Centralizers on semiprime rings

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Abstract. The main result: Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. Suppose that $T(xyx) = xT(y)x$ holds for all $x, y \in R$. In this case T is a centralizer.

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This research has been motivated by the work of Brešar [3] and Zalar [7]. Throughout, R will represent an associative ring with center $Z(R)$. A ring R is n-torsion free, where n is an integer, in case $nx = 0$, $x \in R$ implies $x = 0$. As usual the commutator $xy - yx$ will be denoted by $[x, y]$. We shall use basic commutator identities $[xy, z] = [x, z]y + x[y, z]$ and $[x, yz] = [x, y]z + y[x, z]$. Recall that R is prime if $aRb = (0)$ implies $a = 0$ or $b = 0$, and is semiprime if $aRa = (0)$ implies $a = 0$. An additive mapping $D : R \to R$ is called a derivation if $D(xy) = D(x)y + xD(y)$ holds for all pairs $x, y \in R$ and is called a Jordan derivation in case $D(x^2) = D(x)x + xD(x)$ is fulfilled for all $x \in R$. A derivation D is inner in case there exists $a \in R$ such that $D(x) = [a, x]$ holds for all $x \in R$. Every derivation is a Jordan derivation. The converse is in general not true. A classical result of Herstein ([5]) asserts that any Jordan derivation on a 2-torsion free prime ring is a derivation. A brief proof of Herstein's result can be found in [1]. Cusack ([4]) has generalized Herstein's result to 2-torsion free semiprime rings (see also [2] for an alternative proof). An additive mapping $T : R \to R$ is called a left (right) centralizer in case $T(xy) = T(x)y$ $(T(xy) = xT(y))$ holds for all $x, y \in R$. We follow Zalar [7] and call T a centralizer in case T is both a left and right centralizer. In case R has an identity element $T : R \to R$ is a left (right) centralizer iff T is of the form $T(x) = ax$ $(T(x) = xa)$ for some fixed element $a \in R$. An additive mapping $T : R \to R$ is called a left (right) Jordan centralizer in case $T(x^2) = T(x)x$ $(T(x^2) = xT(x))$ holds for $x \in R$. Following ideas from [2], Zalar ([7]) has proved that any left (right) Jordan centralizer on a 2-torsion free semiprime ring is a left (right) centralizer. In our recent paper ([6]) we prove that in case we have an additive mapping $T : R \to R$, where R is a 2-torsion free

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semiprime ring, satisfying the relation $2T(x^2) = T(x)x + xT(x)$ for all $x \in R$, then T is a centralizer. An additive mapping $D : R \to R$, where R is an arbitrary ring, is a Jordan triple derivation in case $D(xyx) = D(x)yx + xD(y)x + xyD(x)$ holds for all pairs $x, y \in R$. One can easily prove that any Jordan derivation is a Jordan triple derivation (see [1]). Brešar ([3]) has proved that any Jordan triple derivation on 2-torsion free semiprime ring is a derivation.

If $T : R \to R$ is a centralizer, where R is an arbitrary ring, then T satisfies the relation

$$T(xyx) = xT(y)x, \quad x, y \in R. \tag{1}$$

It seems natural to ask whether the converse is true. More precisely, we are asking whether an additive mapping T on a ring R satisfying relation (1) is a centralizer. It is our aim in this paper to prove that the answer is affirmative in case R is a 2-torsion free semiprime ring. The proof of the theorem below is rather long, but it is elementary in the sense that it requires no specific knowledge concerning semiprime ring theory in order to follow the proof.

Theorem 1. *Let R be a 2-torsion free semiprime ring and let $T : R \to R$ be an additive mapping. Suppose that $T(xyx) = xT(y)x$ holds for all pairs $x, y \in R$. In this case T is a centralizer.*

For the proof of the result above we shall need the lemma below, which is suggested by Lemma 4 in [2].

Lemma 1. *Let R be a semiprime ring. Suppose that the relation $axb + bxc = 0$ holds for all $x \in R$ and some $a, b, c \in R$. In this case $(a + c)xb = 0$ is satisfied for all $x \in R$.*

PROOF: Putting xby for x in the relation below

$$axb + bxc = 0, \quad x \in R, \tag{2}$$

we obtain

$$axbyb + bxbyc = 0, \quad x, y \in R. \tag{3}$$

On the other hand right multiplication by yb of (2) gives

$$axbyb + bxcyb = 0, \quad x, y \in R. \tag{4}$$

Subtracting (4) from (3) we obtain

$$bx(byc - cyb) = 0, \quad x, y \in R. \tag{5}$$

The substitution ycx for x in (5) gives

$$bycx(byc - cyb) = 0, \quad x, y \in R. \tag{6}$$

Left multiplication by cy of (5) gives

$$cybx(byc - cyb) = 0, \quad x, y \in R. \tag{7}$$

Subtracting (7) from (6) we obtain

$$(byc - cyb)x(byc - cyb) = 0, \quad x, y \in R,$$

which gives $byc = cyb$, $y \in R$. Therefore bxc can be replaced by cxb in (2), which gives $(a + c)xb = 0$, $x \in R$. The proof is complete. □

Proof of Theorem 1: We intend to prove the relation

$$[T(x), x] = 0, x \in R. \tag{8}$$

For the proof of the above relation we shall need the weaker relation below

$$[[T(x), x], x] = 0, \quad x \in R, \tag{9}$$

Putting $x + z$ for x in relation (1) (linearization), we obtain

$$T(xyz + zyx) = xT(y)z + zT(y)x, \quad x, y, z \in R. \tag{10}$$

Putting $y = x$ and $z = y$ in (10) one obtains

$$T(x^2y + yx^2) = xT(x)y + yT(x)x, \quad x, y \in R. \tag{11}$$

For $z = x^3$ relation (10) reduces to

$$T(xyx^3 + x^3yx) = xT(y)x^3 + x^3T(y)x, \quad x, y \in R. \tag{12}$$

Putting xyx for y in (11) we obtain

$$T(xyx^3 + x^3yx) = xT(x)xyx + xyxT(x)x, \quad x, y \in R. \tag{13}$$

The substitution $x^2y + yx^2$ for y in relation (1) gives

$$T(xyx^3 + x^3yx) = xT(x^2y + yx^2)x, \quad x, y \in R,$$

which gives because of (11)

$$T(x^3yx+xyx^3)=x^2T(x)yx+xyT(x)x^2,\quad x,y\in R. \tag{14}$$

Combining (13) with (14) we arrive at

$$x[T(x),x]yx-xy[T(x),x]x=0,\quad x,y\in R. \tag{15}$$

From the above relation and Lemma 1 it follows that

$$[[T(x),x],x]yx=0,\quad x,y\in R. \tag{16}$$

Let y be $y[T(x),x]$ in (16). We have

$$[[T(x),x],x]y[T(x),x]x=0,\quad x,y\in R. \tag{17}$$

Right multiplication of (16) by $[T(x),x]$ gives

$$[[T(x),x],x]yx[T(x),x]=0,\quad x,y\in R. \tag{18}$$

Subtracting (18) from (17) one obtains $[[T(x),x],x]y[[T(x),x],x]=0$, $x,y\in R$, and (9) follows by semiprimeness of R. The next step is the relation

$$x[T(x),x]x=0,\quad x\in R. \tag{19}$$

The linearization of (9) gives

$$\begin{aligned}[[T(x),x],y]+[[T(x),y],x]+[[T(y),y],x]+[[T(y),x],y]\\ +[[T(x),y],y]=0,\quad x,y\in R.\end{aligned}$$

Putting $-x$ for x in the above relation and comparing the relation so obtained with the above relation we arrive at

$$[[T(x),x],y]+[[T(x),y],x]+[[T(y),x],x]=0,\quad x,y\in R. \tag{20}$$

Putting xyx for y in (20) and using (1), (9) and (20) we obtain

$$\begin{aligned}0&=[[T(x),x],xyx]+[[T(x),xyx],x]+[[xT(y)x,x],x]\\ &=x[[T(x),x],y]x+[[T(x),x]yx+x[T(x),y]x+xy[T(x),x],x]\\ &+[x[T(y),x]x,x]=x[[T(x),x],y]x+[T(x),x][y,x]x+x[[T(x),y],x]x\\ &+x[y,x][T(x),x]+x[[T(y),x],x]x=[T(x),x][y,x]x+x[y,x][T(x),x]\\ &=[T(x),x]yx^2-x^2y[T(x),x]+xyx[T(x),x]-[T(x),x]xyx.\end{aligned}$$

We have therefore $[T(x),x]yx^2 - x^2y[T(x),x] + xyx[T(x),x] - [T(x),x]xyx = 0$, $x, y \in R$, which reduces because of (9) and (15) to

$$[T(x),x]yx^2 - x^2y[T(x),x] = 0, \quad x, y \in R.$$

Left multiplication of the above relation by x gives

$$x[T(x),x]yx^2 - x^3y[T(x),x] = 0, \quad x, y \in R.$$

One can replace in the above relation, according to (15), $x[T(x),x]yx$ by $xy[T(x),x]x$, which gives

$$xy[T(x),x]x^2 - x^3y[T(x),x] = 0, \quad x, y \in R. \tag{21}$$

Left multiplication of the above relation by $T(x)$ gives

$$T(x)xy[T(x),x]x^2 - T(x)x^3y[T(x),x] = 0, \quad x, y \in R. \tag{22}$$

The substitution $T(x)y$ for y in (21) leads to

$$xT(x)y[T(x),x]x^2 - x^3T(x)y[T(x),x] = 0, \quad x, y \in R. \tag{23}$$

Subtracting (23) from (22) we obtain

$$[T(x),x]y[T(x),x]x^2 - [T(x),x^3]y[T(x),x] = 0, \quad x, y \in R.$$

From the above relation and Lemma 1 it follows that

$$([T(x),x^3] - [T(x),x]x^2)y[T(x),x] = 0, \quad x, y \in R,$$

which reduces to

$$(x[T(x),x]x + x^2[T(x),x])y[T(x),x] = 0, \quad x, y \in R.$$

Relation (9) makes it possible to write $[T(x),x]x$ instead of $x[T(x),x]$, which means that $x^2[T(x),x]$ can be replaced by $x[T(x),x]x$ in the above relation. Thus we have

$$x[T(x),x]xy[T(x),x] = 0, \quad x, y \in R.$$

Right multiplication of the above relation by x and substitution yx for y gives finally

$$x[T(x),x]xyx[T(x),x]x = 0, \quad x, y \in R,$$

whence relation (19) follows. Next we prove the relation

$$x[T(x),x]=0,\quad x\in R. \tag{24}$$

The substitution yx for y in (15) gives because of (19)

$$x[T(x),x]yx^2=0,\quad x,y\in R. \tag{25}$$

Putting $yT(x)$ for y in the above relation we obtain

$$x[T(x),x]yT(x)x^2=0,\quad x,y\in R. \tag{26}$$

Right multiplication of (25) by $T(x)$ gives

$$x[T(x),x]yx^2T(x)=0,\quad x,y\in R. \tag{27}$$

Subtracting (27) from (26) we obtain $x[T(x),x]y[T(x),x^2]=0$, $x,y\in R$ which can be written in the form

$$x[T(x),x]y([T(x),x]x+x[T(x),x])=0,\quad x,y\in R.$$

According to (9) one can replace $[T(x),x]x$ in the relation above by $x[T(x),x]$, which gives $x[T(x),x]yx[T(x),x]=0$, $x,y\in R$, whence relation (24) follows. From (9) and (24) it follows that

$$[T(x),x]x=0,\quad x\in R.$$

From the above relation one obtains (see how relation (20) was obtained from (9))

$$[T(x),x]y+[T(x),y]x+[T(y),x]x=0,\quad x,y\in R.$$

Right multiplication of the above relation by $[T(x),x]$ gives because of (24)

$$[T(x),x]y[T(x),x]=0,\quad x,y\in R,$$

which implies (8). Our next task is to prove the relation

$$T(xy+yx)=T(y)x+xT(y),\quad x,y\in R. \tag{28}$$

In order to prove the above relation we need the relations below

$$xA(x,y)x=0,\quad x,y\in R, \tag{29}$$

$$[A(x,y),x]=0,\quad x,y\in R, \tag{30}$$

where $A(x,y)$ stands for $T(xy+yx)-T(y)x-xT(y)$. Let us first prove relation (29). The substitution $xy+yx$ for y in (1) gives

$$T(x^2yx+xyx^2)=xT(xy+yx)x, \quad x,y\in R. \tag{31}$$

On the other hand we obtain by putting $z=x^2$ in (10)

$$T(x^2yx+xyx^2)=xT(y)x^2+x^2T(y)x, \quad x,y\in R. \tag{32}$$

By comparing (31) and (32) we arrive at (29). From (29) one obtains (see how (20) was obtained from (9))

$$xA(x,y)z+xA(z,y)x+zA(x,y)x=0, \quad x,y,z\in R.$$

Right multiplication of the above relation by $A(x,y)x$ gives because of (29)

$$xA(x,y)zA(x,y)x=0, \quad x,y,z\in R. \tag{33}$$

Let us prove relation (30). The linearization of (8) gives

$$[T(x),y]+[T(y),x]=0, \quad x,y\in R. \tag{34}$$

Putting $xy+yx$ for y in the above relation and using (8) we obtain $0=[T(x),xy+yx]+[T(xy+yx),x]=x[T(x),y]+[T(x),y]x+[T(xy+yx),x]$. Thus we have

$$[T(xy+yx),x]+x[T(x),y]+[T(x),y]x=0, \quad x,y\in R.$$

According to (34) one can replace $[T(x),y]$ by $-[T(y),x]$ in the above relation. We have therefore $[T(xy+yx),x]-x[T(y),x]-[T(y),x]x=0$, which can be written in the form $[T(xy+yx)-T(y)x-xT(y),x]=0$. The proof of relation (30) is therefore complete. Relation (30) makes it possible to replace in (33) $xA(x,y)$ by $A(x,y)x$. Thus we have

$$A(x,y)xzA(x,y)x=0, \quad x,y\in R, \tag{35}$$

whence it follows that

$$A(x,y)x=0, \quad x,y\in R. \tag{36}$$

Of course we have also

$$xA(x,y)=0, \quad x,y\in R. \tag{37}$$

The linearization of (36) with respect to x gives

$$A(x,y)z + A(z,y)x = 0, \quad x,y,z \in R.$$

Right multiplication of the above relation by $A(x,y)$ gives because of (37)

$$A(x,y)zA(x,y) = 0, \quad x,y,z \in R,$$

which gives $A(x,y) = 0$, $x,y \in R$. The proof of relation (28) is therefore complete. In particular for $y = x$ relation (30) reduces to

$$2T(x^2) = T(x)x + xT(x), \quad x \in R.$$

Combining the above relation with (8) we arrive at

$$T(x^2) = T(x)x, \quad x \in R,$$

and

$$T(x^2) = xT(x), \quad x \in R.$$

By Proposition 1.4 in [7] it follows that T is a left and also right centralizer, which completes the proof of the theorem. □

Putting (1) $y = x$ in relation (1) we obtain

$$T(x^3) = xT(x)x, \quad x \in R. \tag{38}$$

The question arises whether in a 2-torsion free semiprime ring the above relation implies that T is a centralizer. Unfortunately, we were unable to answer this question in general. However, we succeeded in proving that the answer is affirmative in case R has an identity element.

Theorem 2. *Let R be a 2-torsion free semiprime ring with an identity element and let $T : R \to R$ be an additive mapping. Suppose that $T(x^3) = xT(x)x$ holds for all $x \in R$. In this case T is a centralizer.*

Proof: Putting $x+1$ for x in relation (38), where 1 denotes the identity element, one obtains after some calculations

$$3T(x^2) + 2T(x) = T(x)x + xT(x) + xax + ax + xa, \quad x \in R,$$

where a stands for $T(1)$. Putting $-x$ for x in the relation above and comparing the relation so obtained with the above relation we obtain

$$6T(x^2) = 2T(x)x + 2xT(x) + 2xax, \quad x, \in R, \tag{39}$$

and

$$(40) \qquad 2T(x) = ax + xa, \quad x \in R.$$

We intend to prove that $a \in Z(R)$. According to (40) one can replace $2T(x)$ on the right side of (39) by $ax+xa$ and $6T(x^2)$ on the left side by $3ax^2+3x^2a$, which gives after some calculation

$$ax^2 + x^2a - 2xax = 0, \quad x \in R.$$

The above relation can be written in the form

$$(41) \qquad [[a, x], x] = 0, \quad x \in R.$$

The linearization of the above relation gives

$$(42) \qquad [[a, x], y] + [[a, y], x] = 0, \quad x, y \in R.$$

Putting xy for y in (42) we obtain because of (41) and (42) $0 = [[a, x], xy] + [[a, xy], x] = [[a, x], x]y+x[[a, x], y]+[[a, x], y]+x[[a, y], x] = x[[a, x], y]+[[a, x], x]y+ [a, x][y, x] + x[[a, y], x] = [a, x][y, x]$. Thus we have

$$[a, x][y, x] = 0, x, y \in R.$$

The substitution ya for y in the above relation gives

$$[a, x]y[a, x] = 0, \quad x, y \in R,$$

whence it follows $a \in Z(R)$, which reduces (40) to the form $T(x) = ax$, $x \in R$. The proof of the theorem is complete. □

We conclude with the following conjecture: Let R be a semiprime ring with suitable torsion restrictions. Suppose there exists an additive mapping $T : R \to R$ such that $T(x^{m+n+1}) = x^m T(x)x^n$ holds for all $x \in R$ where $m \geq 1$, $n \geq 1$ are some integers. In this case T is a centralizer.

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