nonloop quasigroup that satisfies the left loop property and $(Z_3, \circ)$ is not a MLLP quasigroup.*

Proof: Clearly $(Z_3, \circ)$ is a quasigroup. But $(Z_3, \circ)$ is not a loop since 0 is a right identity for $(Z_3, \circ)$ but not a left identity. Next we define $y \backslash x = 2(x - y)$. Then a straightforward computation shows that $(Z_3, \circ, \backslash)$ is a LLP quasigroup by Lemma 2.1. Defining $x/y = x - 2y$, one checks that $(Z_3, \circ, /)$ does not satisfy (6) and hence $(Z_3, \circ)$ is not a MLLP quasigroup. □

With Example 4.1 in mind it is easy to construct a nonloop quasigroup that satisfies MLLP (i.e. (6)) but not LLP. In fact we have the following statement whose proof is just the "mirror" of the one of Example 4.1.

Example 4.2. *Let Z_3 be the field of integers modulo 3. Then $(Z_3, \star)$ with $x \star y = 2x + y$ is a nonloop quasigroup that satisfies MLLP but $(Z_3, \star)$ is not a LLP quasigroup.*

Proof: $(Z_3, \star)$ is easily seen to be a quasigroup with 0 as its left but not right identity. Defining $x/y = 2(x - y)$ and $y \backslash x = x - 2y$, we see that $(Z_3, \star)$ satisfies (6) but fails (5). Thus, by Lemma 2.1, $(Z_3, \star)$ is a MLLP nonloop quasigroup that is not LLP. □

Note that the quasigroups constructed in Examples 4.1 and 4.2 are *unipotent*, that is $x^2 = y^2$ for all x, y in Z_3.

We observe that the constructions in Examples 4.1 and 4.2 above could be generalized. Such a generalization produces a set of nonloop quasigroups satisfying either the left loop property or its mirror. We actually have the following assertion whose proof is directly deduced from the ones of Examples 4.1 and 4.2.

Example 4.3. *Let p be a prime, $p > 2$, and a, b some positive integers. Let $(Z_p, .)$ be a groupoid with the operation product defined by $x \,.\, y = ax + by$. Then for $a = 1$ and $1 < b < p$, $(Z_p, .)$ is a LLP nonloop quasigroup that is not MLLP and for $1 < a < p$ and $b = 1$, $(Z_p, .)$ is a MLLP nonloop quasigroup that is not LLP.* □

A nonloop quasigroup with the right alternative law (RALT) is constructed as follows ([9, Lemma 2.3]): let $(Z_6, .)$ be a groupoid with $x.y = x + f(y)$, where f

is defined by $f(0) = 0$, $f(1) = 4$, $f(2) = 5$, $f(3) = 3$, $f(4) = 1$, $f(5) = 2$. Then $(Z_6, .)$ is the desired quasigroup. Next we define $x\backslash y = f(y - x)$ on $(Z_6, .)$. Then observing that f is its own inverse, we easily see that $(Z_6, .)$ satisfies (5). Now, since the right alternative law and the left alternative law (LALT) are mirrors of each other, we deduce that the mirror of $(Z_6, .)$ has both the properties (6) and LALT. Thus, with our Lemma 2.1 in mind, we get the following

Example 4.4. *There are nonloop quasigroups satisfying each of the pairs LLP-RALT and MLLP-LALT.*

□

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