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The property (β) of Orlicz-Bochner sequence spaces

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Abstract. A characterization of property (β) of an arbitrary Banach space is given. Next it is proved that the Orlicz-Bochner sequence space $l_\Phi(X)$ has the property (β) if and only if both spaces l_Φ and X have it also. In particular the Lebesgue-Bochner sequence space $l_p(X)$ has the property (β) iff X has the property (β). As a corollary we also obtain a theorem proved directly in [5] which states that in Orlicz sequence spaces equipped with the Luxemburg norm the property (β), nearly uniform convexity, the drop property and reflexivity are in pairs equivalent.

Keywords: Orlicz-Bochner space, property (β), Orlicz space

Classification: 46E30, 46E40, 46B20

1. Introduction

Let $(X, \|\cdot\|)$ be a real Banach space, $B(X)$ and $S(X)$ be the closed unit ball, unit sphere of X, respectively. For any subset A of X, we denote by $\mathrm{conv}(A)$ the convex hull of A.

The Banach space $(X, \|\cdot\|)$ is *uniformly convex* ($X \in (\mathbf{UC})$ for short), if for each $\epsilon > 0$ there is $\delta > 0$ such that for $x, y \in S(X)$ the inequality $\|x - y\| > \epsilon$ implies $\left\|\frac{1}{2}(x+y)\right\| < 1 - \delta$ (see [4]).

Define for any $x \notin B(X)$ the *drop* $D(x, B(X))$ determined by x by

$$D(x, B(X)) = \mathrm{conv}(\{x\} \cup B(X)).$$

A Banach space X has the *drop property* ($X \in (\mathbf{D})$) if for every closed set C disjoint with $B(X)$ there exists an element $x \in C$ such that $D(x, B(X)) \cap C = \{x\}$.

Recall that for any subset C of X, the *Kuratowski measure of non-compactness* of C is the infimum $\alpha(C)$ of those $\epsilon > 0$ for which there is a covering of C by a finite number of sets of diameter less then ϵ. Rolewicz in [20] has proved that X is uniformly convex iff for any $\epsilon > 0$ there exists $\delta > 0$ such that $1 < \|x\| < 1 + \delta$ implies $\mathrm{diam}(D(x, B(X)) \setminus B(X)) < \epsilon$. In connection with this he has introduced in [21] the following property.

A Banach space X has the *property* $(\boldsymbol{\beta})$ ($X \in (\boldsymbol{\beta})$ for short) if for any $\epsilon > 0$ there exists $\delta > 0$ such that

$$\alpha(D(x, B(X)) \setminus B(X)) < \epsilon$$

whenever $1 < \|x\| < 1 + \delta$.

We say that a sequence $\{x_n\} \subset X$ is ϵ-*separated* for some $\epsilon > 0$ if

$$\operatorname{sep}(x_n) = \inf\{\|x_n - x_m\| : n \neq m\} > \epsilon.$$

The following characterization of the property $(\boldsymbol{\beta})$ is very useful (see [14]):

A Banach space X has the property $(\boldsymbol{\beta})$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that for each element $x \in B(X)$ and each sequence (x_n) in $B(X)$ with $\operatorname{sep}(x_n) \geq \epsilon$ there is an index k for which

$$\left\| \frac{x + x_k}{2} \right\| \leq 1 - \delta.$$

A Banach space is said to be *nearly uniformly convex* ($X \in (\mathbf{NUC})$) if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that for every sequence $\{x_n\} \subseteq B(X)$ with $\operatorname{sep}(x_n) > \epsilon$, we have $\operatorname{conv}(\{x_n\}) \cap (1-\delta)B(X) \neq \emptyset$.

The following implications are true in any Banach space

$$(\mathbf{UC}) \Rightarrow (\boldsymbol{\beta}) \Rightarrow (\mathbf{NUC}) \Rightarrow (\mathbf{D}) \Rightarrow (\mathbf{Rfx}),$$

where $(\mathbf{Rfx})$ denotes the reflexivity (see [9], [17] and [21]). Any of them cannot be reversed in general. However the uniform convexity and the property $(\boldsymbol{\beta})$ are equivalent in Orlicz-Lorentz function spaces and the property $(\boldsymbol{\beta})$ and reflexivity are equivalent in Orlicz sequence spaces (see [5] and [12]).

The Banach space X is said to have *uniformly Kadec-Klee property* ($X \in (\mathbf{UKK})$ for short) if for every $\epsilon > 0$ there exists $\delta \in (0,1)$ such that

$$(\mathbf{UKK}): \qquad \left.\begin{array}{c} (x_n) \subset B(X) \\ x_n \xrightarrow{w} x \\ \operatorname{sep}(x_n) \geq \epsilon \end{array}\right\} \implies \quad \|x\|_X < 1 - \delta.$$

It is known that $X \in (\mathbf{NUC})$ iff $X \in (\mathbf{UKK})$ and X is reflexive ([9]).

In this paper a characterization of the property $(\boldsymbol{\beta})$ of an arbitrary Banach space is given. This result enables us to consider the property $(\boldsymbol{\beta})$ in Orlicz-Bochner sequence spaces $l_\Phi(X)$. One of the fundamental problems in these spaces is the question of whether or not a geometrical property lifts from X to $l_\Phi(X)$. Although the answer to such a question is often expected, the proof of such a response is usually nontrivial. Considerations of that type for various kinds of convexities for different spaces of Bochner type were done by many authors (see for instance [1], [2], [3], [6], [8], [13], [18], [19]). We will prove that the Orlicz-Bochner sequence space $l_\Phi(X)$ has the property $(\boldsymbol{\beta})$ if and only if both spaces l_Φ and X have it also.

Denote by $\mathbb{N}$ and $\mathbb{R}$ the sets of natural and real numbers, respectively.

A map $\Phi : \mathbb{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ is vanishing at 0, even, convex and not identically equal to zero. Let l^0 stand for the space of all real sequences. By the *Orlicz sequence space* we mean

$$l_\Phi = \left\{ x \in l^0 : I_\Phi(cx) = \sum_{i=1}^{\infty} \Phi\left(cx(i)\right) < \infty \quad \text{for some} \quad c > 0 \right\}.$$

We endow l_Φ with the so called *Luxemburg norm* defined by

$$\|x\|_\Phi = \inf \left\{ \epsilon > 0 : I_\Phi \left(\frac{x}{\epsilon} \right) \leq 1 \right\}.$$

For every Orlicz function Φ we define *the complementary function* $\Psi : \mathbb{R} \longrightarrow [0, \infty)$ by the formula

$$\Psi(v) = \sup_{u>0} \{u|v| - \Phi(u)\}$$

for every $v \in \mathbb{R}$. The complementary function Ψ is also an Orlicz function.

We say that the Orlicz function Φ *satisfies the* δ_2*-condition* (we write $\Phi \in \delta_2$) if there exist constants $k_0 > 2$ and $u_0 > 0$ such that

$$0 < \Phi(u_0) < \infty \quad \text{and} \quad \Phi(2u) \leq k_0 \Phi(u) \tag{1}$$

for every $|u| \leq u_0$.

Now, let us define the type of spaces to be considered in this paper. For a real Banach space $\langle X, \|\cdot\|_X \rangle$, denote by $\mathcal{M}(\mathbb{N}, X)$, or just by $\mathcal{M}(X)$, the space of sequences $x = (x_n)$ such that $x_n \in X$ for all $n \in \mathbb{N}$. Define on $\mathcal{M}(X)$ a modular $\widetilde{I_\Phi}(x)$ by the formula

$$\widetilde{I_\Phi}(x) = \sum_{i=1}^{\infty} \Phi\left(\|x(i)\|_X\right).$$

Let

$$l_\Phi(X) = \left\{ x \in \mathcal{M}(X) : x_0 = \left(\|x(i)\|_X\right)_{i=1}^{\infty} \in l_\Phi \right\}.$$

Then $l_\varphi(X)$ equipped with the norm $\|x\| = \|x_0\|_\Phi$ becomes a Banach space which is called *the Orlicz-Bochner sequence space.*

2. Auxiliary lemmas

Lemma 1. *Suppose that* $\Phi \in \delta_2$ *with some constants* u_0 *and* k_0 *defined in* (1). *Then*

$$\lim_{k\to\infty} \left\{ \Phi\left(\left(1 + 1/k\right) u\right) / \Phi\left(u\right) \right\} = 1$$

uniformly for all $|u| \leq u_0$ (Lemma 1.1 in [7]).

Lemma 2. *If $x, y \in X \setminus \{0\}$, then*

$$\|x+y\| \leq \frac{1}{2}\|\hat{x}+\hat{y}\|(\|x\|+\|y\|)+\left(1-\frac{1}{2}\|\hat{x}+\hat{y}\|\right)|\|x\|-\|y\||,$$

where $\hat{x} = x/\|x\|$ (Lemma 1.1 in [8]).

Lemma 3. *If $\Psi \in \delta_2$, then for every $w > 0$ with $0 < \Phi(w) < \infty$ there exist numbers $a = a(w) \in (0,1)$ and $\gamma = \gamma(a(w)) \in (0,1)$ such that*

$$\Phi\left(\frac{u+v}{2}\right) \leq \frac{1}{2}(1-\gamma)(\Phi(u)+\Phi(v)) \tag{2}$$

for all $u \leq w$ and v satisfying $\left|\frac{v}{u}\right| \leq a$.

Proof: We will apply some methods from Lemma 1.1 in [3]. Let $w > 0$ satisfy $0 < \Phi(w) < \infty$. It is well known that

$$\lim_{v\to\infty}\frac{\Psi(v)}{v} = \sup\{u>0 : \Phi(u)<\infty\}.$$

Hence there exists $v_0 = v_0(w)$ such that $0 < \Psi(v_0) < \infty$ and for every $c \in (1,2)$ we get

$$\Phi\left(\frac{c}{2}u\right) = \sup_{v>0}\left\{\frac{c}{2}|u|\,v-\Psi(v)\right\} = \sup_{0<v\leq v_0}\left\{\frac{c}{2}|u|\,v-\Psi(v)\right\}$$

for every $u \leq w$. On the other hand, by $\Psi \in \delta_2$, we obtain that there exists a number $k = k(v_0)$ such that $\Psi(2v) \leq k\Psi(v)$ for every $|v| \leq v_0$. Then, applying Lemma 1, we conclude that there exists a number $\xi \in (1,2)$ such that $\Psi(\xi^2 v) \leq 2\xi\Psi(v)$ for every $|v| \leq v_0$. Hence

$$\begin{aligned}\Phi\left(\frac{\xi}{2}u\right) &= \sup_{v\geq 0}\left\{\frac{\xi}{2}|u|v-\Psi(v)\right\} = \sup_{0<v\leq v_0}\left\{\frac{\xi}{2}|u|v-\Psi(v)\right\}\\ &\leq \sup_{0<v\leq v_0}\left\{\frac{\xi}{2}|u|\,v-\frac{1}{2\xi}\Psi\left(\xi^2 v\right)\right\} \leq \frac{1}{2\xi}\Phi(u)\end{aligned}$$

for every $u \leq w$. Then the proof can be easily finished (see [3]). □

Lemma 4. *Let $\Phi \in \delta_2$. The following assertions are true:*

(a) $\|x_n\| = 1$ *iff* $\widetilde{I_\Phi}(x_n) = 1$;

(b) *for every sequence $(x_n) \in l_\varphi(X)$ we have $\|x_n\| \to 0$ iff $\widetilde{I_\Phi}(x_n) \to 0$;*

(c) *for every $p \in (0,1)$ there exists $q \in (0,1)$ such that the inequality $\widetilde{I_\Phi}(x) \leq 1-p$ implies $\|x\| \leq 1-q$.*

Proof: (a) It was shown in [11].

(b) It is known that $\|x_n\| \to 0$ iff $\widetilde{I_\Phi}(\eta x_n) \to 0$ for any $\eta > 0$. Then, in view of δ_2-condition, one can complete the proof.

(c) The statement in the case $X = \mathbb{R}$ was proved in [10]. For an arbitrary Banach space the proof is similar. □

3. Results

Theorem 1. *A Banach space X has the property ($\boldsymbol{\beta}$) if and only if for every $\epsilon_0 > 0$ there exists $\delta_0 > 0$ such that for each element $x \in X\backslash\{0\}$ and each sequence (x_n) in $X\backslash\{0\}$ with* $\operatorname{sep}\left(\frac{x_n}{\|x_n\|_X}\right) \geq \epsilon_0$ *there is an index k for which*

$$\left\|\frac{x+x_k}{2}\right\|_X \leq \frac{1}{2}\left(\|x\|_X + \|x_k\|_X\right)\left(1 - \frac{2\delta_0 \min\{\|x\|_X, \|x_k\|_X\}}{\|x\|_X + \|x_k\|_X}\right).$$

Proof: **Necessity**. Take $\epsilon_0 > 0$ and $x \in X\backslash\{0\}$. Let the sequence (x_n) in $X\backslash\{0\}$ be such that $\operatorname{sep}\left(\frac{x_n}{\|x_n\|_X}\right) \geq \epsilon_0$. Define $y = \frac{x}{\|x\|_X}$ and $y_n = \frac{x_n}{\|x_n\|_X}$. Then $y, y_n \in B(X)$ and $\operatorname{sep}(y_n) \geq \epsilon_0$. By the property ($\boldsymbol{\beta}$) of X there exist a number $\delta = \delta(\epsilon_0)$ an index k such that $\left\|\frac{y+y_k}{2}\right\|_X \leq 1 - \delta$. Let $\delta_0 = \delta$. If $\|x\|_X \geq \|x_k\|_X$, then

$$\begin{aligned} 1 - \delta_0 \geq \frac{1}{2}\left\|\frac{x}{\|x\|_X} + \frac{x_k}{\|x_k\|_X}\right\|_X &= \left\|\frac{x+x_k}{2\|x_k\|_X} - \frac{x}{2}\left(\frac{1}{\|x_k\|_X} - \frac{1}{\|x\|_X}\right)\right\|_X \\ &\geq \left\|\frac{x+x_k}{2\|x_k\|_X}\right\|_X - \left\|\frac{x}{2}\right\|_X \left|\frac{1}{\|x_k\|_X} - \frac{1}{\|x\|_X}\right|. \end{aligned}$$

Hence a simple computation yields

$$\left\|\frac{x+x_k}{2}\right\|_X \leq \frac{1}{2}\left(\|x\|_X + \|x_k\|_X\right)\left(1 - \frac{2\delta_0 \min\{\|x\|_X, \|x_k\|_X\}}{\|x\|_X + \|x_k\|_X}\right).$$

If $\|x\|_X < \|x_k\|_X$, then the proof is analogous.

Sufficiency. Let $\epsilon > 0$ and $x \in B(X)$. Take a sequence (x_n) in $B(X)$ with $\operatorname{sep}(x_n) \geq \epsilon$. Passing to subsequence, if necessary, we may assume that $\|x_n\|_X \to b, b \in [\epsilon/2, 1]$ and $\|x_n\|_X \geq \epsilon/4$ for every $n \in \mathbb{N}$. Then, applying Lemma 2, we conclude that there exist a number $\epsilon_0 = \epsilon_0(\epsilon) > 0$ and a subsequence $(x_{n_j})_{j=1}^\infty \subset (x_n)_{n=1}^\infty$ such that $\operatorname{sep}\left(\frac{x_{n_j}}{\|x_{n_j}\|_X}\right) \geq \epsilon_0$. Consequently

$$\left\|\frac{x+x_k}{2}\right\|_X \leq \frac{1}{2}\left(\|x\|_X + \|x_k\|_X\right)\left(1 - \frac{2\delta_0 \min\{\|x\|_X, \|x_k\|_X\}}{\|x\|_X + \|x_k\|_X}\right)$$

for some $k \in (n_j)_{j=1}^\infty$. If $\|x\|_X < 1/2$, then $\left\|\frac{x+x_k}{2}\right\|_X \leq \frac{3}{4} = 1 - \frac{1}{4}$. Otherwise, denoting $a = \min\{1/2, \epsilon/4\}$, we get

$$\frac{\min\{\|x\|_X, \|x_k\|_X\}}{\|x\|_X + \|x_k\|_X} = \left(1 + \frac{\max\{\|x\|_X, \|x_k\|_X\}}{\min\{\|x\|_X, \|x_k\|_X\}}\right)^{-1} \geq \frac{1}{1+\frac{1}{a}} = \frac{a}{1+a}.$$

Hence $\left\|\frac{x+x_k}{2}\right\|_X \leq 1 - \frac{2\delta_0 a}{1+a}$. Taking $\delta(\epsilon) = \min\left\{\frac{2\delta_0 a}{1+a}, \frac{1}{4}\right\}$ we can finish the proof.

$\square$

Theorem 2. *The following statements are equivalent:*

(a) $l_\Phi(\mu, X)$ *has the property* $(\boldsymbol{\beta})$;
(b) *both* X *and* l_Φ *have the property* $(\boldsymbol{\beta})$;
(c) X *has the property* $(\boldsymbol{\beta})$ *and* l_Φ *is reflexive;*
(d) X *has the property* $(\boldsymbol{\beta})$, $\Phi \in \delta_2$ *and* $\Psi \in \delta_2$.

Proof: (a) $\Rightarrow$ (b). Since the spaces l_Φ and X are embedded isometrically into $l_\Phi(X)$ and the property $(\boldsymbol{\beta})$ is inherited by subspaces, l_Φ and X have the property $(\boldsymbol{\beta})$.

(b) $\Rightarrow$ (c). The property $(\boldsymbol{\beta})$ implies reflexivity.

(c) $\Rightarrow$ (d). By the reflexivity of l_Φ we conclude that $\Phi \in \delta_2$ and $\Psi \in \delta_2$.

(d) $\Rightarrow$ (a). Assume that X has the property $(\boldsymbol{\beta})$, $\Phi \in \delta_2$ and $\Psi \in \delta_2$. Let $\epsilon > 0$ and $x \in S(l_\Phi(X))$. Take a sequence (x_n) in $S(l_\Phi(X))$ with $\operatorname{sep}(x_n) \geq \epsilon$. By Lemma 4(b) we get that there exists a number $\sigma = \sigma(\epsilon) \in (0,1)$ such that

$$\inf_{n \neq m} \widetilde{I_\Phi}\,(x_n - x_m) \geq \sigma. \tag{3}$$

Denote $b_\Phi = \sup\{u > 0 : \Phi(u) < \infty\}$. Let $w_0 = b_\Phi$ if $\Phi\,(b_\Phi) < 1$, otherwise $w_0 = \Phi^{-1}(1)$. In view of δ_2-condition there exists a number $k > 0$ such that

$$\Phi\,(2u) \leq k\Phi\,(u) \tag{4}$$

for every $|u| \leq w_0$. Take numbers a and γ from Lemma 3 for the number $w = w_0$. Let $l = 1/a$. Then there exists a number k_l such that $\Phi\,(lu) \leq k_l\Phi\,(u)$ for every $|u| \leq w_0$. Consequently

$$\Phi\,(au) \geq \boldsymbol{\beta}\Phi\,(u) \tag{5}$$

for every $|u| \leq w_0/a$, where $\boldsymbol{\beta} = 1/k_l$. Take a number $c > 0$ satisfying

$$c\epsilon < 3\boldsymbol{\beta}\sigma/8k. \tag{6}$$

For every sequence $(y_n)_{n=1}^\infty \subset (x_n)_{n=1}^\infty$ define the sets:

$$A_{(y_n)} = \left\{ i \in \mathbb{N} : \frac{\min\{\|x(i)\|_X, \|y_n(i)\|_X\}}{\max\{\|x(i)\|_X, \|y_n(i)\|_X\}} \geq a \text{ for every } n \in \mathbb{N} \right\},$$

$$B_{(y_n)} = \mathbb{N} \setminus A = \left\{ i \in \mathbb{N} : \frac{\min\{\|x(i)\|_X, \|y_n(i)\|_X\}}{\max\{\|x(i)\|_X, \|y_n(i)\|_X\}} < a \text{ for some } n \in \mathbb{N} \right\}.$$

Note that if $(x_{n_k})_{k=1}^\infty \subset (x_n)_{n=1}^\infty$, then $A_{(x_{n_k})} \supset A_{(x_n)}$ and $B_{(x_{n_k})} \subset B_{(x_n)}$. Moreover for every sequence $(y_n)_{n=1}^\infty \subset (x_n)_{n=1}^\infty$ let

$$M_{(y_n)}(i) = \left\{ n \in \mathbb{N} : \frac{\min\{\|x(i)\|_X, \|y_n(i)\|_X\}}{\max\{\|x(i)\|_X, \|y_n(i)\|_X\}} < a \right\}$$

for every $i \in \mathbb{N}$ and

$$I_{1,(y_n)} = \left\{ i \in \mathbb{N} : \operatorname{card} M_{(y_n)}(i) < \infty \right\} \text{ and } I_{2,(y_n)} = \mathbb{N} \setminus I_1.$$

We divide the proof into two parts.

I. Assume that

$$\widetilde{I_\Phi}\left(x\chi_{B_{(x_n)}} \right) = \sum_{i \in B_{(x_n)}} \Phi\left(\|x(i)\|_X \right) \geq c\epsilon.$$

We will denote $A_{(x_n)} = A$, $B_{(x_n)} = B$, $M_{(x_n)}(i) = M(i)$ for every $i \in \mathbb{N}$, $I_{1,(x_n)} = I_1$, and $I_{2,(x_n)} = I_2$ for short.

1. Suppose that

$$\widetilde{I_\Phi}\left(x\chi_{I_2} \right) \geq c\epsilon. \tag{7}$$

We consider two cases:

a) Assume that there exists a subset $I_{21} \subset I_2$ such that $\widetilde{I_\Phi}\left(x\chi_{I_{21}} \right) \geq c\epsilon/2$ and $\bigcap_{i \in I_{21}} M(i) \neq \emptyset$. Consequently there exists $n_0 \in \mathbb{N}$ such that $n_0 \in \bigcap_{i \in I_{21}} M(i)$. Then, by Lemma 3, we get

$$\sum_{i \in I_{21}} \Phi\left(\left\| \frac{x(i) + x_{n_0}(i)}{2} \right\|_X \right) \leq \sum_{i \in I_{21}} \frac{1}{2}(1-\gamma)\left(\Phi\left(\|x(i)\|_X \right) + \Phi\left(\|x_{n_0}(i)\|_X \right) \right).$$

Denote $p_1 = \frac{\gamma c\epsilon}{4} \in (0,1)$. Thus

$$\widetilde{I_\Phi}\left(\frac{x + x_{n_0}}{2} \right) \leq 1 - \frac{\gamma}{2} \widetilde{I_\Phi}\left(x\chi_{I_{21}} \right) \leq 1 - p_1.$$

Finally, by Lemma 4(c), we get $\left\| \frac{x + x_{n_0}}{2} \right\| \leq 1 - q_1$, where $q_1 \in (0,1)$ depends only on p_1.

b) Assume that for every subset $I \subset I_2$ we have

$$\widetilde{I_\Phi}\left(x\chi_I \right) < c\epsilon/2 \quad \text{or} \quad \bigcap_{i \in I} M(i) = \emptyset. \tag{8}$$

Define

$$J_1 = \left\{ i \in I_2 : \operatorname{card} M'(i) < \infty \right\} \text{ and } J_2 = I_2 \setminus J_1,$$

where

$$M'(i) = M'_{(x_n)}(i) = \left\{ n \in \mathbb{N} : \frac{\min\left\{ \|x(i)\|_X, \|x_n(i)\|_X \right\}}{\max\left\{ \|x(i)\|_X, \|x_n(i)\|_X \right\}} \geq a \right\}$$

for every $i \in \mathbb{N}$. If $\widetilde{I_\Phi}(x\chi_{J_1}) \geq c\epsilon/2$, then there exists a subset $J_{11} \subset J_1$ satisfying card $J_{11} < \infty$ and $\widetilde{I_\Phi}\left(x\chi_{J_{11}}\right) \geq c\epsilon/4$. This case is analogous to 1.a). Hence, in view of (7), we conclude that $\widetilde{I_\Phi}\left(x\chi_{J_2}\right) \geq c\epsilon/2$. Then, by (8), we get $\bigcap_{i\in J_2} M(i) = \emptyset$ and consequently $\bigcup_{i\in J_2} M^{'}(i) = \mathbb{N}$. For every $i \in J_2$ we have card $M(i) = \infty$ and card $M^{'}(i) = \infty$. Take $i_1 \in J_2$. Let $(x_{n_k})_{k=1}^{\infty}$ be a subsequence of $(x_n)_{n=1}^{\infty}$ such that $n_k \in M^{'}(i_1)$ for every $k \in \mathbb{N}$. We obtain $i_1 \in A_{(x_{n_k})}$. Hence $A_{(x_{n_k})} \supset A_{(x_n)}$, $B_{(x_{n_k})} \subset B_{(x_n)}$ and $M_{(x_{n_k})}(i) \subset M_{(x_n)}(i)$ for every $i \in \mathbb{N}$. Furthermore $I_{2,(x_{n_k})} \subset I_{2,(x_n)}$. Thus after a finite number of steps we get a subsequence which satisfies condition II.

2. Suppose that
$$\widetilde{I_\Phi}\left(x\chi_{I_2}\right) < c\epsilon.$$
Hence $\widetilde{I_\Phi}\left(x\chi_{I_1}\right) > 1 - c\epsilon$. We may assume that card $I_1 < \infty$ and $\widetilde{I_\Phi}\left(x\chi_{I_1}\right) \geq 1 - c\epsilon$. Take $i_1 \in I_1$. We have card $M(i_1) < \infty$, so there exists a subsequence $(x_{n_k})_{k=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ such that
$$\frac{\min\left\{\|x(i_1)\|_X, \|x_{n_k}(i_1)\|_X\right\}}{\max\left\{\|x(i_1)\|_X, \|x_{n_k}(i_1)\|_X\right\}} \geq a$$
for every $k \in \mathbb{N}$. For $i_2 \in I_1$ we can find a subsequence $(x_{n_{k_j}})_{j=1}^{\infty} \subset (x_{n_k})_{k=1}^{\infty}$ such that
$$\frac{\min\left\{\|x(i_2)\|_X, \left\|x_{n_{k_j}}(i_2)\right\|_X\right\}}{\max\left\{\|x(i_2)\|_X, \left\|x_{n_{k_j}}(i_2)\right\|_X\right\}} \geq a$$
for every $j \in \mathbb{N}$. In such a way we construct a sequence $(z_n)_{n=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ satisfying
$$\frac{\min\left\{\|x(i)\|_X, \|z_n(i)\|_X\right\}}{\max\left\{\|x(i)\|_X, \|z_n(i)\|_X\right\}} \geq a$$
for every $n \in \mathbb{N}$ and $i \in I_1$. But $\widetilde{I_\Phi}\left(x\chi_{I_1}\right) \geq 1 - c\epsilon$ and $I_1 \subset A_{(z_n)}$, so this situation is considered in case II.

II. Suppose that

$$\widetilde{I_\Phi}\left(x\chi_{A_{(x_{n_k})}}\right) = \sum_{i\in A_{(x_{n_k})}} \Phi\left(\|x(i)\|_X\right) > 1 - c\epsilon \tag{9}$$

for some subsequence $(x_{n_k})_{k=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$. We may assume that card $A_{(x_{n_k})} < \infty$ and still $\widetilde{I_\Phi}\left(x\chi_{A_{(x_{n_k})}}\right) \geq 1 - c\epsilon$. Denote for simplicity (x_{n_k}) by (x_n). We divide this case into two parts.

a) Suppose that there exists a subsequence $(x_{n_k})_{k=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ such that

$$\widetilde{I_\Phi}\left(2x_{n_k}\chi_{B_{(x_n)}}\right) \geq \sigma/2 \tag{10}$$

for every $k \in \mathbb{N}$. Denote for short $B_{(x_n)} = B$. Define $B_k = \{i \in B : n_k \in M(i)\}$. Suppose that for every $k \in \mathbb{N}$ we have $B_k = \emptyset$. Then

$$\frac{\min\left\{\|x(i)\|_X, \|x_{n_k}(i)\|_X\right\}}{\max\left\{\|x(i)\|_X, \|x_{n_k}(i)\|_X\right\}} \geq a$$

for every $i \in B$ and $k \in \mathbb{N}$. Hence $A_{(x_{n_k})} = \mathbb{N}$ and this situation is considered in case II.b). Thus we may assume that there exists $k_0 \in \mathbb{N}$ such that $B_{k_0} \neq \emptyset$. We will prove that

$$\widetilde{I_\Phi}\left(2x_{n_{k_0}}\chi_{B_{k_0}}\right) \geq \sigma/8. \tag{11}$$

If $B \setminus B_{k_0} = \emptyset$, then $B_{k_0} = B$ and (11) holds trivially. Let $B \setminus B_{k_0} \neq \emptyset$. Suppose conversely that $\widetilde{I_\Phi}\left(2x_{n_{k_0}}\chi_{B_{k_0}}\right) < \sigma/8$. Then, in view of (4) and (10), we get $\widetilde{I_\Phi}\left(x_{n_{k_0}}\chi_{B\setminus B_{k_0}}\right) > 3\sigma/8k$. Moreover

$$B \setminus B_{k_0} = \left\{ i \in B : \frac{\min\left\{\|x(i)\|_X, \left\|x_{n_{k_0}}(i)\right\|_X\right\}}{\max\left\{\|x(i)\|_X, \left\|x_{n_{k_0}}(i)\right\|_X\right\}} \geq a \right\}.$$

Consequently, by (5) and (9), we obtain

$$\begin{aligned} c\epsilon \geq \widetilde{I_\Phi}\left(x\chi_B\right) \geq \widetilde{I_\Phi}\left(x\chi_{B\setminus B_{k_0}}\right) \geq \widetilde{I_\Phi}\left(ax_{n_{k_0}}\chi_{B\setminus B_{k_0}}\right) \\ \geq \boldsymbol{\beta}\widetilde{I_\Phi}\left(x_{n_{k_0}}\chi_{B\setminus B_{k_0}}\right) \geq \frac{3\boldsymbol{\beta}\sigma}{8k}, \end{aligned}$$

but this is a contradiction with (6), so (11) is proved. On the other hand, by Lemma 3, we get

$$\begin{aligned} &\sum_{i\in B_{k_0}} \Phi\left(\left\|\frac{x(i)+x_{n_{k_0}}(i)}{2}\right\|_X\right) \\ &\qquad \leq \sum_{i\in B_{k_0}} \frac{1}{2}(1-\gamma)\left(\Phi\left(\|x(i)\|_X\right) + \Phi\left(\left\|x_{n_{k_0}}(i)\right\|_X\right)\right). \end{aligned}$$

Hence

$$\widetilde{I_\Phi}\left(\frac{x+x_{n_{k_0}}}{2}\right) \leq 1 - \frac{\gamma}{2}\widetilde{I_\Phi}\left(x_{n_{k_0}}\chi_{B_{k_0}}\right) \leq 1 - p_2,$$

where $p_2 = \frac{\gamma\sigma}{16k}$. Finally, by Lemma 4(c), we conclude $\left\|\frac{x+x_{n_{k_0}}}{2}\right\| \le 1 - q_2$, where $q_2 \in (0,1)$ depends only on p_2.

b) Assume that there exists a subsequence $(x_{n_k})_{k=1}^{\infty} \subset (x_n)_{n=1}^{\infty}$ such that

$$\widetilde{I_\Phi}\left(2x_{n_k}\chi_{B_{(x_{n_k})}}\right) < \sigma/2 \tag{12}$$

for every $k \in \mathbb{N}$. Denote still this subsequence (x_{n_k}) by (x_n), $A_{(x_{n_k})} = A$ and $B_{(x_{n_k})} = B$. We will show that

$$\inf_{n \ne m} \widetilde{I_\Phi}\left((x_n - x_m)\chi_A\right) \ge \sigma/2. \tag{13}$$

Indeed, if not, then, by (3) and (12), for some $n \ne m$ we would get

$$\begin{aligned}\sigma \le \widetilde{I_\Phi}\left(x_n - x_m\right) &= \widetilde{I_\Phi}\left((x_n - x_m)\chi_A\right) + \widetilde{I_\Phi}\left((x_n - x_m)\chi_B\right)\\ &< \frac{\sigma}{2} + \frac{1}{2}\widetilde{I_\Phi}\left(2x_n\chi_B\right) + \frac{1}{2}\widetilde{I_\Phi}\left(2x_m\chi_B\right) < \sigma,\end{aligned}$$

a contradiction, so (13) is true. Take $\lambda \in \mathbb{R}$ such that

$$0 < \lambda < \sigma/8. \tag{14}$$

For every $n \ne m$ there exists $i_0 \in A$ satisfying $\|x_n(i_0) - x_m(i_0)\|_X \ge \lambda \|x(i_0)\|_X$. Indeed, if not, then $\frac{\sigma}{2} \le \widetilde{I_\Phi}\left((x_n - x_m)\chi_A\right) < \lambda$ for some $n \ne m$. But this is a contradiction with (14). Moreover, we will prove that the following condition holds:

(+) there exist a subset $A_0 \subset A$ and a subsequence $(z_n) \subset (x_n)$ such that

$$\|z_n(i) - z_m(i)\|_X \ge \lambda \|x(i)\|_X \quad \text{for all } n \ne m,\ i \in A_0 \text{ and}$$

$$\|z_n(i) - z_m(i)\|_X < \lambda \|x(i)\|_X \quad \text{for every } n \ne m \text{ and } i \in A \setminus A_0.$$

Denote by F_A the family of all nonempty subsets of the set A. We have card $A < \infty$. Hence card $F_A < \infty$.

1. Consider the element x_1 and the sequence $(x_n)_{n=2}^{\infty}$. Then there exist a subsequence $\left(x_n^{(1)}\right)_{n=1}^{\infty} \subset (x_n)_{n=2}^{\infty}$ and a subset $A_1 \in F_A$, such that

$$\left\|x_1(i) - x_n^{(1)}(i)\right\|_X \ge \lambda \|x(i)\|_X \quad \text{for every } n \in \mathbb{N},\ i \in A_1 \text{ and}$$

$$\left\|x_1(i) - x_n^{(1)}(i)\right\|_X < \lambda \|x(i)\|_X \quad \text{for every } i \in A \setminus A_1 \text{ and } n \in \mathbb{N}.$$

Denote $y_1^{(1)} = x_1$ and $y_{n+1}^{(1)} = x_n^{(1)}$ for every $n \in \mathbb{N}$.

2. Consider the element $x_1^{(1)}$ and the sequence $\left(x_n^{(1)}\right)_{n=2}^{\infty}$. Then there exist a subsequence $\left(x_n^{(2)}\right)_{n=1}^{\infty} \subset \left(x_n^{(1)}\right)_{n=2}^{\infty}$ and a subset $A_2 \in F_A$ such that

$$\left\|x_1^{(1)}(i) - x_n^{(2)}(i)\right\|_X \geq \lambda \left\|x(i)\right\|_X \quad \text{for every } n \in \mathbb{N},\ i \in A_2 \text{ and}$$

$$\left\|x_1^{(1)}(i) - x_n^{(2)}(i)\right\|_X < \lambda \left\|x(i)\right\|_X \quad \text{for every } i \in A \setminus A_2 \text{ and } n \in \mathbb{N}.$$

Denote $y_1^{(2)} = x_1^{(1)}$ and $y_{n+1}^{(2)} = x_n^{(2)}$ for every $n \in \mathbb{N}$. Taking the next steps we conclude that there exists a set $A_0 \in F_A$, a sequence $(j_k)_{k=1}^{\infty}$ of natural numbers and a sequence of subsequences $\left(y_n^{(j_k)}\right)_{n=1}^{\infty}$, $k = 1, 2, \ldots$ such that

$$\left(y_n^{(j_1)}\right)_{n=1}^{\infty} \supset \left(y_n^{(j_2)}\right)_{n=1}^{\infty} \supset \ldots$$

and for every $k \in \mathbb{N}$ we get

$$\left\|y_1^{(j_k)}(i) - y_n^{(j_k)}(i)\right\|_X \geq \lambda \left\|x(i)\right\|_X \quad \text{for every } n \in \mathbb{N}, n \geq 2,\ i \in A_0 \text{ and}$$

$$\left\|y_1^{(j_k)}(i) - y_n^{(j_k)}(i)\right\|_X < \lambda \left\|x(i)\right\|_X \quad \text{for every } n \in \mathbb{N},\ n \geq 2,\ i \in A \setminus A_0.$$

Define $z_n = y_1^{(j_n)}$ for every $n \in \mathbb{N}$. In such a way we have constructed the sequence $(z_n)_{n=1}^{\infty}$ satisfying the condition $(+)$. Denote this subsequence still by (x_n). Furthermore, we will prove that

$$\widetilde{I_\Phi}\left(2x_n\chi_{A_0}\right) \geq \sigma/4 \tag{15}$$

for every $n \in \mathbb{N}$ except at most two elements. Suppose conversely that $\widetilde{I_\Phi}\left(2x_n\chi_{A_0}\right) < \sigma/4$ for $n \in \{n_1, n_2\}$. By condition $(+)$ we obtain $\left\|x_{n_1}(i) - x_{n_2}(i)\right\|_X < \lambda \left\|x(i)\right\|_X$ for every $i \in A \setminus A_0$. Hence, by (13) and (14), we get

$$\begin{aligned}\frac{\sigma}{2} \leq \widetilde{I_\Phi}\left(\left(x_{n_1} - x_{n_2}\right)\chi_A\right) = \widetilde{I_\Phi}\left(\left(x_{n_1} - x_{n_2}\right)\chi_{A_0}\right) + \widetilde{I_\Phi}\left(\left(x_{n_1} - x_{n_2}\right)\chi_{A\setminus A_0}\right)\\ < \frac{1}{2}\widetilde{I_\Phi}\left(2x_{n_1}\chi_{A_0}\right) + \frac{1}{2}\widetilde{I_\Phi}\left(2x_{n_2}\chi_{A_0}\right) + \lambda < \frac{3\sigma}{8},\end{aligned}$$

which is a contradiction.

Note that $\left\|x(i)\right\|_X > 0$ and $\left\|x_n(i)\right\|_X > 0$ for every $i \in A$ and $n \in \mathbb{N}$. For every $i \in A_0$ define the sequence

$$(y_n(i)) = \left(\frac{x_n(i)}{\left\|x(i)\right\|_X}\right)_{n=1}^{\infty} \subset X.$$

By condition (+) we conclude that for every $i \in A_0$ we have $\operatorname{sep}\{y_n(i)\}_X \geq \lambda$. Moreover $\|y_n(i)\|_X \in [a, 1/a]$ for every $n \in \mathbb{N}$ and $i \in A$. Let $i_1 \in A_0$. Passing to a subsequence if necessary, we may assume that $\lim_{n\to\infty} \|y_n(i_1)\|_X = y_1 \in [a, 1/a]$. Furthermore, applying Lemma 2, we conclude that there exist a number $\lambda_1 = \lambda_1(\lambda, y_1)$ and a subsequence $(y_{n_k})_{k=1}^{\infty}$ of $(y_n)_{n=1}^{\infty}$ such that

$$\operatorname{sep}\left\{y_{n_k}(i_1)/\left\|y_{n_k}(i_1)\right\|_X\right\}_X \geq \lambda_1.$$

Moreover, the function $\lambda_1(\lambda, \cdot)$ is nonincreasing. Let $\lambda_0 = \lambda_1(\lambda, 1/a)$. Then

$$\operatorname{sep}\left\{y_{n_k}(i_1)/\left\|y_{n_k}(i_1)\right\|_X\right\}_X \geq \lambda_0.$$

Take $i_2 \in A_0$ and consider a sequence $(y_{n_k}(i_2))_{k=1}^{\infty}$. Similarly we deduce that there exists a subsequence $(y_{n_{k_j}})_{j=1}^{\infty} \subset (y_{n_k})_{k=1}^{\infty}$ such that

$$\operatorname{sep}\left\{y_{n_{k_j}}(i_2)/\left\|y_{n_{k_j}}(i_2)\right\|_X\right\}_X \geq \lambda_0.$$

Because $\operatorname{card} A < \infty$, so in such a way we can find a sequence $(v_n)_{n=1}^{\infty} \subset (y_n)_{n=1}^{\infty}$ satisfying

$$\operatorname{sep}\left\{v_n(i)/\left\|v_n(i)\right\|_X\right\}_X \geq \lambda_0$$

for every $i \in A_0$. Denote still this subsequence by (y_n). But

$$\operatorname{sep}\left\{y_n(i)/\left\|y_n(i)\right\|_X\right\}_X = \operatorname{sep}\left\{x_n(i)/\left\|x_n(i)\right\|_X\right\}_X.$$

Basing on Theorem 1 take a number $\delta_0 = \delta_0(\lambda_0)$. For every $i \in A_0$ we consider an element $x(i) \in X\backslash\{0\}$ and a sequence $(x_n(i))$ in $X\backslash\{0\}$ with $\operatorname{sep}\left(\frac{x_n(i)}{\|x_n(i)\|_X}\right) \geq \lambda_0$. Hence there exists a number $n_0 = n_0(i) \in \mathbb{N}$ such that

$$\begin{aligned} &\left\|\frac{x(i) + x_{n_0}(i)}{2}\right\|_X \\ &\leq \frac{\|x(i)\|_X + \|x_{n_0}(i)\|_X}{2}\left(1 - \frac{2\delta_0 \min\left\{\|x(i)\|_X, \|x_{n_0}(i)\|_X\right\}}{\|x(i)\|_X + \|x_{n_0}(i)\|_X}\right). \end{aligned} \tag{16}$$

For every $i \in A_0$ and every sequence $(u_n(i))_{n=1}^{\infty} \subset (x_n(i))_{n=1}^{\infty} \subset X$, define

$$N\left(i, (u_n(i))\right) = \left\{n = n(i) \in \mathbb{N} : x(i), u_n(i) \text{ satisfies (16)}\right\}.$$

Let $i_1 \in A_0$. The property $(\boldsymbol{\beta})$ of X implies that $\operatorname{card} N\left(i_1, (x_n(i_1))\right) = \infty$. Thus we can find in X a subsequence $(x_{n_k}(i_1))_{k=1}^{\infty} \subset (x_n(i_1))_{n=1}^{\infty}$ such that $x(i_1), x_{n_k}(i_1)$ satisfies the inequality (16) for every $k \in \mathbb{N}$. Consider the sequence $(x_{n_k}(i_2))_{k=1}^{\infty}$. Similarly $\operatorname{card} N\left(i_2, (x_{n_k}(i_2))\right) = \infty$. Consequently there exists a subsequence $\left(x_{n_{k_j}}(i_2)\right)_{j=1}^{\infty} \subset (x_{n_k}(i_2))_{k=1}^{\infty}$ such that $x(i_2), x_{n_{k_j}}(i_2)$ satisfies the inequality (16) for every $j \in \mathbb{N}$. After a finite number of steps we may construct in $l_\Phi(X)$ a subsequence $(x_m)_{m=1}^\infty \subset (x_n)_{n=1}^\infty$ such that for every $i \in A_0$, $x(i), x_m(i)$ satisfies the inequality (16) for every $m \in \mathbb{N}$. Because of the fact that

$$\frac{\min\{\|x(i)\|_X, \|x_m(i)\|_X\}}{\max\{\|x(i)\|_X, \|x_m(i)\|_X\}} \geq a \text{ for every } m \in \mathbb{N} \text{ and } i \in A$$

we obtain

$$\left\|\frac{x(i)+x_m(i)}{2}\right\|_X \leq \frac{1}{2}\left(\|x(i)\|_X + \|x_m(i)\|_X\right)(1-\alpha),$$

for every $m \in \mathbb{N}$ and $i \in A_0$, where $\alpha = \frac{2\delta_0 a}{1+a}$. Then

$$\sum_{i\in A_0} \Phi\left(\left\|\frac{x(i)+x_m(i)}{2}\right\|_X\right) \leq \sum_{i\in A_0} \frac{1}{2}(1-\alpha)\left(\Phi\left(\|x(i)\|_X\right) + \Phi\left(\|x_m(i)\|_X\right)\right)$$

for every $m \in \mathbb{N}$. Applying (15), it is easy to finish the proof in the same way as in the case II.a). □

Remark. It is worth to mention that the property ($\boldsymbol{\beta}$) does not lift from X into $L_\Phi(X)$ in the case when L_Φ is a function Orlicz space. It is enough to consider the Lebesgue-Bochner space $L_p(\mu, X)$ when $1 < p < \infty$ and μ is the Lebesgue measure on $[0,1]$. Then if X is not uniformly convex, then $L_p(\mu, X)$ has not even the uniformly Kadec Klee property (Theorem 3.4.9 in [16]). Moreover, if $L_\Phi(X) \in (\boldsymbol{\beta})$, then obviously $L_\Phi \in (\boldsymbol{\beta})$ and $X \in (\boldsymbol{\beta})$. But $L_\Phi \in (\boldsymbol{\beta})$ iff $L_\Phi \in$ (**UC**) (see [5]). If we additionally assume that $X \in$ (**UC**), then $L_\Phi(X) \in$ (**UC**) (Theorem 3.4.3 in [16]).

As an immediate consequence of Theorem 2, we get the following characterization of the property ($\boldsymbol{\beta}$) in Orlicz sequence spaces with the Luxemburg norm proved directly in [5].

Corollary 1. *Let Φ be an Orlicz function. The following statements are equivalent:*

(a) *l_Φ has the property ($\boldsymbol{\beta}$);*
(b) *l_Φ is (**NUC**);*
(c) *l_Φ has the property (**D**);*
(d) *Φ and Ψ satisfy the δ_2-condition, i.e. l_Φ is reflexive.*

PROOF: It is enough to apply Theorem 2 with $X = \mathbb{R}$ which is uniformly convex, so it has also the property ($\boldsymbol{\beta}$). □

Corollary 2. *The Lebesgue-Bochner sequence space $l^p(X)$ $(1 < p < \infty)$ has the property ($\boldsymbol{\beta}$) iff X has the property ($\boldsymbol{\beta}$).*

PROOF: The sequence space l_p is an Orlicz sequence space generated by the Orlicz function $\Phi(u) = |u|^p$ satisfying all the assumptions of Theorem 2. □

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