# Pointwise convergence and the Wadge hierarchy

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*Abstract.* We show that if $X$ is a $\Sigma^1_1$ separable metrizable space which is not $\sigma$-compact then $C^*_p(X)$, the space of bounded real-valued continuous functions on $X$ with the topology of pointwise convergence, is Borel-$\Pi^1_1$-complete. Assuming projective determinacy we show that if $X$ is projective not $\sigma$-compact and $n$ is least such that $X$ is $\Sigma^1_n$ then $C_p(X)$, the space of real-valued continuous functions on $X$ with the topology of pointwise convergence, is Borel-$\Pi^1_n$-complete. We also prove a simultaneous improvement of theorems of Christensen and Kechris regarding the complexity of a subset of the hyperspace of the closed sets of a Polish space.

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## 1. Introduction

In this paper we study, in the framework of descriptive set theory, the space $C_p(X)$ of real-valued continuous functions on a separable metrizable space $X$ equipped with the topology of pointwise convergence. Let $C^*_p(X)$ be the space of all continuous bounded real-valued functions on $X$ with the topology inherited from $C_p(X)$. In particular we are interested in classifying $C_p(X)$ and $C^*_p(X)$ within the Wadge hierarchy.

If $A \subseteq X$ and $B \subseteq Y$ are subsets of two Polish (i.e., separable completely metrizable) spaces then we say that *$A$ is Wadge reducible to $B$*, and write $A \leq_{\mathrm{W}} B$, if there exists a continuous function (called a *reduction of $A$ to $B$*) $f : X \to Y$ such that $f^{-1}(B) = A$. The preordering $\leq_{\mathrm{W}}$ obviously induces an equivalence relation on the class of all subsets of Polish spaces, whose equivalence classes are called the *Wadge degrees*, and a partial ordering on the Wadge degrees, which is called the *Wadge hierarchy*. The Wadge degree of a set is a measure of its complexity. For background information on the Wadge hierarchy, which has been first studied in [16], we refer the reader to [8].

Let $\mathbf{\Gamma}$ denote any of the classes of subsets of Polish spaces $\mathbf{\Sigma}^0_\alpha$, $\mathbf{\Pi}^0_\alpha$ (where $\alpha$ is a nonzero countable ordinal), $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ (where $n$ is finite nonzero) — see [8] for definitions: in particular $\mathbf{\Sigma}^1_1$ is the class of all continuous images of Polish spaces. In this case write $\check{\mathbf{\Gamma}}$ for the class where $\mathbf{\Pi}$ and $\mathbf{\Sigma}$ are interchanged. If

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$\mathbf{\Gamma} \supseteq \mathbf{\Pi}^0_2$ we say that a separable metrizable space is in $\mathbf{\Gamma}$ if it is in $\mathbf{\Gamma}$ in one, and hence by Lavrentev's theorem ([11, p. 431]) in all, of its completions. A separable metrizable space is Borel (resp. projective) if it is Borel (resp. projective) in one (any) of its completions. We say that a set $A$ (or a space $X$) is $\mathbf{\Gamma}$-*hard* if for every zero-dimensional Polish space $Z$, and every $B \subseteq Z$ which is in $\mathbf{\Gamma}$, we have $B \leq_{\mathrm{W}} A$. If a $\mathbf{\Gamma}$-hard set is also in $\mathbf{\Gamma}$ then we say that it is $\mathbf{\Gamma}$-*complete*. It is obvious that the collection of all $\mathbf{\Gamma}$-complete sets is a Wadge degree. A set is *true* $\mathbf{\Gamma}$ if it is in $\mathbf{\Gamma}$ but not in $\check{\mathbf{\Gamma}}$.

A problem immediately arises in trying to study the Wadge degrees of $C_p(X)$ and $C^*_p(X)$: when $X$ is uncountable the topologies of $C_p(X)$ and $C^*_p(X)$ are not first countable and hence not metrizable; therefore these spaces cannot be viewed as subsets of a Polish space. Thus the study of $C_p(X)$ and $C^*_p(X)$ does not appear to be amenable to the standard techniques of descriptive set theory. However in descriptive set theory one is often interested in the Borel structure of a space and studies *measurable spaces* $(X, \mathsf{S})$, where $\mathsf{S}$ is a $\sigma$-algebra of subsets of $X$. If $\tau$ is a topology on a set $X$ we denote by $\mathsf{B}(\tau)$ the $\sigma$-algebra of the Borel (with respect to $\tau$) subsets of $X$. Following the terminology of [8] we say that a measurable space $(X, \mathsf{S})$ is *standard Borel* if there is a Polish topology $\tau$ on $X$ such that $\mathsf{S} = \mathsf{B}(\tau)$. The measurable space $(X, \mathsf{S})$ is $\mathbf{\Sigma}^1_n$ (resp. $\mathbf{\Pi}^1_n$) if there is a Polish space $(Y, \tau)$ containing $X$ such that $X$ is $\mathbf{\Sigma}^1_n$ (resp. $\mathbf{\Pi}^1_n$) in $Y$ and $\mathsf{S} = \{ B \cap X \mid B \in \mathsf{B}(\tau) \}$.

When we deal with measurable spaces we define the analogue of the Wadge hierarchy by requiring the reduction to be Borel; in this case if $A \subseteq X$ and $B \subseteq Y$ are subsets of two measurable spaces we say that *$A$ is Borel-Wadge reducible to $B$* and write $A \leq_{\mathrm{B}} B$. The Borel-Wadge degrees and the Borel-Wadge hierarchy are defined in the obvious way. For subsets of a measurable space and $\mathbf{\Gamma}$ either $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$ the notions of Borel-$\mathbf{\Gamma}$-hard and Borel-$\mathbf{\Gamma}$-complete set (which we abbreviate by B-$\mathbf{\Gamma}$-hard and B-$\mathbf{\Gamma}$-complete) are also defined in the obvious way, using the Borel-Wadge hierarchy. (Here the B-$\mathbf{\Gamma}$-complete sets form a Borel-Wadge degree.)

Any standard Borel space $(X, \mathsf{S})$ admits a zero-dimensional Polish topology whose Borel structure is $\mathsf{S}$ (see [8, Exercise 13.5]). Therefore if a subset $A$ of a Polish space $X$ is $\mathbf{\Gamma}$-complete for $\mathbf{\Gamma}$ either $\mathbf{\Sigma}^1_n$ or $\mathbf{\Pi}^1_n$, then $A$ is B-$\mathbf{\Gamma}$-complete when $X$ is endowed with its Borel structure. It is a remarkable result of Kechris ([9]) that every B-$\mathbf{\Sigma}^1_1$-complete subset of a Polish space is also $\mathbf{\Sigma}^1_1$-complete, and the same holds with $\mathbf{\Pi}^1_1$ in place of $\mathbf{\Sigma}^1_1$.

It is also immediate that every (B-) $\mathbf{\Gamma}$-complete set is true $\mathbf{\Gamma}$. When $\mathbf{\Gamma}$ is $\mathbf{\Sigma}^0_\alpha$ or $\mathbf{\Pi}^0_\alpha$ the converse holds ([8, Exercise 24.20]). This also holds for $\mathbf{\Sigma}^1_n$ and $\mathbf{\Pi}^1_n$ assuming projective determinacy (henceforth denoted by $\mathsf{PD}$). As usual this works level by level, so that for $n = 1$ only $\mathbf{\Sigma}^1_1$ determinacy is needed. On the other hand this cannot be proved in $\mathsf{ZFC}$ without additional assumptions: if $V = L$ then there exists a true $\mathbf{\Pi}^1_1$ set which is not $\mathbf{\Pi}^1_1$-complete. The main point of this paper is to prove within $\mathsf{ZFC}$ results which were easily derivable using determinacy assumptions from known facts: indeed we will show within $\mathsf{ZFC}$ that some spaces, already known to be true $\mathbf{\Pi}^1_1$, are in fact B-$\mathbf{\Pi}^1_1$-complete.

To be more specific, we will study $C_p(X)$ and $C^*_p(X)$ with the $\sigma$-algebra of their Borel subsets (in some sense forgetting the topological structure), as done by Christensen in [2]. This can be viewed as the study of functors $X \mapsto C_p(X)$ and $X \mapsto C^*_p(X)$ from the category of separable metrizable spaces to the category of measurable spaces. (Similar functors — with the nicer property that the range is again the category of separable metrizable spaces — are $X \mapsto \mathsf{K}(X)$ and $X \mapsto \mathsf{P}(X)$, where $\mathsf{K}(X)$ is the space of compact subsets of $X$ with the Vietoris topology and $\mathsf{P}(X)$ is the space of probability Borel measures on $X$ with the topology of weak convergence: these functors have been studied in [10] and [3], respectively.) It turns out (see Lemma 2.2) that when $X$ is separable metrizable, $C_p(X)$ with its Borel structure is (isomorphic to) a subset of a standard Borel space; hence the study of the Borel-Wadge degrees of $C_p(X)$ and $C^*_p(X)$ does make sense. Therefore what we really study are functors from the category of separable metrizable spaces to the Borel-Wadge hierarchy, i.e., the class of all Borel-Wadge degrees.

Recall that a (non-necessarily metrizable) topological space is *analytic* if it is empty or the continuous image of the Baire space, and it is *$\sigma$-compact* if it can be written as a countable union of compact subspaces.

Let $X$ be a separable metrizable space. The following facts are known:

(A) $C_p(X)$ is analytic if and only if $X$ is $\sigma$-compact;

(B) $C^*_p(X)$ (or, equivalently, the space of continuous functions with values in the interval $[0,1]$) is analytic if and only if $X$ is $\sigma$-compact.

(A) is essentially contained in a result of Christensen ([2, Theorem 3.7]), while (B) follows from a result of Dobrowolski and Marciszewski ([5, Theorem 6.2]), combined with the observation contained in our Lemma 2.2.

It is immediate that a metrizable space is analytic iff it is $\mathbf{\Sigma}^1_1$; similarly if $(X,\tau)$ is a topological space, $(X,\tau)$ is analytic iff $(X,\mathsf{B}(\tau))$ is $\mathbf{\Sigma}^1_1$. If $X$ is a $\mathbf{\Sigma}^1_1$ space $C_p(X)$ and $C^*_p(X)$ are easily seen to be $\mathbf{\Pi}^1_1$ spaces and hence the above results imply that if $X$ is not $\sigma$-compact then they are true $\mathbf{\Pi}^1_1$. We will prove that if $X$ is a $\mathbf{\Sigma}^1_1$ space which is not $\sigma$-compact then $C_p(X)$ and $C^*_p(X)$ are B-$\mathbf{\Pi}^1_1$-complete.

Throughout the paper we follow (as above) the notation and terminology of [8] and refer to this textbook for all standard facts on descriptive set theory.

Here is a short list of complete sets (see [8]) which will be used to establish the hardness of various spaces (obviously if $A$ is $\mathbf{\Gamma}$-hard and $A \leq_{\mathrm{W}} B$ then $B$ is also $\mathbf{\Gamma}$-hard).

**Example 1.1.**

$$C_0 = \left\{ (x_n) \in [0,1]^{\mathbb{N}} \mid \lim x_n = 0 \right\} \quad \text{is } \mathbf{\Pi}^0_3\text{-complete}$$

$$C = \left\{ (x_n) \in [0,1]^{\mathbb{N}} \mid \lim x_n \text{ exists} \right\} \quad \text{is } \mathbf{\Pi}^0_3\text{-complete}$$

$$\mathrm{WF} = \{ T \in \mathrm{Tr} \mid T \text{ is well-founded} \} \quad \text{is } \mathbf{\Pi}^1_1\text{-complete}$$

where Tr is the Polish space of all trees $T \subseteq \mathbb{N}^{<\mathbb{N}}$ with the topology inherited from $2^{\mathbb{N}^{<\mathbb{N}}}$.

With respect to the last example of 1.1 we will use the following notation. The set of all finite sequences of natural numbers is denoted by $\mathbb{N}^{<\mathbb{N}}$. For $s \in \mathbb{N}^{<\mathbb{N}}$, length$(s)$ denotes the number of elements of $s$, and if $n <$ length$(s)$ we write $s(n)$ for the $(n+1)$-th element of $s$. If also $t \in \mathbb{N}^{<\mathbb{N}}$ we write $s \subseteq t$ to mean that $s$ is an initial segment of $t$, i.e., length$(t) \leq$ length$(s)$ and $t(n) = s(n)$ for every $n <$ length$(t)$. If $i \in \mathbb{N}$, $s^\frown \langle i \rangle$ is the sequence $t$ with length$(t) =$ length$(s) + 1$, $s \subseteq t$ and $t(\text{length}(s)) = i$. The unique sequence of length 0 is denoted by $\langle\rangle$. If $\alpha \in \mathbb{N}^{\mathbb{N}}$ is an infinite sequence of natural numbers $\alpha \restriction n$ is the finite initial segment of $\alpha$ with length $n$. A set $T \subseteq \mathbb{N}^{<\mathbb{N}}$ is a tree if $s \in T$ and $t \subseteq s$ implies $t \in T$ and in this case $[T] = \{\, \alpha \in \mathbb{N}^{\mathbb{N}} \mid \forall n(\alpha \restriction n \in T) \,\}$, so that $T \in$ WF if and only if $[T] = \emptyset$.

We now explain the organization of the paper. In Section 2 we will introduce our approach to the problem and prove some basic facts. In Section 3 we study the Borel-Wadge degree of $C_p(X)$ and $C_p^*(X)$ when $X$ is $\mathbf{\Sigma}^1_1$. In Section 4, using set-theoretic assumptions beyond $\mathsf{ZFC}$, we study the Borel-Wadge degree of $C_p(X)$ when $X$ lies arbitrarily high in the projective hierarchy. As a technical tool in the study of $C_p^*(X)$ we proved a result which is interesting in its own right: it is a simultaneous improvement of theorems of Christensen and Kechris regarding the complexity of a subset of the hyperspace of the closed sets of a Polish space. This result is not needed in the current version of the proofs, but we include it in Section 5.

The literature on $C_p(X)$ is enormous: some references which touch upon matters related to our approach are [4], [12] (where $C_p(X)$ is studied as a subset of $\mathbb{R}^X$) and [6], [1], [13] (where for countable $X$, $C_p(X)$ is studied as a topological space per se).

## 2. Some basic facts and constructions

The following notation will be useful in discussing $C_p(X)$ (and $C_p^*(X)$: see the remark at the end of the section). First of all let us use $C(X)$ to denote the set of all real-valued continuous functions on $X$. If $f \in C(X)$, $A \subseteq X$ and $\varepsilon > 0$ let

$$U_f(A; \varepsilon) = \{\, g \in C(X) \mid \forall x \in A \; |f(x) - g(x)| < \varepsilon \,\}.$$

If $A = \{x_1, \ldots, x_n\}$ is finite we write $U_f(x_1, \ldots, x_n; \varepsilon)$. With this notation the topology of pointwise convergence on $C(X)$ is defined by basic open sets of the form $U_f(x_1, \ldots, x_n; \varepsilon)$; obviously the sets of the form $U_f(x; \varepsilon)$ form a sub-basis. This topology is the one $C(X)$ inherits from $\mathbb{R}^X$, and $C(X)$ equipped with this topology is denoted by $C_p(X)$. Another topology on $C(X)$ that will be sometimes useful is the compact-open topology, whose basic open sets have the form $U_f(K; \varepsilon)$ where $K \subseteq X$ is compact. If $X$ is locally compact Polish the compact-open topology on $C(X)$ is Polish, while if $X$ is not locally compact the compact-open topology on $C(X)$ is not even first-countable (see [7, Exercise 3.4.E.d]).

A simple but useful observation is that if $\tau$ and $\tau'$ are topologies on the same set $X$ and $\tau' \subseteq \mathsf{B}(\tau)$ then $\mathsf{B}(\tau') \subseteq \mathsf{B}(\tau)$. Thus to show that the Borel sets of two topologies coincide it suffices to show that every open set in one topology is Borel with respect to the other topology. If the topologies are hereditarily Lindelöf this needs only to be checked for members of a sub-basis for each topology.

The topology of $C_p(X)$ is coarser than the compact-open topology on $C(X)$. On the other hand every basic open set of the compact-open topology is $\mathbf{\Sigma}^0_2$ in $C_p(X)$:

$$g \in U_f(K;\varepsilon) \iff \exists n\, \forall x \in K\ |f(x) - g(x)| \leq \varepsilon - 2^{-n}.$$

Therefore the Borel subsets of $C_p(X)$ coincide with the Borel subsets in the compact-open topology and if $X$ is locally compact $C_p(X)$ is standard Borel.

A useful approach (already used e.g. in [15] and [5]) to the study of $C_p(X)$ is to generalize the case of countable $X$ to a generic separable $X$ by fixing a countable dense set $D \subseteq X$ and considering the space

$$\tilde{C}_p(X;D) = \left\{ (r_a)_{a \in D} \in \mathbb{R}^D \mid \exists f \in C(X)\ \forall a \in D\ f(a) = r_a \right\},$$

with the topology induced by $\mathbb{R}^D$. The space $\tilde{C}_p(X;D)$ is clearly homeomorphic to the space $C(X)$ with the topology induced by the restriction function $C(X) \to \mathbb{R}^D$, $f \mapsto f \restriction D$. We denote $C(X)$ with this topology by $C_p(X;D)$. A sub-basis for $C_p(X;D)$ consists of all sets of the form $U_f(a;\varepsilon)$ with $a \in D$.

As far as the Borel structure is concerned the choice of $D$ is immaterial:

**Lemma 2.1.** *If $X$ is separable metrizable and $D, D' \subseteq X$ are countable dense then $C_p(X;D)$ and $C_p(X;D')$ have the same Borel sets.*

Proof: It suffices to show that any sub-basic open set of $C_p(X;D')$ is Borel in $C_p(X;D)$. Let $d$ be a compatible metric. If $a' \in D'$ and $\varepsilon > 0$ then for $g \in C(X)$ we have

$$g \in U_f(a';\varepsilon) \iff \exists m\, \forall n\, \exists a \in D\ (d(a,a') < 2^{-n}\ \&\ g \in U_f(a;\varepsilon - 2^{-m})),$$

so that $U_f(a';\varepsilon)$ is $\mathbf{\Sigma}^0_3$ in $C_p(X;D)$. □

The two approaches to the classification of $C_p(X)$ we just described (considering only the Borel structure or using a countable dense set) are strictly related, as the next lemma shows:

**Lemma 2.2.** *If $X$ is separable metrizable and $D \subseteq X$ is countable dense then $C_p(X;D)$ and $C_p(X)$ have the same Borel sets.*

Proof: The topology of $C_p(X;D)$ is coarser than the topology of $C_p(X)$, for any $D$. Vice versa, since $C_p(X)$ is hereditarily Lindelöf (because $X$ is second countable, see [7, Exercise 3.8.D]), any open set $U$ of $C_p(X)$ is countable union of basic open sets: let $U = \bigcup_n U_{f_n}(x_0^n, \dots, x_{k_n}^n; \varepsilon_n)$. Let $D' \subseteq X$ be countable dense with $x_i^n \in D'$ for all $n$ and $i \leq k_n$. It is immediate that $U$ is open in $C_p(X; D')$: by Lemma 2.1 $U$ is Borel in $C_p(X; D)$ for any countable dense $D \subseteq X$. □

Since $C_p(X; D)$ and $\tilde{C}_p(X; D)$ are homeomorphic, we have that if $\tilde{C}_p(X; D)$ is Borel (resp. $\mathbf{\Pi}^1_n$-complete, $\mathbf{\Sigma}^1_n$-complete) as a subset of the Polish space $\mathbb{R}^D$ for some (any) countable dense $D \subseteq X$, then $C_p(X)$ is standard Borel (resp. B-$\mathbf{\Pi}^1_n$-complete, B-$\mathbf{\Sigma}^1_n$-complete). Thus our goal will be to establish the Wadge degree of $\tilde{C}_p(X; D)$.

If $X$ is projective an upper bound for the complexity of $\tilde{C}_p(X; D)$ (and hence of $C_p(X)$ as a Borel space) is easily obtained:

**Lemma 2.3.** *Let $X$ be separable metrizable and $D \subseteq X$ be countable dense. If $X$ is $\mathbf{\Sigma}^1_n$ then $\tilde{C}_p(X; D)$ is $\mathbf{\Pi}^1_n$ in $\mathbb{R}^D$.*

Proof: Let $d$ be a compatible metric for $X$. It is immediate that $(r_a)_{a \in D} \in \mathbb{R}^D$ belongs to $\tilde{C}_p(X; D)$ if and only if

$$\forall x \in X\, \forall m\, \exists k\, \forall a, b \in D\, (\max(d(a, x), d(b, x)) < 2^{-k} \implies |r_a - r_b| \leq 2^{-m})$$

which is easily seen to be $\mathbf{\Pi}^1_n$. □

For $D \subseteq X$ countable dense, define $C^*_p(X; D)$ and $\tilde{C}^*_p(X; D)$ in the obvious way, considering only continuous bounded real-valued functions on $X$. Lemmas 2.1, 2.2, and 2.3 hold also in the case of bounded functions (with the same proofs). Dobrowolski and Marciszewski proved ([5, Theorem 6.2]) that if $X$ is a $\mathbf{\Sigma}^1_1$ space which is not $\sigma$-compact then $\tilde{C}^*_p(X; D)$ (or even $\tilde{C}_p(X; D) \cap [0, 1]^D$) is true $\mathbf{\Pi}^1_1$: the forward direction of (B) of Section 1 follows from this and the bounded version of Lemma 2.2.

## 3. $C_p(X)$ when $X$ is $\mathbf{\Sigma}^1_1$

We begin our investigation of $C_p(X)$ for a $\mathbf{\Sigma}^1_1$ space $X$ from the case where $X$ is "small", i.e., $\sigma$-compact. In this case $\tilde{C}_p(X; D)$ is actually Borel (this is the same as [5, Lemma 6.1] and (i) $\implies$ (ii) of [15, Theorem 2.1], but we include a proof here for the sake of completeness):

**Lemma 3.1.** *If $X$ is metrizable $\sigma$-compact and $D \subseteq X$ is countable dense then $\tilde{C}_p(X; D)$ and $\tilde{C}^*_p(X; D)$ are $\mathbf{\Pi}^0_3$ in $\mathbb{R}^D$.*

Proof: Since $\tilde{C}^*_p(X; D) = \tilde{C}_p(X; D) \cap \bigcup_k [-k, k]^D$ and $\bigcup_k [-k, k]^D$ is $\mathbf{\Sigma}^0_2$ it suffices to prove the result for $\tilde{C}_p(X; D)$.

Let $X = \bigcup_n K_n$ where every $K_n$ is compact and let $d$ be a compatible metric on $X$. Let $(r_a)_{a \in D} \in \mathbb{R}^D$. We will show that $(r_a)_{a \in D} \in \tilde{C}_p(X; D)$ if and only if

$$\forall n\, \forall m\, \exists k\, \forall x \in K_n\, \forall a, b \in D\, (\max(d(a, x), d(b, x)) < 2^{-k} \implies |r_a - r_b| \leq 2^{-m}).$$

This completes the proof for $\tilde{C}_p(X;D)$ because for every $n$, $m$ and $k$

$$\left\{ (r_a) \mid \forall x \in K_n \, \forall a, b \in D \, (\max(d(a,x), d(b,x)) < 2^{-k} \implies |r_a - r_b| \leq 2^{-m}) \right\}$$

is closed.

To prove the forward direction of the equivalence suppose $(r_a)_{a \in D} \in \tilde{C}_p(X;D)$ and let $f \in C(X)$ be such that $f(a) = r_a$ for every $a \in D$. Fix $n$ and $m$: for every $x \in X$ let $k(x)$ be the least $k$ such that $|f(x) - f(x')| \leq 2^{-m-1}$ for every $x' \in X$ with $d(x,x') < 2^{-k}$. We claim that for every $n$ the set $\{ k(x) \mid x \in K_n \} \subseteq \mathbb{N}$ is bounded. If this was not the case then for some $n$ and every $k$ there exist $x_k \in K_n$ and $x'_k \in X$ with $d(x_k, x'_k) < 2^{-k}$ and $|f(x_k) - f(x'_k)| > 2^{-m-1}$. Since $K_n$ is compact we may suppose that $\lim x_k = x$ for some $x \in K_n$. Thus we have also $\lim x'_k = x$ and, by continuity of $f$, we would have $\lim |f(x_k) - f(x'_k)| = 0$, which is not the case. This proves the claim and for every $n$ there exists $k = \max \{ k(x) \mid x \in K_n \}$. Then for all $x \in K_n$ and $a, b \in D$ with $\max(d(a,x), d(b,x)) < 2^{-k}$ we have $|r_a - r_b| \leq |f(a) - f(x)| + |f(x) - f(b)| \leq 2^{-m-1} + 2^{-m-1} = 2^{-m}$.

For the backward direction we need to define $f \in C(X)$ such that $f(a) = r_a$ for every $a \in D$. Given $x \in X$, fix $n$ such that $x \in K_n$: for every $m$ there exist $k$ and a closed interval $I^x_m \subset \mathbb{R}$ of length $\leq 2^{-m}$ such that for all $a \in D$ with $d(a,x) < 2^{-k}$ we have $r_a \in I^x_m$. We may assume that $I^x_{m+1} \subseteq I^x_m$, so that $\bigcap_m I^x_m$ is a singleton, whose only element we define to be $f(x)$. Given $x$ and $m$ let $k$ be as above: if $d(x,x') < 2^{-k}$ then $f(x') \in I^x_m$ and $|f(x) - f(x')| \leq 2^{-m}$. Therefore $f$ is continuous at $x$. Clearly if $x = a \in D$ we have $f(x) = r_a$. □

Barring trivial cases the complexity of $\tilde{C}_p(X;D)$ cannot be lower than the one obtained in Lemma 3.1 (see [4, Theorem 6] for a similar result concerning $C_p(X)$):

**Lemma 3.2.** *If $X$ is separable metrizable not discrete and $D \subseteq X$ is countable dense then $\tilde{C}_p(X;D)$ and $\tilde{C}^*_p(X;D)$ are $\mathbf{\Pi}^0_3$-hard in $\mathbb{R}^D$.*

Proof: Let $w \in X$ be a point which is not isolated. We will assume that $w \notin D$ and show that $C \leq_{\mathrm{W}} \tilde{C}_p(X;D)$ and $C \leq_{\mathrm{W}} \tilde{C}^*_p(X;D)$. If $w \in D$ it is easy to modify our proof so that it shows that $C_0 \leq_{\mathrm{W}} \tilde{C}_p(X;D)$ and $C_0 \leq_{\mathrm{W}} \tilde{C}^*_p(X;D)$. (Both $C$ and $C_0$ are defined in 1.1.)

Since $w$ is not isolated let $(a_n) \subseteq D$ be a sequence converging to $w$. Let $s_n = d(w, a_n) > 0$; we may assume that $s_n > s_{n+1}$ for every $n$.

Given $(z_k) \in [0,1]^{\mathbb{N}}$ for every $a \in D$ set

$$r^{(z_k)}_a = \begin{cases} z_0 & \text{if } d(a,w) \geq s_0; \\ t z_n + (1-t) z_{n+1} & \text{if } s_{n+1} \leq d(a,w) \leq s_n \\ & \text{and } d(a,w) = s_{n+1} + t(s_n - s_{n+1}). \end{cases}$$

The map $(z_k) \mapsto (r^{(z_k)}_a)$ from $[0,1]^{\mathbb{N}}$ to $[0,1]^D$ is continuous because the value of each $r^{(z_k)}_a$ depends continuously on at most two $z_k$'s. Moreover $(r^{(z_k)}_a) \in \tilde{C}_p(X;D)$

if and only if $(r_a^{(z_k)}) \in \tilde{C}_p^*(X;D)$ if and only if $(z_k) \in c$. To see this notice that we can define $f = f^{(z_k)} \in C(X \setminus \{w\})$ such that $f(a) = r_a^{(z_k)}$ for every $a \in D$ by setting

$$f(x) = \begin{cases} z_0 & \text{if } d(x,w) \geq s_0; \\ tz_n + (1-t)z_{n+1} & \text{if } s_{n+1} \leq d(x,w) \leq s_n \\ & \text{and } d(x,w) = s_{n+1} + t(s_n - s_{n+1}). \end{cases}$$

That the sequence $(r_a^{(z_k)})$ belongs to $\tilde{C}_p(X;D)$ is equivalent to the fact that $f$ can be extended to $w$: clearly this can be done if and only if $\lim z_k$ exists (notice that $f(a_n) = r_{a_n}^{(z_k)} = z_n$), and in this case $(r_a^{(z_k)}) \in \tilde{C}_p^*(X;D)$. □

We will now consider the case where $X$ is "large", i.e., not $\sigma$-compact. Okunev ([15]) proved that in this case $\tilde{C}_p(X;D)$ is true $\mathbf{\Pi}_1^1$ and Dobrowolski and Marciszewski ([5]) extended the same result to $\tilde{C}_p^*(X;D)$: we improve both results proving $\mathbf{\Pi}_1^1$-completeness.

**Theorem 3.3.** *Let $X$ be a $\mathbf{\Sigma}_1^1$ space which is not $\sigma$-compact. Then for any $D \subseteq X$ countable dense $\tilde{C}_p(X;D) \cap [-1,1]^D$ is $\mathbf{\Pi}_1^1$-complete.*

Proof: By Lemma 2.3 $\tilde{C}_p(X;D)$ is $\mathbf{\Pi}_1^1$, and therefore $\tilde{C}_p(X;D) \cap [-1,1]^D$ is also $\mathbf{\Pi}_1^1$. Thus we need to prove that $\tilde{C}_p(X;D) \cap [-1,1]^D$ is $\mathbf{\Pi}_1^1$-hard in $[-1,1]^D$.

Let $Z$ be a metric compactification of $X$. Since $X$ is $\mathbf{\Sigma}_1^1$ and not $\sigma$-compact, $X \setminus D$ is $\mathbf{\Sigma}_1^1$ and there is no $\mathbf{\Sigma}_2^0$ subset of $Z$ separating $X \setminus D$ from $Z \setminus X$. By the strengthening of Hurewicz's theorem due to Kechris, Louveau and Woodin ([10, Theorem 4] or [8, Theorem 21.22]) there exists a set $F \subseteq X \setminus D$ which is closed in $X$ and homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

Define a complete compatible metric on $F$ and extend it to a compatible metric $d$ on the whole of $X$: this can be done because $F$ is closed in $X$ (see e.g. [7, Exercise 4.5.21.c]). When we speak of the diameter of a subset of $X$ we will use the metric $d$.

Our goal is to show $\mathrm{WF} \leq_{\mathrm{W}} \tilde{C}_p(X;D) \cap [-1,1]^D$. To this end we will define a continuous map $\mathrm{Tr} \to [-1,1]^D$, $T \mapsto (r_a^T)_{a \in D}$. The $r_a^T$'s are defined via a Lusin scheme $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ in $X$ with the $F_s$ closed with non-empty interior. The Lusin scheme will be such that the set $\left\{ x \in X \mid \exists \alpha \in \mathbb{N}^{\mathbb{N}}\ x \in \bigcap_n F_{\alpha \restriction n} \right\}$ is contained in $F$, is closed in $X$, and is homeomorphic to $\mathbb{N}^{\mathbb{N}}$.

To be precise, we will define a sequence $(F_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of closed subsets of $X$ and a sequence $(a_s)_{s \in \mathbb{N}^{<\mathbb{N}}}$ of elements of $D$ with the following properties:

(i) $F_{\langle\rangle} = X$;

(ii) $\mathrm{diam}(F_s) < 2^{-\mathrm{length}(s)}$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ with $\mathrm{length}(s) > 0$;

(iii) $\mathrm{Int}(F_s) \cap F \neq \emptyset$ and hence $\mathrm{Int}(F_s) \neq \emptyset$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ (where $\mathrm{Int}(A)$ is the interior of $A$);

(iv) $\bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle} \subseteq \mathrm{Int}(F_s)$ and hence $\partial F_s \cap \bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle} = \emptyset$ for every $s \in \mathbb{N}^{<\mathbb{N}}$ (where $\partial A$ is the boundary of $A$);

(v) $a_s \in F_s \setminus \bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle}$ for every $s \in \mathbb{N}^{<\mathbb{N}}$;

(vi) for every $k$ and $x \in X$ there exists an open neighborhood $U$ of $x$ such that $U \cap F_s \neq \emptyset$ for at most one $s \in \mathbb{N}^{<\mathbb{N}}$ with $\mathrm{length}(s) = k$.

We now show how we can complete the proof once these two sequences are defined. First of all notice that by (iv) and (vi) if $F_s \cap F_t \neq \emptyset$ then either $s \subseteq t$ or $t \subseteq s$. (vi) implies also that $\bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle}$ is closed for every $s \in \mathbb{N}^{<\mathbb{N}}$. Moreover for every $\alpha \in \mathbb{N}^{\mathbb{N}}$ we have that $\bigcap_{n\in\mathbb{N}} F_{\alpha\restriction n}$ is a singleton whose element belongs to $F$: (ii) implies that the intersection contains at most one element, by (iii) there is an element in $F \cap F_{\alpha\restriction n}$ for every $n$, by (ii) and (iv) the sequence of these elements is $d$-Cauchy, and since $d$ is complete on $F$ it has a limit which is in $F$ and belongs to $\bigcap_{n\in\mathbb{N}} F_{\alpha\restriction n}$. This and (i) imply that for every $x \in X \setminus F$ there exists a longest sequence $s \in \mathbb{N}^{<\mathbb{N}}$ such that $x \in F_s$.

For every $s \in \mathbb{N}^{<\mathbb{N}}$ let us define a continuous function $g_s : F_s \to [-1, 1]$: start by letting $g_s(x) = (-1)^{\mathrm{length}(s)}$ if $x \in \{a_s\} \cup \partial F_s$ and $g_s(x) = (-1)^{\mathrm{length}(s)+1}$ if $x \in \bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle}$; then complete the definition using Tietze extension theorem. This can be done because the two explicit definitions are not in conflict by (iv) and (v) and the set where $g_s$ has been defined explicitly is the union of two closed sets by the observation made above.

Now we define the map $\mathrm{Tr} \to [-1, 1]^D$, $T \mapsto (r^T_a)_{a\in D}$ and show that $T \in \mathrm{WF}$ if and only if $(r^T_a)_{a\in D} \in \tilde{C}_p(X; D)$. Given $T \in \mathrm{Tr}$ and $a \in D$ let $r^T_a = g_s(a)$, where $s \in \mathbb{N}^{<\mathbb{N}}$ is the longest sequence such that $s \in T$ and $a \in F_s$: by the remark made above (and since $F \subseteq X \setminus D$) this definition is meaningful. The continuity of the map $T \mapsto (r^T_a)_{a\in D}$ is immediate.

If $T \in \mathrm{WF}$ then define $f^T : X \to [-1, 1]$ by $f^T(x) = g_s(x)$, where $s \in \mathbb{N}^{<\mathbb{N}}$ is the longest sequence such that $s \in T$ and $x \in F_s$. Since $T \in \mathrm{WF}$ the definition is meaningful for every $x$ and clearly $f^T$ extends $(r^T_a)_{a\in D}$. To check that $f^T$ is continuous on $X$ (and hence $(r^T_a)_{a\in D} \in \tilde{C}_p(X; D)$) fix $x \in X$ and let $s \in \mathbb{N}^{<\mathbb{N}}$ be the longest sequence such that $s \in T$ and $x \in F_s$. If $x \in \mathrm{Int}(F_s)$ then $f^T = g_s$ on

$$\mathrm{Int}(F_s) \setminus \bigcup \left\{ F_{s^\frown\langle i\rangle} \mid i \in \mathbb{N} \;\&\; x \notin F_{s^\frown\langle i\rangle} \right\}$$

which is a neighborhood of $x$: therefore $f^T$ is continuous at $x$. If $x \notin \mathrm{Int}(F_s)$ then $x \in \partial F_s$. Let $U$ be a neighborhood of $x$ such that $U \cap F_t = \emptyset$ for every $t \neq s$ with $\mathrm{length}(t) = \mathrm{length}(s)$; $\mathrm{length}(s) > 0$ (because $\partial F_{\langle\rangle} = \emptyset$) and by (iv) we may assume that $U \subseteq \mathrm{Int}(F_{s'})$ where $s' = s \restriction (\mathrm{length}(s) - 1)$. Then $f^T = g_s$ on $U \cap F_s$ and $f^T = g_{s'}$ on $U \setminus F_s$. Since the definition of $g_s$ implies that $g_s = (-1)^{\mathrm{length}(s)} = g_{s'}$ on $\partial F_s$, $f^T$ is continuous on $U$.

If $T \notin \mathrm{WF}$ let $\alpha \in [T]$ and let $x \in F$ be the unique element of $\bigcap_{n\in\mathbb{N}} F_{\alpha\restriction n}$. To show that $(r^T_a)_{a\in D} \notin \tilde{C}_p(X; D)$ it suffices to show that any extension of $(r^T_a)_{a\in D}$ is not continuous at $x$. Let $f$ be such an extension: by (v) for every $n \in \mathbb{N}$ we have $a_{\alpha\restriction n} \in F_{\alpha\restriction n} \setminus \bigcup_{i\in\mathbb{N}} F_{\alpha\restriction n^\frown\langle i\rangle}$ and $\alpha \restriction n \in T$, so that $f(a_{\alpha\restriction n}) = r^T_{a_{\alpha\restriction n}} = g_{\alpha\restriction n}(a_{\alpha\restriction n}) = (-1)^n$, while $\lim_{n\to\infty} a_{\alpha\restriction n} = x$ by (ii); these two facts imply that $f$ is not continuous at $x$.

To complete the proof it suffices to show that $(F_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ and $(a_s)_{s\in\mathbb{N}^{<\mathbb{N}}}$ satisfying (i)–(vi) do exist.

$F_{\langle\rangle}$ is defined by (i).

Suppose we have defined $F_s$ satisfying (ii) and (iii) for every $s$ with length$(s) = k$ and that (vi) holds for $k$. By (iii) and the density of $D$, for every such $s$ there exists $a_s \in D \cap \mathrm{Int}(F_s)$. $F \cap \mathrm{Int}(F_s)$ is a relatively open nonempty subset of $F$, which is homeomorphic to $\mathbb{N}^\mathbb{N}$: therefore $F \cap \mathrm{Int}(F_s)$ contains a sequence $(x^s_i)$ of distinct points which has no accumulation point. For every $i$ we can choose inductively $F_{s^\frown\langle i\rangle}$ to be a closed neighborhood of $x^s_i$ such that $F_{s^\frown\langle i\rangle} \subseteq \mathrm{Int}(F_s)$, $\mathrm{diam}(F_{s^\frown\langle i\rangle}) < 2^{-k-1-i}$, $a_s \notin F_{s^\frown\langle i\rangle}$, and $F_{s^\frown\langle i\rangle} \cap F_{s^\frown\langle j\rangle} = \emptyset$ for every $j < i$.

(iv) and (v) are satisfied by $s$ and (ii) and (iii) by each $s^\frown\langle i\rangle$. To check that (vi) holds for $k+1$ let $x \in X$: since (vi) holds for $k$ there exists an open neighborhood $U$ of $x$ such that which intersects at most one $F_s$ with length$(s) = k$. If $U \cap F_s \neq \emptyset$ for some $s$ with length$(s) = k$ we need to show that some open neighborhood of $x$ intersects at most one $F_{s^\frown\langle i\rangle}$; if this were not the case then $x$ would be an accumulation point of $\bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle}$ not belonging to $\bigcup_{i\in\mathbb{N}} F_{s^\frown\langle i\rangle}$: since $\lim \mathrm{diam}(F_{s^\frown\langle i\rangle}) = 0$, $x$ would be an accumulation point of $\{\, x^s_i \mid i \in \mathbb{N} \,\}$, against the choice of $\{\, x^s_i \mid i \in \mathbb{N} \,\}$. □

The crucial idea in the preceding proof is due to the anonymous referee. In an earlier draft we could only prove Theorem 3.3 under various specific assumptions, e.g. when $X = \mathbb{N}^\mathbb{N}$ or $X$ is an infinite-dimensional separable Banach space or, in the case of $C_p(X)$, when $X$ is Borel.

The complexity of the Borel space $C_p(X)$ for $X$ separable metrizable and $\mathbf{\Sigma}^1_1$ is now completely classified and we obtain the following result, which is an improvement of (A) and (B):

**Corollary 3.4.** *Let $X$ be a $\mathbf{\Sigma}^1_1$ space. If $X$ is $\sigma$-compact then $C_p(X)$ and $C^*_p(X)$ are standard Borel. If $X$ is not $\sigma$-compact then $C_p(X)$ and $C^*_p(X)$ are $B$-$\mathbf{\Pi}^1_1$-complete.*

Proof: Fix $D \subseteq X$ countable dense. A subset of a Polish space (in our case $\mathbb{R}^D$) is standard Borel if and only if it is Borel. Hence Lemma 3.1 and Theorem 3.3 imply that the statements hold with $\tilde{C}_p(X;D)$ and $\tilde{C}^*_p(X;D)$ in place of $C_p(X)$ and $C^*_p(X)$. The space $\tilde{C}_p(X;D)$ is homeomorphic to $C_p(X;D)$ and, by Lemma 2.2, the latter is Borel isomorphic to $C_p(X)$; the same applies to the bounded versions of these spaces and the proof is complete. □

## 4. $C_p(X)$ when $X$ is projective

We now study the Borel-Wadge degree of $C_p(X)$ when $X$ is projective but not $\mathbf{\Sigma}^1_1$. To carry out this study we need (as usual) some additional set-theoretic assumption. We will assume projective determinacy ($\mathsf{PD}$), but (as usual) the proof works level by level, so that for $\mathbf{\Sigma}^1_{n+1}$ spaces, only $\mathbf{\Sigma}^1_n$ determinacy is needed.

To compute the Wadge degree of $\tilde{C}_p(X;D)$ we will use the following lemma:

**Lemma 4.1.** *Let $Z$ be a metrizable space, $X \subseteq Z$ and $D \subseteq X$ be countable dense in $X$. Let $W = Z \setminus D$ and $Y = Z \setminus X$. Then $\mathsf{K}(Y) \leq_{\mathrm{W}} \tilde{C}_p(X;D)$, where $\mathsf{K}(Y)$ is viewed as a subset of $\mathsf{K}(W)$.*

Proof: Let $d$ be a compatible metric for $Z$. Given $K \in \mathsf{K}(W)$ and $a \in D$ let $r^K_a = 1/d(a,K)$ (notice that $K \cap D = \emptyset$ and hence $d(a,K) > 0$). It is clear that the map $K \mapsto (r^K_a)_{a \in D}$ from $\mathsf{K}(W)$ to $\mathbb{R}^D$ is continuous.

If $K \in \mathsf{K}(Y)$ then $K \cap X = \emptyset$ and the function $f : X \to \mathbb{R}$ defined by $f(x) = 1/d(x,K)$ is continuous and extends $(r^K_a)$. Thus $(r^K_a) \in \tilde{C}_p(X;D)$.

If $K \notin \mathsf{K}(Y)$ then there exists $x \in X \cap K$. Notice that $x \notin D$ and let $(a_n) \subseteq D$ be a sequence converging to $x$. Clearly $\lim r^K_{a_n} = +\infty$ and hence every $f : X \to \mathbb{R}$ such that $f(a) = r^K_a$ for every $a \in D$, is not continuous at $x$. Hence $(r^K_a) \notin \tilde{C}_p(X;D)$. $\square$

Lemma 4.1 shows that the Wadge degree of $\tilde{C}_p(X;D)$ is at least as high as the Wadge degree of $\mathsf{K}(Y)$: the latter has been studied by Kechris, Louveau and Woodin ([10, Lemma 1]), who obtained (their proof goes through even if the ambient space is neither compact nor zero-dimensional):

**Lemma 4.2.** *Let $Z$ be a Polish space and $Y \subseteq Z$, so that $\mathsf{K}(Y) \subseteq \mathsf{K}(Z)$.*

1. *If $Y$ is $\mathbf{\Sigma}^0_2$-hard then $\mathsf{K}(Y)$ is $\mathbf{\Pi}^1_1$-hard.*
2. *If $Y$ is $\mathbf{\Sigma}^1_n$-hard then $\mathsf{K}(Y)$ is $\mathbf{\Pi}^1_{n+1}$-hard.*

Under $\mathsf{PD}$ Lemma 4.2.2 (together with the observation that if $X$ is $\mathbf{\Pi}^1_n$ then $\mathsf{K}(X)$ is also $\mathbf{\Pi}^1_n$) yields a complete classification of the Wadge degree of $\mathsf{K}(X)$ for $X$ projective and not $\mathbf{\Pi}^1_1$. (This classification is not explicitly stated in [10], but see [3, p. 8].) In a similar vein we prove the following theorem that, coupled with Corollary 3.4, completely classifies the Borel-Wadge degree of $C_p(X)$ for $X$ a projective space.

**Theorem 4.3.** *Assume $\mathsf{PD}$. Let $X$ be a projective space which is not $\sigma$-compact and $D \subseteq X$ be countable dense. If $n \geq 1$ is least such that $X$ is $\mathbf{\Sigma}^1_n$ then $\tilde{C}_p(X;D)$ and $C_p(X)$ are B-$\mathbf{\Pi}^1_n$-complete.*

Proof: If $X$ is $\mathbf{\Sigma}^1_1$ the results have already been proved in Theorem 3.3, so we may assume that $n > 1$. The second statement follows from the first by Lemma 2.2, so it suffices to prove that $\tilde{C}_p(X;D)$ is true $\mathbf{\Pi}^1_n$.

Let $Z$ be a completion of $X$: $X$ is $\mathbf{\Sigma}^1_n$ but not $\mathbf{\Sigma}^1_{n-1}$ as a subset of $Z$. The first fact implies, by Lemma 2.3, that $\tilde{C}_p(X;D) \subseteq \mathbb{R}^D$ is $\mathbf{\Pi}^1_n$. Let $W = Z \setminus D$ which is a Polish space. The set $X \cap W$ is $\mathbf{\Sigma}^1_n$ and not $\mathbf{\Sigma}^1_{n-1}$ in $W$ and hence $Y = W \setminus X$ is not $\mathbf{\Pi}^1_{n-1}$ in $W$. Under $\mathsf{PD}$ this implies that $Y$ is $\mathbf{\Sigma}^1_{n-1}$-hard and hence by Lemma 4.2.2, $\mathsf{K}(Y) \subseteq \mathsf{K}(W)$ is $\mathbf{\Pi}^1_n$-hard. By Lemma 4.1 $\tilde{C}_p(X;D)$ is $\mathbf{\Pi}^1_n$-hard, which implies that it is not $\mathbf{\Sigma}^1_n$. □

In the proof of Theorem 4.3 we used Lemma 4.1 and hence unbounded functions, and one can ask the following question:

**Problem 4.4.** Assume the hypothesis of Theorem 4.3. Is $C^*_p(X)$ $B$-$\mathbf{\Pi}^1_n$-complete?

## 5. A result on the Effros Borel structure

In this section we prove a result which is unrelated to the other results of the paper, but was used in one of the proofs alluded to after the proof of Theorem 3.3. Since it is interesting in its own right we include it here. It improves a theorem of Christensen ([2, Theorem 3.8]) by showing that the subset of the hyperspace of the closed sets defined below (Theorem 5.1) is B-$\mathbf{\Pi}^1_1$-complete (Christensen showed that it is true $\mathbf{\Pi}^1_1$). A specific case of our result (when $X$ is a separable Hilbert space and $P$ its open unit ball) was proved by Kechris ([8, Theorem 27.6]) by a construction which is much more direct that the one we need to prove the theorem in full generality.

Given a topological space $X$ denote by $\mathsf{F}(X)$ the hyperspace of the closed subsets of $X$ equipped with the Effros Borel structure (see e.g. [8, Section 12.C]).

**Theorem 5.1.** *Let $X$ be a Polish space and $P \subseteq X$ be $\mathbf{\Pi}^1_1$. If $\overline{P} \setminus P$ is not $\sigma$-compact then $\{ F \in \mathsf{F}(\overline{P}) \mid F \subseteq P \}$ is B-$\mathbf{\Pi}^1_1$-complete as a subset of the standard Borel space $\mathsf{F}(\overline{P})$ (and a fortiori as a subset of the standard Borel space $\mathsf{F}(X)$).*

Proof: The fact that $\{ F \in \mathsf{F}(\overline{P}) \mid F \subseteq P \}$ is $\mathbf{\Pi}^1_1$ is straightforward. To prove that it is B-$\mathbf{\Pi}^1_1$-hard we define a Lusin scheme in $X$.

Clearly $\overline{P} \setminus P$ is $\mathbf{\Sigma}^1_1$ not $\sigma$-compact and, by imitating the proof of the version of Hurewicz theorem in [8, Corollary 21.19], we obtain that there exists $N \subseteq \overline{P} \setminus P$ which is relatively closed in $\overline{P} \setminus P$ and homeomorphic to $\mathbb{N}^{\mathbb{N}}$. Let us write $N = \{ x_\alpha \mid \alpha \in \mathbb{N}^{\mathbb{N}} \}$ (where $\alpha \mapsto x_\alpha$ is the homeomorphism) and let, for every $s \in \mathbb{N}^{<\mathbb{N}}$, $N_s = \{ x_\alpha \mid s \subset \alpha \}$: $N_s$ is clopen in $N$.

Let $d$ be a complete compatible metric for $X$: we will compute the diameter of subsets of $X$ according to $d$. Fix also a bijection $\# : \mathbb{N}^{<\mathbb{N}} \to \mathbb{N}$.

We will define a sequence $\{ U_s \mid s \in \mathbb{N}^{<\mathbb{N}} \}$ of open subsets of $X$ with the following properties:

(i) $N_s \subseteq U_s$;
(ii) $U_s \subseteq \left\{ x \in X \mid d(x, N) < 2^{-\#(s)} \right\}$;
(iii) $\operatorname{diam}(U_s) \leq 2 \cdot \operatorname{diam}(N_s)$;
(iv) $U_t \subseteq U_s \iff s \subseteq t$;
(v) $U_{s^\frown\langle i\rangle} \cap U_{s^\frown\langle j\rangle} = \emptyset$ if $i \neq j$;
(vi) $\overline{U_{s^\frown\langle i\rangle}} \cap N_{s^\frown\langle j\rangle} = \emptyset$ if $i \neq j$.

To start the construction it suffices to define $U_{\langle\rangle}$ satisfying (i)–(iii). Now suppose we have defined $U_s$ and $U_{s^\frown\langle j\rangle}$ for every $j < i$ and we want to define $U_{s^\frown\langle i\rangle}$.

Since $N_{s^\frown\langle i\rangle}$ is clopen within $N$ there exists an open (within $X$) set $V_{s^\frown\langle i\rangle}$ such that $V_{s^\frown\langle i\rangle}\cap N = N_{s^\frown\langle i\rangle}$. By shrinking $V_{s^\frown\langle i\rangle}$ we may assume that $V_{s^\frown\langle i\rangle}\subseteq U_s\cap\left\{x\in X \mid d(x,N)<2^{-\#(s^\frown\langle i\rangle)}\right\}$ and $\operatorname{diam}(V_{s^\frown\langle i\rangle})\le 2\cdot\operatorname{diam}(N_{s^\frown\langle i\rangle})$. Let $W_{s^\frown\langle i\rangle} = V_{s^\frown\langle i\rangle}\setminus\bigcup_{j<i}\overline{U_{s^\frown\langle j\rangle}}$. Since (vi) is satisfied for all $j<i$, so that $\overline{U_{s^\frown\langle j\rangle}}\cap N_{s^\frown\langle i\rangle}=\emptyset$, we have $W_{s^\frown\langle i\rangle}\cap N = N_{s^\frown\langle i\rangle}$. Now let

$$U_{s^\frown\langle i\rangle}=\left\{x\in W_{s^\frown\langle i\rangle}\mid d(x,N_{s^\frown\langle i\rangle})<d(x,X\setminus W_{s^\frown\langle i\rangle})\right\},$$

which insures that (i) and (vi) are satisfied by $U_{s^\frown\langle i\rangle}$ and completes the construction.

For every $s\in\mathbb{N}^{<\mathbb{N}}$ pick $x_s\in P\cap U_s$. Given $T\in\mathrm{Tr}$ let $F_T=\overline{\{x_s\mid s\in T\}}$. The map $\mathrm{Tr}\to\mathsf{F}(\overline{P})$, $T\mapsto F_T$, is Borel and we claim that $T\notin\mathrm{WF}$ if and only if $F_T\not\subseteq P$, so that $\mathrm{WF}\le_{\mathrm{B}}\left\{F\in\mathsf{F}(\overline{P})\mid F\subseteq P\right\}$.

If $T\notin\mathrm{WF}$ let $\alpha\in[T]$. Since $\lim_{k\to\infty}\operatorname{diam}(N_{\alpha\restriction k})=0$ we have $\lim_{k\to\infty}x_{\alpha\restriction k}=x_\alpha$. Thus $x_\alpha\in F_T\cap(\overline{P}\setminus P)$ and $F_T\not\subseteq P$.

Now suppose $F_T\not\subseteq P$ and let $x\in F_T\cap(\overline{P}\setminus P)$. There exists a sequence $(s_k)\subseteq T$ such that $\lim_{k\to\infty}x_{s_k}=x$. By condition (ii) of the construction $d(x,N)=0$ and, since $N$ is closed in $\overline{P}\setminus P$, $x\in N$. Hence $x=x_\alpha$ for some $\alpha\in\mathbb{N}^{\mathbb{N}}$. If $\alpha\notin[T]$ choose $n$ such that $\alpha\restriction n\notin T$. Since $U_s\cap U_{\alpha\restriction n}\neq\emptyset$ only for finitely many $s\in T$, if $k$ is large enough we have $x_{s_k}\notin U_{\alpha\restriction n}$: this contradicts $\lim_{k\to\infty}x_{s_k}=x_\alpha$. Therefore $\alpha\in[T]$ and $T\notin\mathrm{WF}$. $\square$

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