On compactness of solutions to the compressible isentropic Navier-Stokes equations when the density is not square integrable

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Abstract. We show compactness of bounded sets of weak solutions to the isentropic compressible Navier-Stokes equations in three space dimensions under the hypothesis that the adiabatic constant $\gamma > 3/2$.

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1. Introduction

In this paper, we generalize the result of Lions [9] concerning compactness of bounded sets of weak solutions to the Navier-Stokes equations of a compressible isentropic fluid flow:

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{1.1}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda + \mu)\nabla \ \operatorname{div} \mathbf{u} + a \nabla \varrho^\gamma = 0, \tag{1.2}$$

where the density $\varrho = \varrho(t, x)$ and the velocity $\mathbf{u} = \mathbf{u}(t, x)$ are functions of the time $t \in (0, T)$ and the spatial coordinate $x \in \Omega$. Here $\Omega \subset R^3$ is a bounded domain with regular boundary on which $\mathbf{u}$ satisfies the standard no-slip boundary conditions:

$$\mathbf{u}|_{\partial\Omega} = 0. \tag{1.3}$$

Formally, multiplying the second equation by $\mathbf{u}$ and integrating by parts one obtains the energy inequality:

$$\frac{d}{dt} E(t) + \int_\Omega \mu |\nabla \mathbf{u}|^2 + (\lambda + \mu) |\operatorname{div} \mathbf{u}|^2 \ dx \leq 0 \tag{1.4}$$

where

$$E(t) = E[\varrho, \mathbf{u}](t) = \int_\Omega \frac{1}{2} \varrho |\mathbf{u}|^2 + \frac{a}{\gamma - 1} \varrho^\gamma \ dx.$$

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We assume $a > 0$, $\gamma > 1$, $\mu > 0$, and $\lambda + 2/3\,\mu \geq 0$ throughout the whole text.

In what follows, we shall deal with *finite energy weak solutions* of the problem (1.1)–(1.3), specifically, ϱ, $\mathbf{u}$ will comply with the following hypotheses:

- ϱ, $\mathbf{u} = [u^1, u^2, u^3]$ satisfy
$$\varrho \geq 0,\ \ \varrho \in L^\infty(0,T; L^\gamma(\Omega)),\ \ u^i \in L^2(0,T; W_0^{1,2}(\Omega)),\ \ i = 1,2,3;$$
- the energy E is locally integrable on $(0,T)$ and the energy inequality (1.4) holds in $\mathcal{D}'(0,T)$;
- the equations (1.1), (1.2) are satisfied in $\mathcal{D}'((0,T)\times\Omega)$ and, in addition, (1.1) holds in the sense of renormalized solutions, i.e.,

$$\begin{cases} \partial_t b(\varrho) + \operatorname{div}(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\ \operatorname{div}\mathbf{u} = 0 \ \text{ in } \ \mathcal{D}'((0,T)\times\Omega) \\ \text{for any } \ b \in C^1(R) \ \text{ such that } \ |b'(z)z| + |b(z)| \leq c \ \text{ for all } \ z \in R. \end{cases} \tag{1.5}$$

Our main result reads as follows:

Theorem 1.1. *Let $\gamma > \frac{3}{2}$ and $\Omega \subset R^3$ be a bounded Lipschitz domain. Assume ϱ_n, $\mathbf{u}_n$ is a sequence of finite energy weak solutions of the problem* (1.1)–(1.3) *on the set $(0,T)\times\Omega$ satisfying*

$$\operatorname*{ess\,lim\,sup}_{t\to 0+} E[\varrho_n, \mathbf{u}_n](t) \leq E_0 \ \ \textit{uniformly in} \ \ n = 1,2,\ldots, \tag{1.6}$$

and

$$\varrho_n(0) \to \varrho_0 \ \ \textit{in} \ \ L^1(\Omega). \tag{1.7}$$

Then there exists a subsequence (*not relabeled*) *such that*

$$\varrho_n \to \varrho \ \ \textit{in} \ \ L^1((0,T)\times\Omega) \ \ \textit{and} \ \ C([0,T]; L^\gamma_{weak}(\Omega)),$$
$$\mathbf{u}_n \to \mathbf{u} \ \ \textit{weakly in} \ \ L^2(0,T; [W_0^{1,2}(\Omega)]^3)$$

where ϱ, $\mathbf{u}$ is a finite energy weak solution of the problem (1.1)–(1.3).

Theorem 1.1 generalizes a similar result of Lions [9] where the adiabatic exponent satisfies the restriction

$$\gamma \geq \frac{9}{5}. \tag{1.8}$$

Using (1.8) together with estimates analogous to those presented in Section 3 below, one can prove that the sequence ϱ_n is bounded in $L^2((0,T)\times\Omega)$ and so is its weak limit ϱ. This in turn implies that the limit functions ϱ, $\mathbf{u}$ represent a renormalized solution of (1.1) in the sense of DiPerna and Lions [3], i.e., they satisfy (1.5). Such a result is far from being obvious under the sole hypothesis $\gamma > 3/2$.

The rest of the paper is devoted to the proof of Theorem 1.1. In Section 2, we review some basic properties of renormalized solutions, in particular, we shall show that (1.5) holds, in fact, on the whole set $(0,T) \times R^3$ provided ϱ, $\mathbf{u}$ are prolonged to be zero out of Ω.

In Section 3, we introduce a generalized inverse of the divergence operator and obtain further L^p-estimates of the density analogous to those presented in [6] and by Lions [10].

The limit passage for $n \to \infty$ is carried out in Sections 4 and 5. Similarly as in [9] and [5], the main result (Proposition 5.1) asserts a sort of weak continuity of the quantity $a\varrho^\gamma - (\lambda + 2\mu)\ \mathrm{div}\,\mathbf{u}$ called the effective viscous flux. Here we give a simple proof based on Div-Curl Lemma of compensated compactness.

The main novelty of the present paper is the proof of strong convergence of the density in $L^1((0,T) \times \Omega)$ under the sole hypothesis $\gamma > 3/2$. This is done in Sections 6–8. The present approach is based on the cut-off operators introduced in [4] and [5]. More specifically, we consider a family of functions

$$T_k(z) = k\ T(\frac{z}{k}) \ \text{ for } \ z \in R,\ k = 1, 2, \dots$$

where $T \in C^\infty(R)$ is chosen so that

$$T(z) = z \text{ for } |z| \le 1,\ T(z) = 2 \text{ for } z \ge 3,\ T \text{ concave on } [0,\infty),\ T(-z) = -T(z).$$

The main idea, formulated in Proposition 6.1, is to show that

$$\limsup_{n\to\infty} \|T_k(\varrho_n) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)} \le c(E_0) \tag{1.9}$$

where the constant c is independent of k. This is an estimate in the spirit of Jiang and Zhang [8] where the authors prove the existence of weak solutions to the problem (1.1)–(1.3) with radially symmetric initial data. They show, roughly speaking, that neither the sequence ϱ_n nor its weak limit ϱ is square integrable but the amplitude of possible oscillations is.

The relation (1.9) is then used in Section 7 to prove that the limits ϱ, $\mathbf{u}$ satisfy (1.5). The proof of Theorem 1.1 is completed in Section 8 by showing the strong convergence of the densities ϱ_n.

2. Basic properties of renormalized solutions

Lemma 2.1. *Let ϱ, $\mathbf{u}$ be a finite energy weak solution of the problem* (1.1)–(1.3)
.

Then, prolonging ϱ, $\mathbf{u}$ to be zero outside Ω we have

$$\partial_t b(\varrho) + \mathrm{div}\,(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho))\ \mathrm{div}\,\mathbf{u} = 0 \ \ \textit{in} \ \ \mathcal{D}'((0,T) \times R^3) \tag{2.1}$$

for any b as in (1.5).

PROOF: We have to show

$$\int_0^T \int_{R^3} b(\varrho)\varphi_t + b(\varrho)\mathbf{u}.\nabla\varphi + (b(\varrho) - b'(\varrho)\varrho) \ \operatorname{div} \mathbf{u} \ \varphi \ dx \ dt = 0$$

for all $\varphi \in \mathcal{D}((0,T)\times R^3)$. To this end, consider a sequence of functions $\phi_m \in \mathcal{D}(\Omega)$ such that

$$(2.2) \qquad \begin{cases} 0 \le \phi_m \le 1, \ \phi_m(x) = 1 \ \text{ for all } \ x \ \text{ such that } \ \operatorname{dist}[x, \partial\Omega] \ge \dfrac{1}{m}, \\ |\nabla\phi_m(x)| \le 2m \ \text{ for all } \ x \in \Omega. \end{cases}$$

Now, we have

$$\int_0^T \int_{R^3} b(\varrho)\varphi_t \ dx \ dt = \int_0^T \int_\Omega b(\varrho)(\phi_m\varphi)_t + b(\varrho)(1-\phi_m)\varphi_t \ dx \ dt,$$

$$\int_0^T \int_{R^3} b(\varrho)\mathbf{u}.\nabla\varphi \ dx \ dt = \\ \int_0^T \int_\Omega b(\varrho)\mathbf{u}.\nabla(\phi_m\varphi) + b(\varrho)(1-\phi_m)\mathbf{u}.\nabla\varphi - b(\varrho)\mathbf{u}.\nabla\phi_m\varphi \ dx \ dt,$$

and

$$\int_0^T \int_{R^3} (b'(\varrho)\varrho - b(\varrho)) \ \operatorname{div} \mathbf{u} \ \varphi \ dx \ dt = \\ \int_0^T \int_\Omega \Big(b'(\varrho)\varrho - b(\varrho)\Big) \ \operatorname{div} \mathbf{u}\Big(\phi_m\varphi + (1-\phi_m)\varphi\Big) \ dx \ dt.$$

Since ϱ, $\mathbf{u}$ satisfy (1.5), one has

$$\int_0^T \int_\Omega b(\varrho)(\phi_m\varphi)_t + b(\varrho)\mathbf{u}.\nabla(\phi_m\varphi) + (b(\varrho) - b'(\varrho)\varrho) \ \operatorname{div} \mathbf{u} \ \phi_m \ \varphi \ dx \ dt = 0;$$

whence it is enough to show

$$(2.3) \qquad \int_0^T \int_\Omega b(\varrho)\mathbf{u}.\nabla\phi_m \ \varphi \ dx \ dt \to 0 \ \text{ as } \ m \to \infty.$$

The velocity components u^i, $i = 1,2,3$ belong to $L^2(0,T; W_0^{1,2}(\Omega))$ and, consequently,

$$|\mathbf{u}| \ \operatorname{dist}^{-1}[x, \partial\Omega] \in L^2(0,T; L^2(\Omega)).$$

On the other hand, by virtue of (2.2),

$$\mathrm{dist}[x, \partial\Omega] |\nabla \phi_m| \to 0 \ \text{ in } \ L^p(\Omega) \ \text{ for any } \ 1 \le p < \infty,$$

yielding (2.3). $\square$

With the conclusion of Lemma 2.1 at hand, we can regularize the equation (2.1) in the spirit of DiPerna and Lions [3]. Introducing a regularizing sequence $\vartheta_\varepsilon(x)$, one obtains

$$\partial_t S_\varepsilon[b(\varrho)] + \mathrm{div}\,(S_\varepsilon[b(\varrho)]\mathbf{u}) + S_\varepsilon\Big[(b'(\varrho)\varrho - b(\varrho))\,\mathrm{div}\,\mathbf{u}\Big] = r_\varepsilon \tag{2.4}$$

where $S_\varepsilon[v] = \vartheta_\varepsilon * v$. By virtue of [9, Lemma 2.3], we have

$$r_\varepsilon \to 0 \ \text{ in } \ L^2(0,T; L^2(R^3)) \ \text{ as } \ \varepsilon \to 0+ \tag{2.5}$$

since b is uniformly bounded.

3. More about integrability of the density

For any finite energy weak solution of (1.1), (1.2), the pressure term $p(\varrho) = a\varrho^\gamma$ belongs *a priori* only to the space $L^\infty(0,T; L^1(\Omega))$. We shall show that one can control possible concentration effects up to the boundary. To this end, we introduce the operator $\mathcal{B} = [\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3]$ enjoying the properties:

- $$\mathcal{B} : \Big\{ g \in L^p(\Omega) \mid \int_\Omega g = 0 \Big\} \mapsto [W_0^{1,p}(\Omega)]^3$$

  is a bounded linear operator, i.e.,

  $$\|\mathcal{B}[g]\|_{W_0^{1,p}(\Omega)} \le c(p)\|g\|_{L^p(\Omega)} \ \text{ for any } \ 1 < p < \infty;$$

- the function $\mathbf{v} = \mathcal{B}[g]$ solves the problem

  $$\mathrm{div}\,\mathbf{v} = g \ \text{ in } \ \Omega, \ \mathbf{v}|_{\partial\Omega} = 0;$$

- if, moreover, g can be written in the form $g = \mathrm{div}\,\mathbf{h}$ for a certain $\mathbf{h} \in [L^r(\Omega)]^3$, $\mathbf{h}.\mathbf{n}|_{\partial\Omega} = 0$, then

  $$\|\mathcal{B}[g]\|_{L^r(\Omega)} \le c(r)\|\mathbf{h}\|_{L^r(\Omega)}$$

  for arbitrary $1 < r < \infty$.

The operator $\mathcal{B}$ was introduced by Bogovskii [1]. A complete proof of the above mentioned properties may be found in Galdi [7, Theorem 3.3] or Borchers and Sohr [2, Proof of Theorem 2.4].

At this stage we can use the operator $\mathcal{B}$ to construct multipliers of the form

$$\varphi_i(t,x) = \psi(t)\mathcal{B}_i\Big[S_\varepsilon[b(\varrho)] - \oint_\Omega S_\varepsilon[b(\varrho)]\ dx\Big],\ i = 1,2,3,\ \psi \in \mathcal{D}(0,T)$$

where S_ε are the smoothing operators introduced in (2.4) and $\oint_\Omega v\,\mathrm{d}x = \frac{1}{|\Omega|}\int_\Omega v\,\mathrm{d}x$.

The functions φ_i are smooth with respect to the x-variable while $\partial_t\varphi_i$ are bounded in $L^2(0,T;W_0^{1,2}(\Omega))$ in view of (2.4), (2.5). Consequently, the quantities φ_i, $i = 1,2,3$ may be used as test functions for the equations (1.2) and, after a bit lengthy but straightforward computation where (2.4) is taken into account, one arrives at the following formula:

$$a\int_0^T\int_\Omega \psi\varrho^\gamma S_\varepsilon[b(\varrho)]\ \mathrm{d}x\ \mathrm{d}t = \tag{3.1}$$

$$\int_0^T \psi\Big(\int_\Omega a\varrho^\gamma\ \mathrm{d}x\Big)\Big(\oint_\Omega S_\varepsilon[b(\varrho)]\ \mathrm{d}x\Big)\,\mathrm{d}t + (\lambda+\mu)\int_0^T\int_\Omega \psi\ S_\varepsilon[b(\varrho)]\ \operatorname{div}\mathbf{u}\ \mathrm{d}x\ \mathrm{d}t -$$

$$\int_0^T\int_\Omega \psi_t\ \varrho u^i\ \mathcal{B}_i\Big\{S_\varepsilon[b(\varrho)] - \oint_\Omega S_\varepsilon[b(\varrho)]\mathrm{d}x\Big\}\ \mathrm{d}x\ \mathrm{d}t +$$

$$\int_0^T\int_\Omega \psi\Big(\mu\partial_{x_j}u^i - \varrho u^i u^j\Big)\ \partial_{x_j}\mathcal{B}_i\Big\{S_\varepsilon[b(\varrho)] - \oint_\Omega S_\varepsilon[b(\varrho)]\ \mathrm{d}x\Big\}\ \mathrm{d}x\ \mathrm{d}t +$$

$$\int_0^T\int_\Omega \psi\varrho u^i\,\mathcal{B}_i\Big\{S_\varepsilon\Big[(b(\varrho) - b'(\varrho)\varrho)\operatorname{div}\mathbf{u}\Big] - \oint_\Omega S_\varepsilon\Big[(b(\varrho) - b'(\varrho)\varrho)\operatorname{div}\mathbf{u}\Big]\,\mathrm{d}x\Big\}\,\mathrm{d}x\,\mathrm{d}t +$$

$$\int_0^T\int_\Omega \psi\varrho u^i\ \mathcal{B}_i\Big\{r_\varepsilon - \oint_\Omega r_\varepsilon\ \mathrm{d}x\Big\}\ \mathrm{d}x - \int_0^T\int_\Omega \psi\varrho u^i\ \mathcal{B}_i\Big\{\operatorname{div}\Big(S_\varepsilon[b(\varrho)]\mathbf{u}\Big)\Big\}\ \mathrm{d}x\ \mathrm{d}t$$

(the summation convention has been used).

Now, making use of (2.5), we can pass to the limit for $\varepsilon \to 0$ in (3.1). Moreover, approximating the function $z \mapsto z^\theta$ by a sequence of functions b_n satisfying (1.5), we deduce the following result (see [6] for details):

Proposition 3.1. *Let $\gamma > \frac{3}{2}$ and let ϱ, $\mathbf{u}$ be a finite energy weak solution of the problem* (1.1)–(1.3) *such that*

$$\operatorname{ess}\limsup_{t\to 0+} E(t) \le E_0.$$

Then there exist $\theta > 0$, depending only on γ, and $c = c(T,E_0)$, such that

$$\int_0^T\int_\Omega \varrho^{\gamma+\theta}\ dx\ dt \le c(T,E_0).$$

Remark. It can be shown (cf. Lions [9]) that the optimal value of θ is $\theta = \frac{2}{3}\gamma - 1$. Thus for $\gamma \ge 9/5$, one gets $\gamma + \theta \ge 2$.

4. The limit passage

The uniform energy estimates induced by (1.4) and the hypothesis (1.6) together with (1.1), (1.2) yield

$$\begin{aligned} &\varrho_n \to \varrho \ \text{ in } \ C([0,T]; L^{\gamma}_{weak}(\Omega)), \\ &\mathbf{u}_n \to \mathbf{u} \ \text{ weakly in } \ L^2(0,T; [W^{1,2}_0(\Omega)]^3), \end{aligned} \tag{4.1}$$

$$\varrho_n \mathbf{u}_n \to \varrho \mathbf{u} \ \text{ in } \ C([0,T]; L^{\frac{2\gamma}{\gamma+1}}_{weak}(\Omega)), \tag{4.2}$$

and, by virtue of Proposition 3.1,

$$\varrho^{\gamma} \to \overline{\varrho^{\gamma}} \ \text{ weakly in } \ L^{\frac{\gamma+\theta}{\gamma}}((0,T)\times\Omega)$$

passing to subsequences as the case may be. Moreover (4.1) together with (4.2) imply

$$\varrho u_n^i u_n^j \to \varrho u^i u^j, \ i,j = 1,2,\dots \ \text{ in, say, } \ \mathcal{D}'((0,T)\times\Omega)$$

and, consequently, ϱ, $\mathbf{u}$ satisfy

$$\partial_t \varrho + \operatorname{div}(\varrho \mathbf{u}) = 0, \tag{4.3}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}(\varrho \mathbf{u} \otimes \mathbf{u}) - \mu \Delta \mathbf{u} - (\lambda+\mu)\nabla \ \operatorname{div} \mathbf{u} + a \nabla \overline{\varrho^{\gamma}} = 0 \tag{4.4}$$

in $\mathcal{D}'((0,T)\times\Omega)$. Thus the only thing to prove is the strong convergence of ϱ_n in L^1 or, equivalently, $\overline{\varrho^{\gamma}} = \varrho^{\gamma}$.

By virtue of Lemma 2.1, we have

$$\partial_t T_k(\varrho_n) + \operatorname{div}(T_k(\varrho_n)\mathbf{u}_n) + (T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)) \ \operatorname{div} \mathbf{u}_n = 0 \quad \text{in } \ \mathcal{D}'((0,T)\times R^3) \tag{4.5}$$

where T_k are the cut-off functions introduced in Section 1.

Passing to the limit for $n \to \infty$ we obtain

$$\partial_t \overline{T_k(\varrho)} + \operatorname{div}(\overline{T_k(\varrho)}\mathbf{u}) + \overline{(T_k'(\varrho)\varrho - T_k(\varrho))\operatorname{div} \mathbf{u}} = 0 \ \text{ in } \ \mathcal{D}'((0,T)\times R^3) \tag{4.6}$$

where

$$(T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)) \ \operatorname{div} \mathbf{u}_n \to \overline{(T_k'(\varrho)\varrho - T_k(\varrho)) \ \operatorname{div} \mathbf{u}} \ \text{ weakly in } \ L^2((0,T)\times\Omega)$$

and

$$T_k(\varrho_n) \to \overline{T_k(\varrho)} \ \text{ in } \ C([0,T]; L^p_{weak}(\Omega)) \ \text{ for all } \ 1 \le p < \infty. \tag{4.7}$$

5. The effective viscous flux

We shall investigate the properties of the quantity $a\varrho^\gamma - (\lambda + 2\mu)\operatorname{div}\mathbf{u}$ called usually the effective viscous flux. It turns out that it is "more regular" than its components, in particular, it exhibits certain weak continuity. This is the crucial property used in the proof of existence of weak solutions as presented in Lions [9].

Proposition 5.1. *Under the hypotheses of Theorem 1.1, we have*

$$\lim_{n\to\infty}\int_0^T\int_\Omega \psi\phi\Big(a\varrho_n^\gamma - (\lambda+2\mu)\operatorname{div}\mathbf{u}_n\Big)T_k(\varrho_n)\,dx\,dt =$$
$$\int_0^T\int_\Omega \psi\phi\Big(a\overline{\varrho^\gamma} - (\lambda+2\mu)\operatorname{div}\mathbf{u}\Big)\overline{T_k(\varrho)}\,dx\,dt$$

for any $\psi \in \mathcal{D}(0,T)$, $\phi \in \mathcal{D}(\Omega)$.

Remark. Similar assertion with $T_k(\varrho)$ replaced by ϱ^θ may be found in Lions [9]. Here we give a different proof based on Div-Curl Lemma.

PROOF: Consider the operators

$$\mathcal{A}_j[v] = \Delta^{-1}\partial_{x_j}(v),\ j = 1,2,3, \quad \text{specifically,}$$
$$\mathcal{A}_j[v] = \mathcal{F}^{-1}\Big\{\frac{-\mathrm{i}\xi_j}{|\xi|^2}\mathcal{F}\{v\}(\xi)\Big\},\ j = 1,2,3,$$

where $\mathcal{F}$ denotes the Fourier transform.

By means of the Mikhlin multiplier theorem, we have

$$\|\partial_{x_i}\mathcal{A}_j[v]\|_{L^p(\Omega)} \leq c(p)\|v\|_{L^p(R^3)} \quad \text{for any} \quad 1 < p < \infty$$

and

$$\|\mathcal{A}_i[v]\|_{L^q(\Omega)} \leq c(q,r)\|v\|_{L^r(R^3)}$$

where $r \leq q \leq \frac{3r}{3-r}$ if $1 < r < 3$, q arbitrary finite if $r = 3$, $q = \infty$ for $r > 3$.

Now, we use the quantities

$$\varphi_i(t,x) = \psi(t)\phi(x)\mathcal{A}_i[T_k(\varrho_n)],\ \psi \in \mathcal{D}(0,T),\ \phi \in \mathcal{D}(\Omega),\ i = 1,2,3$$

as test functions for (1.2) (as always, ϱ_n is prolonged by zero outside Ω):

$$
\begin{aligned}
(5.1)\qquad & \int_0^T\int_\Omega \psi\phi\Big[a\varrho_n^\gamma-(\lambda+2\mu)\ \operatorname{div}\mathbf{u}_n\Big]T_k(\varrho_n)\ dx\ dt=\\
& \int_0^T\int_\Omega \psi\Big[(\lambda+\mu)\operatorname{div}\mathbf{u}_n-a\varrho_n^\gamma\Big]\partial_{x_i}\phi\ \mathcal{A}_i[T_k(\varrho_n)]\ dx\ dt+\\
& \mu\int_0^T\int_\Omega \psi\Big\{\nabla\phi.\nabla u_n^i\ \mathcal{A}_i[T_k(\varrho_n)]-u_n^i\ \partial_{x_j}\phi\ \partial_{x_j}\mathcal{A}_i[T_k(\varrho_n)]\Big\}\ dx\ dt+\\
& \mu\int_0^T\int_\Omega \psi\mathbf{u}_n.\nabla\phi\ T_k(\varrho_n)\ dx\ dt-\\
& \int_0^T\int_\Omega \phi\varrho_n u_n^i\Big\{\partial_t\psi\ \mathcal{A}_i[T_k(\varrho_n)]+\psi\mathcal{A}_i[(T_k(\varrho_n)-T_k'(\varrho_n)\varrho_n)\operatorname{div}\mathbf{u}_n]\Big\}\ dx\ dt-\\
& \int_0^T\int_\Omega \psi\ \varrho_n u_n^i u_n^j\ \partial_{x_j}\phi\ \mathcal{A}_i[T_k(\varrho_n)]\ dx\ dt+\\
& \int_0^T\int_\Omega \psi u_n^i\Big\{T_k(\varrho_n)\mathcal{R}_{i,j}[\phi\varrho_n u_n^j]-\phi\varrho_n u_n^j\mathcal{R}_{i,j}[T_k(\varrho_n)]\Big\}\ dx\ dt
\end{aligned}
$$

where the operators $\mathcal{R}_{i,j}$ are defined as

$$\mathcal{R}_{i,j}[v]=\mathcal{F}^{-1}\Big\{\frac{\xi_i\xi_j}{|\xi|^2}\mathcal{F}\{v\}(\xi)\Big\}.$$

Here, we have used the summation convention and (4.5).

Analogously, we can repeat the above arguments considering the equations (4.4), (4.6) and the test functions

$$\varphi_i(t,x)=\psi\phi\mathcal{A}_i[\overline{T_k(\varrho)}],\ i=1,2,3$$

to deduce

$$
\begin{aligned}
(5.2)\qquad & \int_0^T\int_\Omega \psi\phi\Big[a\overline{\varrho^\gamma}-(\lambda+2\mu)\ \operatorname{div}\mathbf{u}\Big]\overline{T_k(\varrho)}\ dx\ dt=\\
& \int_0^T\int_\Omega \psi\Big[(\lambda+\mu)\operatorname{div}\mathbf{u}-a\overline{\varrho^\gamma}\Big]\partial_{x_i}\phi\ \mathcal{A}_i[\overline{T_k(\varrho)}]\ dx\ dt+\\
& \mu\int_0^T\int_\Omega \psi\Big\{\nabla\phi\ \nabla u^i\ \mathcal{A}_i[\overline{T_k(\varrho)}]-u^i\ \partial_{x_j}\phi\ \partial_{x_j}\mathcal{A}_i[\overline{T_k(\varrho)}]+\mathbf{u}.\nabla\phi\ \overline{T_k(\varrho)}\Big\}\ dx\ dt-\\
& \int_0^T\int_\Omega \phi\overline{\varrho}\ u^i\Big\{\partial_t\psi\ \mathcal{A}_i[\overline{T_k(\varrho)}]+\psi\mathcal{A}_i[\overline{(T_k(\varrho)-T_k'(\varrho)\varrho)\ \operatorname{div}\mathbf{u}}]\Big\}\ dx\ dt-\\
& \int_0^T\int_\Omega \psi\ \overline{\varrho}\ u^i u^j\ \partial_{x_j}\phi\ \mathcal{A}_i[\overline{T_k(\varrho)}]\ dx\ dt+\\
& \int_0^T\int_\Omega \psi u^i\Big\{\overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho\ u^j]-\phi\varrho\ u^j\mathcal{R}_{i,j}[\overline{T_k(\varrho)}]\Big\}\ dx\ dt.
\end{aligned}
$$

It can be proved that all the terms on the right-hand side of (5.1) converge to their counterparts in (5.2) which yields the desired conclusion. Of course, the hardest term is the last integral in (5.1), (5.2) respectively, i.e., one has to show:

$$\int_0^T \int_\Omega \psi u_n^i \Big\{ T_k(\varrho_n)\mathcal{R}_{i,j}[\phi\varrho_n u_n^j] - \phi\varrho_n u_n^j \mathcal{R}_{i,j}[T_k(\varrho_n)] \Big\} \, dx \, dt \to \tag{5.3}$$
$$\int_0^T \int_\Omega \psi u^i \Big\{ \overline{T_k(\varrho)}\mathcal{R}_{i,j}[\phi\varrho u^j] - \phi\varrho u^j \mathcal{R}_{i,j}[\overline{T_k(\varrho)}] \Big\} \, dx \, dt.$$

In view of (4.1), (4.2), and (4.7), the relation (5.3) is a consequence of the following assertion:

Lemma 5.1. *Suppose*

$$v_n \to v \ \text{ weakly in } \ L^p(R^3), \ w_n \to w \ \text{ weakly in } \ L^q(R^3)$$

where $1/p + 1/q = 1/r < 1$.

Then

$$v_n \mathcal{R}_{i,j}[w_n] - w_n \mathcal{R}_{i,j}[v_n] \to v\mathcal{R}_{i,j}[w] - w\mathcal{R}_{i,j}[v] \ \text{ weakly in } \ L^r(R^3), \ i,j = 1,2,3.$$

Proof of Lemma 5.1: It is easy to see that the conclusion of Lemma 5.1 is a particular case of a more general statement:

$$\sum_{i,j=1}^3 v_n^i \mathcal{R}_{i,j}[w_n^j] - w_n^j \mathcal{R}_{i,j}[v_n^i] \to \sum_{i,j=1}^3 v^i \mathcal{R}_{i,j}[w^j] - w^j \mathcal{R}_{i,j}[v^i] \ \text{ in } \ \mathcal{D}'(R^3). \tag{5.4}$$

provided $\mathbf{v}_n = [v_n^1, v_n^2, v_n^3]$, $\mathbf{w}_n = [w_n^1, w_n^2, w_n^3]$ are sequences of vector functions satisfying

$$\mathbf{v}_n \to \mathbf{v} \ \text{ weakly in } \ [L^p(R^3)]^3, \ \mathbf{w}_n \to \mathbf{w} \ \text{ weakly in } \ [L^q(R^3)]^3.$$

Indeed, Lemma 5.1 follows from (5.4) taking $\mathbf{v}_n = v_n \mathbf{e}_i$, $\mathbf{w}_n = w_n \mathbf{e}_j$ where $\mathbf{e}_i$, $i = 1,2,3$ is the orthogonal basis of R^3.

To show (5.4), one can use the symmetry $\mathcal{R}_{i,j} = \mathcal{R}_{j,i}$ to deduce

$$\sum_{i,j=1}^3 v_n^i \mathcal{R}_{i,j}[w_n^j] - w_n^j \mathcal{R}_{i,j}[v_n^i] =$$
$$\sum_{i=1}^3 \Big[\Big(v_n^i - (\sum_{k=1}^3 \mathcal{R}_{i,k}[v_n^k]) \Big) (\sum_{j=1}^3 \mathcal{R}_{i,j}[w_n^j]) \Big] -$$
$$\sum_{j=1}^3 \Big[\Big(w_n^j - (\sum_{k=1}^3 \mathcal{R}_{k,j}[w_n^k]) \Big) (\sum_{i=1}^3 \mathcal{R}_{i,j}[v_n^i]) \Big] =$$
$$U_n . V_n - X_n . Y_n$$

where

$$\text{div } U_n = \sum_{i=1}^{3} \partial_{x_i} \Big(v_n^i - (\sum_{k=1}^{3} \mathcal{R}_{i,k}[v_n^k]) \Big) =$$

$$\text{div } X_n = \sum_{j=1}^{3} \partial_{x_j} \Big(w_n^j - (\sum_{k=1}^{3} \mathcal{R}_{j,k}[w_n^k]) \Big) = 0$$

and

$$V_n = \nabla(\Delta^{-1} \sum_{j=1}^{3} \partial_{x_j} w_n^j),\ Y_n = \nabla(\Delta^{-1} \sum_{i=1}^{3} \partial_{x_i} v_n^i),\ \text{i.e., } \mathrm{curl}(V_n) = \mathrm{curl}(Y_n) = 0.$$

Consequently, it is possible to use the $L^p - L^q$ version of Div-Curl Lemma (see e.g. Yi [11]) to conclude

$$U_n.V_n \to U.V,\ X_n.Y_n \to X.Y \quad \text{in} \quad \mathcal{D}'(R^3)$$

where

$$U^i = \Big(v^i - (\sum_{k=1}^{3} \mathcal{R}_{i,k}[v^k]) \Big),\ V^i = \sum_{j=1}^{3} \mathcal{R}_{i,j}[w^j],$$

$$X^j = \Big(w^j - (\sum_{k=1}^{3} \mathcal{R}_{j,k}[w^k]) \Big),\ Y^j = \sum_{i=1}^{3} \mathcal{R}_{j,i}[v^i],\ i,j = 1,2,3.$$

□

We have proved Proposition 5.1. □

6. The amplitude of oscillations

The main result of this section is inspired by the paper of Jiang and Zhang [8].

Proposition 6.1. *Under the hypotheses of Theorem* 1.1, *let* ϱ *be a weak limit of the sequence* ϱ_n.

Then

$$\limsup_{n\to\infty} \|T_k(\varrho_n) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)} \le c(E_0)$$

where the constant $c(E_0)$ *is independent of* k.

PROOF: One has

$$\lim_{n\to\infty} \int_0^T \int_\Omega \varrho_n^\gamma T_k(\varrho_n) - \overline{\varrho^\gamma}\ \overline{T_k(\varrho)}\ dx\ dt = \tag{6.1}$$
$$\lim_{n\to\infty} \int_0^T \int_\Omega (\varrho_n^\gamma - \varrho^\gamma)(T_k(\varrho_n) - T_k(\varrho))\ dx\ dt +$$
$$\int_0^T \int_\Omega (\overline{\varrho^\gamma} - \varrho^\gamma)(T_k(\varrho) - \overline{T_k(\varrho)})\ dx\ dt \geq$$
$$\lim_{n\to\infty} \int_0^T \int_\Omega (\varrho_n^\gamma - \varrho^\gamma)(T_k(\varrho_n) - T_k(\varrho))\ dx\ dt \geq$$
$$\limsup_{n\to\infty} \int_0^T \int_\Omega |T_k(\varrho_n) - T_k(\varrho)|^{\gamma+1}\ dx\ dt$$

as $z \mapsto z^\gamma$ is convex, T_k concave on $[0,\infty)$, and

$$(z^\gamma - y^\gamma)(T_k(z) - T_k(y)) \geq |T_k(z) - T_k(y)|^{\gamma+1} \quad \text{for all} \quad z, y \geq 0.$$

On the other hand,

$$\lim_{n\to\infty} \int_0^T \int_\Omega \operatorname{div} \mathbf{u}_n\ T_k(\varrho_n) - \operatorname{div} \mathbf{u}\ \overline{T_k(\varrho)}\ dx\ dt = \tag{6.2}$$
$$\lim_{n\to\infty} \int_0^T \int_\Omega \Big(T_k(\varrho_n) - T_k(\varrho) + T_k(\varrho) - \overline{T_k(\varrho)} \Big) \operatorname{div} \mathbf{u}_n\ dx\ dt \leq$$
$$2 \sup_n \| \operatorname{div} \mathbf{u}_n \|_{L^2((0,T)\times\Omega)} \limsup_{n\to\infty} \| T_k(\varrho_n) - T_k(\varrho) \|_{L^2((0,T)\times\Omega)}.$$

The relations (6.1), (6.2) combined with Proposition 5.1 yield the desired conclusion.

□

7. The renormalized solutions

Proposition 7.1. *Under the hypotheses of Theorem* 1.1, *the limit functions* ϱ, $\mathbf{u}$ *solve* (4.3) *in the sense of renormalized solutions, i.e.,*

$$\partial_t b(\varrho) + \operatorname{div}(b(\varrho)\mathbf{u}) + (b'(\varrho)\varrho - b(\varrho)) \operatorname{div} \mathbf{u} = 0 \tag{7.1}$$

holds in $\mathcal{D}'((0,T)\times R^3)$ *for any* $b \in C^1(R)$, $|b'(z)z| + |b(z)| \leq c$ *provided* ϱ, $\mathbf{u}$ *are set zero outside* Ω.

PROOF: It is enough to prove (7.1) for any b satisfying, in addition to the above hypotheses,

$$b'(z) = 0 \quad \text{for all} \quad z \quad \text{large enough, say, } z \geq M$$

where M is a certain constant. The rest follows by a simple density argument.

Regularizing (4.6) one gets

$$\partial_t S_\varepsilon[\overline{T_k(\varrho)}] + \operatorname{div}(S_\varepsilon[\overline{T_k(\varrho)}]\mathbf{u}) + S_\varepsilon[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\ \operatorname{div}\mathbf{u}}] = r_\varepsilon \tag{7.2}$$

where $r_\varepsilon \to 0$ in $L^2(0,T;L^2(R^3))$ for any fixed k.

Multiplying (7.2) by $b'(S_\varepsilon[\overline{T_k(\varrho)}])$ and letting $\varepsilon \to 0$ we deduce

$$\begin{gathered}\partial_t b(\overline{T_k(\varrho)}) + \operatorname{div}(b(\overline{T_k(\varrho)})\mathbf{u}) + \Big(b'(\overline{T_k(\varrho)})\overline{T_k(\varrho)} - b(\overline{T_k(\varrho)})\Big)\ \operatorname{div}\mathbf{u} = \\ b'(\overline{T_k(\varrho)})[\overline{(T_k(\varrho) - T_k'(\varrho)\varrho)\operatorname{div}\mathbf{u}}]\end{gathered} \tag{7.3}$$

in $\mathcal{D}'((0,T)\times R^3)$.

At this stage, the idea is to pass to the limit in (7.3) for $k \to \infty$. We have

$$\overline{T_k(\varrho)} \to \varrho \ \text{ as } \ k\to\infty \ \text{ in } \ L^p((0,T)\times\Omega) \ \text{ for any } \ 1 \le p < \gamma$$

since

$$\|\overline{T_k(\varrho)} - \varrho\|_{L^p((0,T)\times\Omega)} \le \liminf_{n\to\infty} \|T_k(\varrho_n) - \varrho_n\|_{L^p((0,T)\times\Omega)}$$

and

$$\|T_k(\varrho_n) - \varrho_n\|^p_{L^p((0,T)\times\Omega)} \le 2^p k^{p-\gamma}\|\varrho_n\|^\gamma_{L^\gamma((0,T)\times\Omega)}. \tag{7.4}$$

Thus (7.3) will imply (7.1) provided we show

$$b'(\overline{T_k(\varrho)})[\overline{(T_k'(\varrho)\varrho - T_k(\varrho)\operatorname{div}\mathbf{u}}] \to 0 \ \text{ in } \ L^1((0,T)\times\Omega) \ \text{ as } \ k\to\infty. \tag{7.5}$$

Denoting

$$Q_{k,M} = \{(t,x)\in(0,T)\times\Omega \mid \overline{T_k(\varrho)} \le M\},$$

we can estimate

$$\begin{gathered}\int_0^T\int_\Omega \left| b'(\overline{T_k(\varrho)})[\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\operatorname{div}\mathbf{u}}]\right|\,dx\,dt \le \\ \sup_{0\le z\le M}|b'(z)|\int\int_{Q_{k,M}}\left|\overline{(T_k'(\varrho)\varrho - T_k(\varrho))\operatorname{div}\mathbf{u}}\right|\,dx\,dt \le \\ \sup_{0\le z\le M}|b'(z)|\ \sup_n\|\mathbf{u}_n\|_{L^2(0,T;W^{1,2}(\Omega))}\ \liminf_{n\to\infty}\|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|_{L^2(Q_{k,M})}.\end{gathered}$$

Now, by interpolation,

$$\begin{gathered}\|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|^2_{L^2(Q_{k,M})} \le \\ \|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|^\alpha_{L^1((0,T)\times\Omega)}\|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|^{(1-\alpha)(\gamma+1)}_{L^{\gamma+1}(Q_{k,M})},\ \alpha = \frac{\gamma-1}{\gamma}\end{gathered} \tag{7.6}$$

where, similarly as in (7.4),

$$\|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|_{L^1((0,T)\times\Omega)} \leq 2^\gamma k^{1-\gamma} \sup_n \|\varrho_n\|^\gamma_{L^\gamma((0,T)\times\Omega)}, \tag{7.7}$$

and

$$\begin{gathered}\|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|_{L^{\gamma+1}(Q_{k,M})} \leq \\ 2\Big(\|T_k(\varrho_n) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)} + \|T_k(\varrho)\|_{L^{\gamma+1}(Q_{k,M})}\Big) \leq \\ 2\Big(\|T_k(\varrho_n) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)} + \|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^{\gamma+1}((0,T)\times\Omega)} + \\ \|\overline{T_k(\varrho)}\|_{L^{\gamma+1}(Q_{k,M})}\Big) \leq \\ 2\|T_k(\varrho_n) - T_k(\varrho)\|_{L^{\gamma+1}((0,T)\times\Omega)} + 2\|T_k(\varrho) - \overline{T_k(\varrho)}\|_{L^{\gamma+1}((0,T)\times\Omega)} + 2M|\Omega|.\end{gathered} \tag{7.8}$$

By virtue of Proposition 6.1 and (7.8), one gets

$$\limsup_{n\to\infty} \|T_k'(\varrho_n)\varrho_n - T_k(\varrho_n)\|_{L^{\gamma+1}(Q_{k,M})} \leq 2c + 2M|\Omega|$$

which, together with (7.6), (7.7), completes the proof of (7.5). □

8. Strong convergence of the density

We introduce a family of functions L_k:

$$L_k(z) = \begin{cases} z\log(z) \quad \text{for} \quad 0 \leq z < k, \\ z\log(k) + z\int_k^z T_k(s)/s^2 \, ds \quad \text{for} \quad z \geq k. \end{cases}$$

Seeing that L_k can be written as

$$L_k(z) = \beta_k z + b_k(z) \tag{8.1}$$

where $|b_k(z)| \leq c(k)$ and $b_k'(z)z - b_k(z) = T_k(z)$ for all $z > 0$, we can combine (1.1), (1.5) to deduce

$$\partial_t L_k(\varrho_n) + \operatorname{div}(L_k(\varrho_n)\mathbf{u}_n) + T_k(\varrho_n) \ \operatorname{div} \mathbf{u}_n = 0 \tag{8.2}$$

and, by virtue of (4.3) and Proposition 7.1,

$$\partial_t L_k(\varrho) + \operatorname{div}(L_k(\varrho)\mathbf{u}) + T_k(\varrho) \ \operatorname{div} \mathbf{u} = 0 \tag{8.3}$$

in $\mathcal{D}'((0,T)\times\Omega)$.

Consequently, we can assume

$$L_k(\varrho_n) \to \overline{L_k(\varrho)} \text{ in } C([0,T]; L^\gamma_{weak}(\Omega)) \tag{8.4}$$

and, approximating $z\log(z) \approx L_k(z)$,

$$\varrho_n \log(\varrho_n) \to \overline{\varrho \log(\varrho)} \text{ in } C([0,T]; L^\alpha_{weak}(\Omega)) \text{ for any } 1 \le \alpha < \gamma.$$

Taking the difference of (8.2) and (8.3) and integrating with respect to t we get

$$\int_\Omega (L_k(\varrho_n) - L_k(\varrho))(t)\phi \, dx = \int_\Omega (L_k(\varrho_n)(0) - L_k(\varrho_0))\phi \, dx +$$
$$\int_0^t \int_\Omega (L_k(\varrho_n)\mathbf{u}_n - L_k(\varrho)\mathbf{u}).\nabla\phi + (T_k(\varrho) \operatorname{div} \mathbf{u} - T_k(\varrho_n) \operatorname{div} \mathbf{u}_n)\phi \, dx \, dt$$

for any $\phi \in \mathcal{D}(\Omega)$. Passing to the limit for $n \to \infty$ and making use of the hypothesis (1.7) together with (8.4), one obtains

$$\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(t)\phi \, dx =$$
$$\int_0^t \int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))\mathbf{u}).\nabla\phi \, dx \, dt +$$
$$\lim_{n\to\infty} \int_0^t \int_\Omega (T_k(\varrho) \operatorname{div} \mathbf{u} - T_k(\varrho_n) \operatorname{div} \mathbf{u}_n)\phi \, dx \, dt$$

Taking $\phi = \phi_m$ the sequence approximating the characteristic function of Ω as in (2.2) and making use of the boundary conditions (1.3), one derives

$$\begin{aligned} &\int_\Omega (\overline{L_k(\varrho)} - L_k(\varrho))(t) \, dx = \\ &\int_0^t \int_\Omega T_k(\varrho) \operatorname{div} \mathbf{u} \, dx \, dt - \lim_{n\to\infty} \int_0^t \int_\Omega T_k(\varrho_n) \operatorname{div} \mathbf{u}_n \, dx \, dt. \end{aligned} \tag{8.5}$$

Observe that the term $\overline{L_k(\varrho)} - L_k(\varrho)$ is bounded in view of (8.1).

Finally, making use of Proposition 5.1 and the monotonicity of the pressure, we cane estimate the right-hand side of (8.5):

$$\begin{aligned} &\int_0^t \int_\Omega T_k(\varrho) \operatorname{div} \mathbf{u} \, dx \, dt - \lim_{n\to\infty} \int_0^t \int_\Omega T_k(\varrho_n) \operatorname{div} \mathbf{u}_n \, dx \, dt \le \\ &\int_0^t \int_\Omega (T_k(\varrho) - \overline{T_k(\varrho)}) \operatorname{div} \mathbf{u} \, dx \, dt. \end{aligned} \tag{8.6}$$

By virtue of Proposition 6.1, the right-hand side of (8.6) tends to zero as $k \to \infty$. Accordingly, one can pass to the limit for $k \to \infty$ in (8.5) to conclude

$$\overline{\varrho \log(\varrho)}(t) = \varrho \log(\varrho)(t) \text{ for all } t \in [0,T]$$

which implies strong convergence of the sequence ϱ_n in $L^1((0,T) \times \Omega)$.

Theorem 1.1 has been proved.

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