Connected Hausdorff subtopologies

Jack Porter

Abstract. A non-connected, Hausdorff space with a countable network has a connected Hausdorff-subtopology iff the space is not-H-closed. This result answers two questions posed by Tkačenko, Tkachuk, Uspenskij, and Wilson [Comment. Math. Univ. Carolinae 37 (1996), 825–841]. A non-H-closed, Hausdorff space with countable π-weight and no connected, Hausdorff subtopology is provided.

Keywords: connected, H-closed, extensions, condensations

Classification: 54C10, 54D05, 54D35

Introduction

Let X be a space. A topology σ on X is a **subtopology** of $\tau(X)$ if $\sigma \subseteq \tau(X)$. The aim of this paper is to determine when a space has a connected, Hausdorff subtopology. Tkačenko, Tkachuk, Uspenskij, and Wilson [TTUW] have established these two results:

(1) A countable infinite Hausdorff space has a connected, Hausdorff subtopology iff it is not H-closed.

(2) A nonconnected, T_3 space with a countable network has a connected, Hausdorff subtopology iff it is not compact.

In this paper we extend (1) and (2) and completely answer two of the questions posed in [TTUW] by proving this result:

Main Theorem. *A nonconnected, Hausdorff space with a countable network has a connected Hausdorff subtopology iff it is not H-closed.*

Examples are provided to show that the hypothesis property of countable network in the main theorem cannot be replaced by a countable π-weight or 2^ω-network. Vermeer [V] defined a Hausdorff space to be **absolute Katětov** if every Hausdorff subtopology has an H-closed subtopology and noted that H-closed spaces are absolute Katětov. He asked if every absolute Katětov is H-closed. We show that a countable Hausdorff space is absolute Katětov iff it is H-closed and provide an example of a non-H-closed space that is absolutely Katětov.

We extend the well-known result that a compact Hausdorff space with a countable network is second countable to this result: if X is an H-closed space with a countable network, then $X(s)$ is second countable. An example of an H-closed space with a countable network is provided which is not second countable.

First some basic definitions (see [PW1]) are provided.

A Hausdorff space X is **H-closed** if whenever Y is a Hausdorff space and X is a subspace of Y, then X is closed in Y. For a Hausdorff space X, this is equivalent to the property that every open ultrafilter on X converges and to the property that for every open cover $\mathcal{C}$ of X, there is a finite subset $\mathcal{D} \subseteq \mathcal{C}$ such that $X = cl_X(\cup\mathcal{D})$. A Hausdorff space X is **almost H-closed** if there is exactly one free open ultrafilter on X.

A space X is **feebly compact** (see 1.11 in [PW1]) if for every countable open cover $\mathcal{C}$ of X, there is a finite subset $\mathcal{D} \subseteq \mathcal{C}$ such that $X = cl_X(\cup\mathcal{D})$. A space is not feebly compact iff there is an infinite locally finite family of pairwise disjoint nonempty open subsets. A Tychonoff space is feebly compact iff it is pseudocompact.

Let X be a Hausdorff space and $\tau(X)(s)$ be the topology generated by the open base $\{int_X cl_X(U) : U \in \tau\}$. It is easy to check that $\tau(X)(s) \subseteq \tau(X)$ and that $(X, \tau(X)(s))$, sometimes denoted as $X(s)$, is also a Hausdorff space. In particular, $\tau(X(s)) = \tau(X)(s)$. A space X is **semiregular** if $\tau(X)(s) = \tau(X)$. The space $X(s)$ is semiregular.

Let X and Y be two spaces. A function $f : X \to Y$ is $\theta-$**continuous** if for each $p \in X$ and open set $U \in \tau(Y)$ such that $f(p) \in U$, there is an open set $V \in \tau(X)$ such that $p \in V$ and $f[cl_X V] \subseteq cl_Y U$.

Here are some known results about H-closed spaces and $\theta-$continuous functions that will be useful in the sequel.

Fact 1. *Let X and Y be Hausdorff spaces and $f : X \to Y$ be a surjection.*

(a) *If X is H-closed and f is $\theta-$continuous, then Y is also H-closed.*
(b) *If X is connected and f is $\theta-$continuous, then Y is also connected.*
(c) *The space X is H-closed iff $X(s)$ is H-closed.*
(d) *If X is H-closed and σ is a Hausdorff subtopology, then $\tau(X(s)) \subseteq \sigma \subseteq \tau(X)$.*
(e) *The space X is connected iff $X(s)$ is connected.*

Note. An easy consequence of Fact 1 is that an H-closed space has a connected Hausdorff subtopology iff it is connected.

Let X and Y be sets and $f : Y \to X$ be a function. For $A \subseteq Y$, define $f^{\#}[A] = \{x \in X : f^{\leftarrow}(x) \subseteq A\}$. Note that for subsets $A, B \subseteq Y$, $f^{\#}[Y \backslash A] = X \backslash f[A]$ and $f^{\#}[A \cap B] = f^{\#}[A] \cap f^{\#}[B]$. The topology on Y generated by $\{f^{\#}[U] : U \in \tau(Y)\}$ is called the $\theta-$**quotient topology** induced by f. The function f is called **irreducible** if for each nonempty open set $U \in \tau(Y)$, there is some $x \in X$ such that $f^{\leftarrow}(x) \subseteq U$.

Fact 2. *Let $f : Y \to X$ be onto and compact where Y is a Hausdorff space and X is a set. Let σ be the $\theta-$quotient topology induced by f. Then:*

(a) *(X, σ) is a Hausdorff space,*
(b) *if X is a space and f is closed, then $\sigma \subseteq \tau(X)$,*

(c) *if f is irreducible, then f is $\theta-$continuous, and*
(d) *if f is irreducible and Y is semiregular, then X is semiregular.*

Application 3. (a) One obtains an easy proof of this result from [TTUW]: *Let Y be a Hausdorff connected extension of a space X. If there is a closed, discrete subset A of X such that $|Y\backslash X| \leq |A|$, then X has a connected, Hausdorff subtopology.*

[If $g : Y\backslash X \to A$ is any one-to-one function, it is straightforward to show that the function $f : Y \to X$ defined by $f(y) = g(y)$ for $y \in Y\backslash X$ and $f(x) = x$ for $x \in X$ is a perfect irreducible surjection; apply Fact 2.]

(b) Let X be a Hausdorff space with a countable π-base $\mathcal{B}$ such that for each $B \in \mathcal{B}$, clB is not feebly compact. By a result in [PW2] we know that X has a connected Hausdorff extension Y such that $Y\backslash X$ is countable. There is an infinite closed discrete subset as X is not countably compact. Applying (a), X has a connected, Hausdorff subtopology.

Let X be a Hausdorff space and let $\Theta X = \{\mathcal{U} : \mathcal{U}$ is an open ultrafilter on $X\}$. For $U \in \tau(X)$, let $O(U) = \{\mathcal{U} : U \in \mathcal{U}\}$. For $U, V \in \tau(X)$, it is easy to verify (see [PW1]) that $O(\varnothing) = \varnothing$, $O(X) = \Theta X, O(U \cap V) = O(U) \cap O(V), O(U \cup V) = O(U) \cup O(V), \Theta X\backslash O(U) = O(X\backslash cl_X U)$, and $O(U) = O(int_X cl_X U)$. ΘX with the topology generated by $\{O(U) : U \in \tau(X)\}$ is an extremally disconnected compact Hausdorff space. The subspace $EX = \{\mathcal{U} \in \Theta X : \mathcal{U}$ is fixed$\}$ is called the **absolute** of X. The function $k : EX \to X$ defined by $k(\mathcal{U})$ is the unique convergent point of $\mathcal{U}$ is called a covering function. The subspace EX is dense in ΘX (in particular, EX is an extremally disconnected Tychonoff space and $\Theta X = \beta EX$), and the covering function $k : EX \to X$ is irreducible, $\theta-$continuous, perfect and onto.

A family $\mathcal{F}$ of subsets of a space X is a **network** if for each open set U and $p \in U$, there is an $F \in \mathcal{F}$ such that $p \in F \subseteq U$. A space X with a countable network $\mathcal{F}$ has a coarser second countable Hausdorff topology. This is verified by first letting $\mathcal{H} = \{(F, G) \in \mathcal{F}^2$: there are disjoint open sets U, V such that $F \subseteq U$ and $G \subseteq V\}$. For $(F, G) \in \mathcal{H}$, let U_{FG}, V_{FG} be disjoint open sets such that $F \subseteq U_{FG}, G \subseteq V_{FG}$. Note that $\mathcal{H}$ is countable and so $\{U_{FG}, V_{FG} : (F, G) \in \mathcal{H}\}$ generates a second countable topology σ on X such that $\sigma \subseteq \tau(X)$. If $p, q \in X$ are distinct points, there are disjoint open sets $U, V \in \tau(X)$ such that $p \in U$ and $q \in V$. So, there are $F, G \in \mathcal{F}$ such that $p \in F \subseteq U$ and $q \in G \subseteq V$. Now, U_{FG}, V_{FG} are disjoint open sets containing p, q respectively.

Thus, the σ is the desired coarser second countable Hausdorff topology.

A key lemma from [TTUW] is needed before we can start the proof of the main result.

Lemma 4 ([TTUW])**.** *A noncompact, separable metrizable space has a separable metrizable subtopology which is nowhere locally compact.*

Proof of the Main Theorem. Suppose X is not H-closed and has a countable network for X. As X is Lindelöf and not H-closed, it follows that X is not feebly compact. Thus, there is a locally finite family $\{U_n : n \in \omega\}$ of pairwise disjoint nonempty open subsets of X. It is easy to verify that $\{O(U_n) : n \in \omega\}$ is a locally finite family of pairwise disjoint nonempty clopen subsets of EX. So, EX is not feebly compact. As EX is Tychonoff, it follows that EX is not pseudocompact and there is a continuous unbounded real-valued function f_0 on EX. There is a countable family $\{V_n : n \in \mathbb{N}\}$ of open subsets of X with the property that if $p, q \in X$ and $p \neq q$, there is some $n \in \mathbb{N}$ such that $p \in V_n$ and $q \notin cl_X V_n$. Now, $EX \cap O(V_n)$ is a clopen subset of EX; let f_n be the continuous real-valued function on EX such that $f_n[EX \cap O(V_n)] = \{0\}$ and $f_n[EX \backslash O(V_n)] = \{1\}$. In particular, for $p, q \in X$ and $p \neq q$, there is some $n \in \mathbb{N}$ such that $f_n[k^{\leftarrow}(p)] = \{0\}$ and $f_n[k^{\leftarrow}(q)] = \{1\}$. The diagonal function $f : EX \to \prod_\omega \mathbb{R}$ defined by $f(y)(n) = f_n(y)$ for $n \in \omega$ is continuous (not necessarily one-to-one), $f[EX]$ is not compact as f_0 is unbounded, and $f[k^{\leftarrow}(p)] \cap f[k^{\leftarrow}(q)] = \varnothing$ for distinct points $p, q \in X$. By Lemma 4, the space $f[EX]$ has a separable metrizable subtopology μ which is nowhere locally compact. By Application 3(b), $(f[EX], \mu)$ has a connected, Hausdorff subtopology σ. Since $f[k^{\leftarrow}(p)] \cap f[k^{\leftarrow}(q)] = \varnothing$ for distinct points $p, q \in X$, it follows there is function $g : f[EX] \to X$ such that $g \circ f = k$, i.e., the following diagram commutes.

$$\begin{tikzcd}
EX \arrow[r, equal] \arrow[d, "f"'] \arrow[dr, phantom, "\circ"] & EX \arrow[d, "k"] \\
(f[X], \mu) \arrow[r, "g"] \arrow[d, "id_{f[X]}"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "id_X"] \\
(f[X], \sigma) \arrow[r, "g"] \arrow[d, "id_{f[X]}"'] \arrow[dr, phantom, "\circ"] & X \arrow[d, "id_X"] \\
(f[X], \sigma) \arrow[r, "g"] & (X, \rho)
\end{tikzcd}$$

Note that $f : EX \to (f[X], \sigma)$ is continuous and for $p \in X$, $g^{\leftarrow}(p) = f[k^{\leftarrow}(p)]$. Thus, $g : (f[X], \sigma) \to X$ is a compact function. Clearly, g is onto. If A is a closed subset of $(f[X], \sigma)$, then $g[A] = k[f^{\leftarrow}[A]]$ is closed in X. So, g is a closed function. If $\varnothing \neq U \in \sigma$, then there is a point $p \in X$ such that $k^{\leftarrow}(p) \subseteq f^{\leftarrow}[U]$. So, $g^{\leftarrow}(p) = f[k^{\leftarrow}(p)] \subseteq f[f^{\leftarrow}[U]] = U$. This shows that $g : (f[X], \sigma) \to X$ is irreducible.

Let ρ be the $\theta-$quotient topology on X induced by $g : (f[X], \sigma) \to X$. By Fact 2, (X, ρ) is a Hausdorff space, $\rho \subseteq \tau(X)$, and $g : (f[X], \sigma) \to (X, \rho)$ is $\theta-$continuous. By Fact 1, (X, ρ) is connected. $\square$

By using the fact that a countable H-closed space has a dense subset of isolated points [PW1], an easy consequence of the above theorem is the following known result which motivated the problem of this manuscript.

Corollary ([TTUW]). *A countable Hausdorff space has a connected, Hausdorff subtopology iff it is not H-closed.*

Notation ([PW1]). Let X be a space and $\mathcal{F}, \mathcal{G}$ be filter bases on X. The notation $\mathcal{F} \leq \mathcal{G}$ means for each $F \in \mathcal{F}$, there is a $G \in \mathcal{G}$ such that $G \subseteq F$, and $\mathcal{F} = \mathcal{G}$ means $\mathcal{F} \leq \mathcal{G}$ and $\mathcal{G} \leq \mathcal{F}$.

Recall [PV] that a Hausdorff space is **Katětov** if it has an H-closed subtopology.

Corollary. *A countable Hausdorff space which is not H-closed has a Hausdorff subtopology which is not Katětov.*

Proof: A countable Hausdorff space which is not H-closed has a connected, Hausdorff subtopology and this subtopology has no isolated points. In particular, this subtopology is not Katětov as a countable H-closed space has a dense set of isolated points. □

Fact 7. *Let X be an almost H-closed space with three pairwise disjoint clopen sets. Let σ be a Hausdorff subtopology of X. Then (X, σ) is not connected and either $\tau(X)(s) \subseteq \sigma$ or (X, σ) is H-closed.*

Proof: Let σ be Hausdorff subtopology of X. If $\tau(X(s)) \subseteq \sigma$, then (X, σ) is not connected as $X(s)$ is not connected by Fact 1(e). Suppose $\tau(X)(s) \not\subseteq \sigma$. Let $\mathcal{U}$ be the free open ultrafilter on X. For each $q \in X$, $\mathcal{F}_q = \{U \in \tau(X)(s) : q \in U\}$ and $\mathcal{G}_q = \{U \in \sigma : q \in U\}$ are open filter bases on X. There is some $r \in X$ such that $\mathcal{F}_r \not\leq \mathcal{G}_r$ and there is some $V \in \mathcal{F}_r$ such that $U \backslash V \neq \varnothing$ for all $U \in \mathcal{G}_r$. There is some $W \in \mathcal{F}_r$ such that $W = int_X cl_X W \subseteq V$. It follows that $\mathcal{V} = \{U \backslash cl_X W : U \in \mathcal{G}_r\}$ is a free open filterbase on X. Thus, $\mathcal{V} \subseteq \mathcal{U}$ and $\mathcal{G}_r \subseteq \mathcal{U}$. If $\mathcal{F}_s \not\leq \mathcal{G}_s$, then a similar argument shows that $\mathcal{G}_s \subseteq \mathcal{U}$. That is, $\mathcal{G}_r$ meets $\mathcal{G}_s$. As (X, σ) is Hausdorff, $\mathcal{G}_r = \mathcal{G}_s$. Assume that (X, σ) is not H-closed. Then there is a free open filter $\mathcal{W}$ on (X, σ). So, $\mathcal{W}$ is a free open filter base on X and $\mathcal{W} \subseteq \mathcal{U}$. So, $\mathcal{G}_r$ meets $\mathcal{W}$, a contradiction. Thus, (X, σ) is H-closed. Of the three pairwise disjoint clopen sets, at least two do not meet $\mathcal{U}$. So, there is a clopen set C such that $r \notin C \notin \mathcal{U}$. As $\mathcal{G}_r \subseteq \mathcal{F}_r \cup \mathcal{U}$, $r \notin cl_\sigma C$. So, C is closed in (X, σ). As $C \in \tau(X)(s)$ and $\mathcal{F}_s \subseteq \mathcal{G}_s$ for all $s \in X \backslash \{r\}$, it follows that $C \in \sigma$. Hence, (X, σ) is not connected. □

Example. (1) One question is whether the main result is true when the "countable network" part of the hypothesis is replaced by "countable π-weight". The Sorgenfrey Line is the usual example of a space with countable π-weight but no countable network. However, the Sorgenfrey Line has a connected Tychonoff subtopology (i.e., the real line is a subtopology). Now, $\beta\omega \backslash \{p\}$ where $p \in \beta\omega \backslash \omega$ is almost H-closed and 0-dimensional. By Fact 7, $\beta\omega \backslash \{p\}$ has no connected Hausdorff subtopology. Also, $\beta\omega \backslash \{p\}$ has weight 2^ω and hence a 2^ω-network. So, the Main Theorem cannot be improved by replacing the hypothesis of "countable network" by "2^ω-network".

(2) Another question is whether the main result is true when the hypothesis of "countable network" is replaced by "cardinality $\leq 2^\omega$". Here is a counterexample: By repeating the proof of 3.5 in [PW2], there is an almost H-closed extension X of ω such that $|X| = \mathfrak{c}$. By Fact 7, X does not have a connected Hausdorff subtopology.

Comment. Vermeer [V] noted that H-closed spaces are absolute Katětov and asked if there were absolute Katětov spaces which were not H-closed. Vermeer's question is re-inforced by the Corollary that the only countable spaces which are absolute Katětov are the H-closed spaces. However, Fact 7 shows that any almost H-closed space is also absolute Katětov.

H-closed plus countable network

Note. A space with a countable network is separable and Lindelöf and has the property that every discrete subspace is countable. A compact Hausdorff space with a countable network is second countable. A natural question is whether an H-closed space with a countable network is second countable. The answer is yes if the space is also semiregular (i.e., minimal Hausdorff) but an example (after the following Fact) shows that an H-closed space with a countable network may not have a countable π-base.

Fact 8. *If X is an H-closed space with a countable network, then $X(s)$ is second countable.*

Proof: Let $\mathcal{C} = \{C_n : n \in \omega\}$ be a countable network for X. Let $\mathcal{C}^2 = \{\langle C_n, C_m\rangle$: there are regular open sets U_{nm} and V_{nm} such that $C_n \subseteq U_{nm}$, $C_m \subseteq V_{nm}$, and $U_{nm} \cap V_{nm} = \varnothing\}$. For each pair $\langle C_n, C_m\rangle \in \mathcal{C}^2$, we select exactly one pair $\langle U_{nm}, V_{nm}\rangle$. Let σ be the topology on X generated by $\{U_{nm}, V_{nm} : \langle C_n, C_m\rangle \in \mathcal{C}^2\}$, and note that (X, σ) is a Hausdorff space with a countable base and $\sigma \subseteq \tau(X)$. As X is H-closed, $\tau(X)(s) \subseteq \sigma$. However, since $\tau(X)(s)$ is generated by the collection of all regular open sets, it follows that $\sigma \subseteq \tau(X)(s)$. That is, $\sigma = \tau(X)(s)$. $\square$

Example. Let $X = [0,1]^2$, $Y = X \backslash ([0,1] \times \{0\})$, σ the usual topology on X, and $\mathcal{S} = \{S \subset Y^\omega$: there is a bijection $f : \omega \to S$ converging to $(0,0)\}$. Note that $\mathcal{S}$ is closed under finite unions. Let $\tau(X)$ denote the topology on X generated by $\sigma \cup \{X \backslash S : S \in \mathcal{S}\}$. Note that $\tau(X)(s) = \sigma$. So, X is H-closed. Let $\mathcal{B}$ be a countable base for (X, σ). Then $\mathcal{B} \cup \{[0, \frac{1}{n}) \times \{0\} : n \in \omega \backslash \{0\}\}$ is a countable network for X. Also, X is not first countable at $(0,0)$. In fact, there is no countable π-base at $(0,0)$. $\square$

References

[PT] Porter J.R., Tikoo M.L., *On Katětov spaces*, Trans. Amer. Math. Soc. **289** (4) (1985), 59–71.

[PV] Porter J.R., Vermeer J., *Space with coarser minimal Hausdorff topologies*, Canad. Math. Bull. **32.4** (1989), 425–433.

[PW1] Porter J.R., Woods R.G., *Extensions and Absolutes of Hausdorff Spaces*, Springer-Verlag, Berlin, 1988.

[PW2] Porter J.R., Woods R.G., *Subspaces of connected spaces*, Topology Appl. **68** (1996), 113–131.

[V] Vermeer J., Private communication, 1984.

[TTUW] Tkačenko M.G., Tkachuk V.V., Uspenskij V.V., Wilson R.G., *In quest of weaker connected topologies*, Comment. Math Univ. Carolinae **37.4** (1996), 825–841.

Department of Mathematics, University of Kansas, Lawrence, Kansas 66045 USA

E-mail: porter@math.ukans.edu

(*Received* December 17, 1999)