On Mazurkiewicz sets

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Abstract. A Mazurkiewicz set M is a subset of a plane with the property that each straight line intersects M in exactly two points. We modify the original construction to obtain a Mazurkiewicz set which does not contain vertices of an equilateral triangle or a square. This answers some questions by L.D. Loveland and S.M. Loveland. We also use similar methods to construct a bounded noncompact, nonconnected generalized Mazurkiewicz set.

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By a Mazurkiewicz set (shortly M-set) we mean a subset X of the plane such that every straight line intersects X in exactly two points. It was constructed in [3] using transfinite induction. The notion was generalized in two directions: to generalized Mazurkiewicz sets (GM-sets) and to sets with the double midset property (DMP-sets). Let us recall that a subset X of the plane is a GM-set if it contains at least two points and each line that separates two points of X intersects X in exactly two points. A subset X of the plane is a DMP-set if it contains at least two points and the perpendicular bisector of every segment joining two points in X intersect X in exactly two points. It follows from the definitions that every M-set is a GM-set and every GM-set is an DMP-set. For more information about these notions see [2]. In the same article the authors ask some questions related to the subject. Here we answer some of them in a more general case constructing, using transfinite induction, an M-set with some additional geometrical properties. Namely, the M-set that does not contain vertices of an equilateral triangle or vertices of a square, and whose image under the inversion with respect to the unit circle is a bounded, noncompact, nonconnected GM-set.

We will need some denotation. The symbol $\mathfrak{c}$ denotes the cardinal number continuum, i.e. the first ordinal number whose cardinality is the cardinality of reals. All the constructions are going to be done in the complex plane $\mathbb{C}$. Given $x, y \in \mathbb{C}$ the symbol $l(x, y)$ denotes the line through x and y if $x \neq y$, and $l(x, x) = \{x\}$. If both x and y are distinct from 0, and $x \neq y$, then $c(x, y)$ is the circle that contains x, y and 0. Moreover we put $c(x, x) = \{x\}$. For a subset $A \subset \mathbb{C}$ we put $L(A) = \bigcup\{l(x, y) : x, y \in A\}$ and $C(A) = \bigcup\{c(x, y) : x, y \in A\}$.

We denote by B the open unit disk in the plane, i.e. $B = \{z \in \mathbb{C} : |z| < 1\}$.

Theorem. *There is an M-set A satisfying the following conditions:*

(1) *A does not contain vertices of an equilateral triangle;*
(2) *A does not contain vertices of a right isosceles triangle;*
(3) *$A \cap \operatorname{cl} B = \emptyset$;*
(4) *any circle that contains 0 and is not contained in $\operatorname{cl} B$ intersects A at exactly two points.*

Proof: Given two different points $a, b \in \mathbb{C}$ define $P(a,b)$ as the set of all points $x \in \mathbb{C}$ such that the triangle with vertices a, b, x is an equilateral one or a right isosceles one. Thus $P(a,b)$ has exactly eight points. In particular we have $P(0,1) = \{i, -i, 1+i, 1-i, \frac{1}{2} + \frac{\sqrt{2}}{2}i, \frac{1}{2} + \frac{-\sqrt{2}}{2}i, \frac{1}{2} + \frac{\sqrt{3}}{2}i, \frac{1}{2} + \frac{-\sqrt{3}}{2}i\}$. Additionally we put $P(x,x) = \emptyset$. For a set $A \subset \mathbb{C}$ let $P(A) = \bigcup\{P(x,y) : x, y \in A\}$.

Let $\{l_\alpha : \alpha \le \mathfrak{c}\}$ be the set of all straight lines in the plane, and let $\{c_\alpha : \alpha \le \mathfrak{c}\}$ be the set of all circles passing through 0 and not contained in $\operatorname{cl} B$. We will define, for $\alpha < \mathfrak{c}$, the set A_α, and $A = \bigcup\{A_\alpha : \alpha < \mathfrak{c}\}$ will be the required M-set. Assume that, for some $\alpha < \mathfrak{c}$, the sets A_β for $\beta < \alpha$, have been defined satisfying the following conditions:

(1_β) $\operatorname{card}(A_\beta) < \mathfrak{c}$;
(2_β) for every $\gamma < \beta$ we have $A_\gamma \subset A_\beta$;
(3_β) $A_\beta \cap l_\beta$ is a two point set;
(4_β) $A_\beta \cap c_\beta$ is a two point set;
(5_β) $A_\beta \cap P(A_\beta) = \emptyset$;
(6_β) A_β contains no three colinear points;
(7_β) there is no circle in the plane that contains three different points of A_β and the point 0;
(8_β) $A_\beta \cap \operatorname{cl} B = \emptyset$.

Put $N_\alpha = \bigcup\{A_\beta : \beta < \alpha\}$. Then

- $\operatorname{card}(P(N_\alpha)) < \mathfrak{c}$,
- $\operatorname{card}(l_\alpha \cap (\bigcup\{l(x,y) : x, y \in N_\alpha\})) < \mathfrak{c}$,
- $\operatorname{card}(c_\alpha \cap (\bigcup\{c(x,y) : x, y \in N_\alpha\})) < \mathfrak{c}$,
- $\operatorname{card}(l_\alpha \cap N_\alpha) \le 2$,
- $\operatorname{card}(c_\alpha \cap N_\alpha) \le 2$.

Thus we can choose points x_α, y_α, z_α, t_α that satisfy the following conditions, where $G_\alpha = \operatorname{cl} B \cup P(N_\alpha) \cup L(N_\alpha) \cup C(N_\alpha)$.

- $x_\alpha, y_\alpha \in l_\alpha \setminus c_\alpha$,
- $z_\alpha, t_\alpha \in c_\alpha \setminus l_\alpha$,
- if $\operatorname{card}(l_\alpha \cap N_\alpha) = 2$, then $\{x_\alpha, y_\alpha\} = l_\alpha \cap N_\alpha$,
- if $\operatorname{card}(c_\alpha \cap N_\alpha) = 2$, then $\{z_\alpha, t_\alpha\} = c_\alpha \cap N_\alpha$,
- if $\operatorname{card}(l_\alpha \cap N_\alpha) = 1$, then $\{x_\alpha\} = l_\alpha \cap N_\alpha$ and $y_\alpha \notin G_\alpha$,
- if $\operatorname{card}(c_\alpha \cap N_\alpha) = 1$, then $\{z_\alpha\} = c_\alpha \cap N_\alpha$ and $t_\alpha \notin G_\alpha$,
- if $\operatorname{card}(l_\alpha \cap N_\alpha) = 0$, then $x_\alpha, y_\alpha \notin G_\alpha$,
- if $\operatorname{card}(c_\alpha \cap N_\alpha) = 0$, then $z_\alpha, t_\alpha \notin G_\alpha$.

Finally put $A_\alpha = N_\alpha \cup \{x_\alpha, y_\alpha, z_\alpha, t_\alpha\}$. One can verify that, by the construction, conditions (1_α)–(8_α) are satisfied. Putting $A = \bigcup\{A_\alpha : \alpha < \mathfrak{c}\}$ we see that A is the required M-set. This finishes the proof. $\square$

Remark 1. In [2, Questions 2 and 3, p. 488] the authors asked if there is a DMP-set in the plane that does not contain vertices of a square (Question 2) and if there is a DMP-set in the plane that does not contain vertices of an equilateral triangle (Question 3). Because every M-set is a DMP-set, the Theorem answers both questions.

Denote by $h : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ the inversion with respect to the unit circle, i.e. $h(z) = 1/\overline{z}$. Observe that $h(h(z)) = z$.

Proposition. *Let A be an M-set that satisfies conditions* (3) *and* (4) *of the Theorem. Then $h(A)$ is a GM-set.*

Proof: First observe that $h(A) \subset B$ by condition (3). Let l be a line that separates two points of $h(A)$. If $0 \in l$, then $h(l \setminus \{0\}) = l \setminus \{0\}$. If $0 \notin l$, then $h(l)$ is a circle passing through 0 and not contained in B. In any case $h(l) \cap A$ is a two point set by (4), and therefore $h(h(l)) \cap h(A) = l \cap h(A)$ is a two point set, as required. $\square$

Remark 2. In [2, Question 6, p. 490] the authors ask the following question. Is there a bounded GM-set which is not a simple closed curve? Is a bounded GM-set necessarily closed? Connected? Since the constructed set $h(A)$ is a bounded GM-set homeomorphic to an M-set, it is neither closed (M-sets are not bounded, so not compact) nor connected (M-sets are zerodimensional, see [1, Theorem 2, p. 553]). Thus the Proposition answers in the negative all of the three questions. It also answers more particular Question 7 and partially Question 8.

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