# On $p$-injectivity, YJ-injectivity and quasi-Frobeniusean rings

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*Dedicated to Professor Carl Faith on his 75-th birthday*

*Abstract.* A new characteristic property of von Neumann regular rings is proposed in terms of annihilators of elements. An ELT fully idempotent ring is a regular ring whose simple left (or right) modules are either injective or projective. Artinian rings are characterized in terms of Noetherian rings. Strongly regular rings and rings whose two-sided ideals are generated by central idempotents are characterized in terms of special annihilators. Quasi-Frobeniusean rings are characterized in terms of $p$-injectivity. Also, a commutative YJ-injective ring with maximum condition on annihilators and finitely generated socle is quasi-Frobeniusean.

*Keywords:* von Neumann regular, $V$-ring, Artinian ring, $p$-injectivity, YJ-injectivity, quasi-Frobeniusean

*Classification:* 16D30, 16D36, 16D50

## Introduction

Von Neumann regular rings, $V$-rings, self-injective rings and generalizations are extensively studied since several years (cf. for example, [1], [3]–[16], [28]–[30]). This sequel of [21] and [24] contains the following results for a ring $A$: (1) $A$ is von Neumann regular iff $A$ is a semi-prime ring such that every finitely generated one-sided ideal is the annihilator of an element of $A$ (Theorem 1); (2) $A$ is strongly regular iff for every $b \in A$, $Ab + I(AbA)$ is a complement right ideal of $A$ (Proposition 3); (3) Left Noetherian rings whose essential right ideals are idempotent two-sided ideals are left Artinian (Proposition 4); (4) Special two-sided ideals are used to characterize rings whose two-sided ideals are generated by central idempotents (Proposition 8), (5) $A$ is quasi-Frobeniusean iff $A$ has a $p$-injective left generator and projective $p$-injective left $A$-modules are injective (Theorem 9); (6) If every simple right $A$-module is flat and every maximal left ideal of $A$ is either injective or a two-sided ideal of $A$, then $A$ is either a left self-injective regular left $V$-ring or strongly regular; (7) If $A$ is commutative, then $A$ is quasi-Frobeniusean iff $A$ is a YJ-injective ring with maximum condition on annihilators and a finitely generated socle (Theorem 11).

Throughout, $A$ denotes an associative ring with identity and $A$-modules are unital. Recall that a left $A$-module $M$ is $p$-injective (resp $f$-injective) if, for any principal (resp. finitely generated) left ideal $I$ of $A$, every left $A$-homomorphism of $I$ into $M$ extends to $A$ (cf. [4, p. 122], [14, p. 340], [17]). $A$ is called left $p$-injective (resp. $f$-injective) if ${}_AA$ is $p$-injective (resp. $f$-injective). Similarly, $p$-injectivity and $f$-injectivity are definited on the right side. ($P$-injectivity is also called principal injectivity in the literature). Following C. Faith, $A$ is called a left $V$-ring if every simple left $A$-module is injective. A well-known theorem of I. Kaplansky asserts that a commutative ring is von Neumann regular iff it is a $V$-ring. In general, von Neumann regular rings need not be $V$-rings and the converse is not true either. A theorem of M. Ikeda-T. Nakayama asserts that $A$ is a left $p$-injective ring iff every principal right ideal of $A$ is a right annihilator. It is well-known that $A$ is von Neumann regular iff every left (right) $A$-module is flat ([4, p. 91]). This remains true if "flat" is replaced by "$p$-injective" ([17]).

Our first result is motivated by [19, Question 1] and [27, Question 1].

**Theorem 1.** *The following conditions are equivalent:*

(1) *$A$ is von Neumann regular;*
(2) *$A$ is a semi-prime ring whose finitely generated one-sided ideals are annihilators of an element of $A$;*
(3) *$A$ is a semi-prime ring such that every finitely generated left ideal is the left annihilator of an element of $A$ and every principal right ideal of $A$ is the right annihilator of an element of $A$.*

Proof: It is clear that (1) implies (2) while (2) implies (3). Assume (3). Let $F$ be a finitely generated left ideal of $A$. Then $F = I(u)$, $u \in A$. Since $uA = r(w)$, $w \in A$, $F = I(uA) = I(r(w)) = I(r(Aw))$. But $Aw$ is a left annihilator which implies that $Aw = I(r(Aw))$. Therefore $F = Aw$ which shows that every finitely generated left ideal of $A$ is principal. Since $A$ is semi-prime, $A$ is left semi-hereditary by [4, Theorem 7.5B] and it follows that every principal left ideal of $A$ is projective. Since every principal right ideal is a right annihilator, by Ikeda-Nakayama's theorem, $A$ is left $p$-injective. Now $A$ being a left *p.p.* ring is equivalent to "every quotient of a $p$-injective left $A$-module is $p$-injective" ([18, Remark 2]). Since $A$ is left $p$-injective, every cyclic left $A$-module is $p$-injective which yields that $A$ is von Neumann regular ([17, Lemma 2]). Thus (3) implies (1). □

In Theorem 1, the term "semi-prime" cannot be omitted (otherwise, any principal ideal quasi-Frobeniusean ring would be von Neumann regular !).

**Remark 1.** If every finitely generated one-sided ideal of $A$ is the annihilator of an element, then $A$ is a left and right IF-ring whose finitely generated one-sided ideals are principal (cf. [4, Theorem 6.9]).

**Remark 2.** The fact that a strongly regular ring is unit-regular follows from a cancellation theorem of G. Ehrlich (cf. [4, Corollary 6.3C] and [5, Corollary 4.2]).

The proof of [17, Proposition 1] shows in an elementary way that this result holds. Note that a left and right $V$-ring whose essential left ideals are two-sided is a unit-regular ring (whose prime factor rings are Artinian).

We may note the following characterization of principal ideal rings in terms of $p$-injectivity.

**Remark 3.** $A$ is a principal left ideal ring iff every finitely generated left ideal of $A$ is principal and every $p$-injective left $A$-module is injective.

Another result on annihilators.

**Proposition 2.** *If every finitely generated left ideal of $A$ is the left annihilator of a finite subset of $A$ and every finitely generated right ideal of $A$ is a right annihilator, then $A$ is left $f$-injective and right $p$-injective.*

Proof: Since every principal one-sided ideal of $A$ is an annihilator, $A$ is a left and right $p$-injective ring by Ikeda-Nakayama's theorem. Let $F, K$ be finitely generated left ideals of $A$. By hypothesis, we have $F = I(U)$, $K = I(V)$, where $U, V$ are finitely generated right ideals of $A$. Then $U = r(I(U))$, $V = r(I(V))$ which imply $r(F) + r(K) = r(I(U)) + r(I(V)) = U + V = r(I(U + V)) = r(I(U) \cap I(V)) = r(F \cap K)$. By Ikeda-Nakayama's theorem, $A$ is left $f$-injective. □

**Question 1.** If $A$ is left $p$-injective such that every finitely generated left ideal of $A$ is the left annihilator of an element of $A$, is $A$ left $f$-injective?

**Remark 4.** In Proposition 2, $\mathrm{Soc}({}_AA) = \mathrm{Soc}(A_A)$.

**Proposition 3.** *The following conditions are equivalent:*

(1) *$A$ is strongly regular;*
(2) *for any $b \in A$, $Ab + I(AbA)$ is a complement right ideal of $A$.*

Proof: Assume (1). For any $b \in A$, $Ab = Ae$, where $e$ is a central idempotent. Then $I(AbA) = I(b) = A(1-e)$ and $Ab + I(AbA) = Ae \oplus A(1-e) = A$. Therefore (1) implies (2) evidently.

Assume (2). Suppose that $c \in A$ such that $(Ac)^2 = 0$. Then $I(AcA)$ is an essential right ideal of $A$. By hypothesis, $I(AcA) = Ac + I(AcA)$ is a complement right ideal of $A$ which proves that $A = I(AcA)$, whence $c = 0$. This proves that $A$ must be semi-prime. It follows that for any $b \in A$, $I = Ab \oplus I(AbA)$ (because $Ab \cap I(Ab) = 0$) is a complement right ideal of $A$. Now there exists a complement right ideal $C$ of $A$ such that $I \oplus C$ is an essential right ideal of $A$. Then $CAb \subseteq C \cap Ab \subseteq C \cap I = 0$ implies that $C \subseteq I(AbA) \subseteq I$ and hence $C \subseteq C \cap I = 0$. Therefore $I$ is an essential right ideal of $A$ which yields $I = A$. Therefore $A = Ab \oplus I(AbA)$ which proves that $A$ is von Neumann regular. Since $AbA \cap I(AbA) = 0$ (in as much as $A$ is semi-prime), then $A = Ab \oplus I(AbA) \subseteq AbA \oplus I(AbA)$ which yields $A = AbA \oplus I(AbA)$. For any $u \in AbA$, $u = v + w$,

$v \in Ab$, $w \in I(AbA)$. Then $u - v \in AbA \cap I(AbA) = 0$ which implies $u = v \in Ab$, proving that $Ab = AbA$ is generated by a central idempotent (because $A$ is semiprime). Thus (2) implies (1).

□

**Question 2.** If $Ab + r(AbA)$ is a complement left ideal of $A$ for every $b \in A$, is $A$ regular, biregular?

We now give a sufficient condition for left Noetherian rings to be left Artinian.

**Proposition 4.** *If $A$ is a left Noetherian ring such that every essential right ideal of $A$ is an idempotent two-sided ideal, then $A$ is left Artinian.*

Proof: Let $B$ be a prime factor ring of $A$. Then every essential right ideal of $B$ is an idempotent two-sided ideal of $B$. For any $0 \neq b \in B$, set $T = BbB$. Let $K$ be a complement right subideal of $T$ such that $E = bB \oplus K$ is an essential right subideal of $T$. Since $T$ is an essential right ideal of $B$ ($B$ being prime), $E$ is an essential right ideal of $B$ which implies that $E$ is an idempotent two-sided ideal of $B$. Now $b \in E = E^2$ implies that $b = \sum_{i=1}^{n}(bb_i + k_i)(bc_i + s_i)$, $b_i, c_i \in B$ and $k_i, s_i \in K$, whence $b - \sum_{i=1}^{n} bb_i(bc_i + s_i) = \sum_{i=1}^{n} k_i(bc_i + s_i) \in bB \cap K = 0$. Then $b \in bBbB$ which proves that every right ideal of $B$ is idempotent. Since every essential right ideal of $B$ is two-sided, then $B$ is von Neumann regular by [1, Theorem 3.1]. Since $B$ is left Noetherian, it is well-known that $B$ must be simple Artinian. If $A$ is prime, then $A$ is Artinian as just seen. If $A$ is not prime, then by [3, Lemma 18.34B], $A$ is left Artinian. This establishes the proposition. □

Note that the ring in Proposition 4 needs not be right duo. The proof of Proposition 4 yields a characterization of Artinian rings.

**Theorem 5.** *The following conditions are equivalent:*

(1) *$A$ is left Artinian;*
(2) *$A$ is a left Noetherian ring such that for any prime factor ring $B$ of $A$, every essential right ideal of $B$ is an idempotent two-sided ideal of $B$.*

An ideal of $A$ will always mean a two-sided ideal of $A$. Recall that $A$ is ELT (resp. ERT) if every essential left (resp. right) ideal of $A$ is an ideal of $A$. As usual, $A$ is called fully (resp. (1) fully left; (2) fully right) idempotent if every ideal (resp. (1) left ideal; (2) right ideal) of $A$ is idempotent.

Note that if $A$ is fully idempotent and every maximal left (with even every maximal right) ideal of $A$ is an ideal, then $A$ needs not be von Neumann regular ([28, Theorem 1]).

**Theorem 6.** *If $A$ is an* ELT *fully idempotent ring, then $A$ is a von Neumann regular ring whose simple right (or left) modules are either projective or injective.*

Proof: Let $B$ be a prime factor ring of $A$. Then $B$ is an ELT fully idempotent ring. The proof of Proposition 4 shows that $B$ is fully left idempotent. By

[1, Theorem 3.1], $B$ is von Neumann regular. Looking carefully at the proof of [1, Theorem 3.1], we see that $A$ is also ERT. Let $M$ be a maximal right ideal of $A$. If $A/M_A$ is not projective, then $M_A$ is essential in $A_A$ which implies that $M$ is an ideal of $A$ and is therefore a maximal left ideal of $A$. For any $y \in M$, $y \in yAyA \subseteq yM$ which implies that the simple left $A$-module $A/M$ is flat ([2, p. 458]). By [21, Lemma 1], $A/M_A$ is injective. This proves that every simple right $A$-module is either injective or projective. Similarly, every simple left $A$-module is either injective or projective. □

**Corollary 7.** *An* ELT *fully idempotent ring is either regular with non-zero socle or strongly regular.*

The next remark is connected with [6, Corollary 6], [18, Lemma 1] and [26, Question 1].

**Remark 5.** If $A$ is an ERT (or ELT) ring whose simple left modules are $p$-injective, then $A$ is regular and every simple one-sided $A$-module is either injective or projective.

**Remark 6.** If $A$ is a semi-prime ELT ring containing an injective maximal left ideal, then $A$ is a left and right self-injective, left and right $V$-ring of bounded index. Consequently, $A$ is left and right FPF by a theorem of S. Page [4, Theorem 5.49].

We now consider a particular class of biregular rings which generalizes simple non-Artinian rings and semi-simple Artinian rings.

We introduce two definitions.

**Definitions.** Let $E, T$ be ideals of $A$, $E \subseteq T$. We say that

(1) $T$ is an essential extension of $E$ (or $E$ is essential in $T$) if $E \cap N \neq 0$ for any non-zero ideal $N$ of $A$ contained in $T$;
(2) $E$ is a complement ideal of $A$ if $E$ has no proper essential extension in $A$.

**Proposition 8.** *The following conditions are equivalent for a ring $A$:*

(1) *every ideal of $A$ is generated by a central idempotent;*
(2) *for every ideal $T$ of $A$, $T + (I(T) \cap r(T))$ is a complement ideal of $A$.*

Proof: Assume (1). Let $T$ be an ideal of $A$. If $I = T + (I(T) \cap r(T))$, since $T = Ae$, where $e$ is a central idempotent, then $I(T) = I(eA) = A(1-e) = (1-e)A = r(T)$ and $A = Ae \oplus A(1-e) = T \oplus (I(T) \cap r(T))$. Therefore (1) implies (2).

Assume (2). Let $T$ be an ideal of $A$ such that $T^2 = 0$. Then $r(T)$ is an essential left ideal of $A$ which implies $r(T)$ is an essential ideal of $A$. Similarly, $I(T)$ is an essential ideal of $A$. Therefore $r(T) \cap I(T)$ is an essential ideal of $A$ which implies that $T + I(T) \cap r(T)$ is an essential ideal of $A$. By hypothesis, $T + (I(T) \cap r(T))$ is a complement ideal of $A$ which yields $T + (I(T) \cap r(T)) = A$.

But $T^2 = 0$ implies that $T \subseteq I(T) \cap r(T)$, whence $A = I(T) \cap r(T)$, yielding $A = I(T) = r(T)$. This implies $T = 0$ and proves that $A$ must be semi-prime. Now for any ideal $U$ of $A$, $U \cap I(U) = 0$ and $I(U) = r(U)$. If $I = U + (I(U) \cap r(U))$, then $I = U + I(U) = U \oplus I(U)$. Suppose that $I$ is not an essential ideal of $A$: there exists a non-zero ideal $N$ of $A$ such that $I \cap N = 0$. Now $NU \subseteq N \cap U \subseteq N \cap I = 0$ which implies that $N \subseteq I(U) = r(U)$, whence $N = N \cap I(U) \subseteq N \cap I = 0$. This contradiction proves that $I$ must be essential in $A$. By hypothesis, $I = A$ which proves that $U$ is generated by a central idempotent (in as much as $A$ is semi-prime). Thus (2) implies (1).

□

The following property of $p$-injectivity seems interesting.

**Remark 7.** If $T$ is an ideal of $A$ such that ${}_A A/T$ is $p$-injective, then the factor ring $A/T$ is left $p$-injective.

Quasi-Frobeniusean rings are now characterized in terms of $p$-injectivity.

**Theorem 9.** *The following conditions are equivalent:*

(1) *$A$ is quasi-Frobeniusean;*
(2) *$A$ is left pseudo-Frobeniusean and projective $p$-injective right $A$-modules are injective;*
(3) *there exists a $p$-injective left generator of $A$-Mod and projective $p$-injective left $A$-modules are injective.*

Proof: (1) implies (2) and (3) by [3, Theorem 24.20].

Assume (2). Since $A$ is left pseudo-Frobeniusean, then every left ideal of $A$ is a left annihilator which implies that $A$ is right $p$-injective. For any projective right $A$-module $P$, there exist $B$, a direct sum of copies of $A_A$, and an epimorphism $p$ of $B$ onto $P_A$. Then $B/\ker p \approx P_A$ implies that $B \approx \ker p \oplus B/\ker p$. Since $B$ is a direct sum of $p$-injective right $A$-modules, then $B_A$ is $p$-injective. Therefore $B/\ker p$ is a $p$-injective right $A$-module and hence $P_A$ is $p$-injective. By hypothesis, $P_A$ is injective. Then (2) implies (1) by [3, Theorem 24.20].

Assume (3). Let $G$ be a $p$-injective left generator of $A$-Mod. For any projective left $A$-module $F$, there exists $C$, a direct sum of copies of ${}_A G$, and an epimorphism $q : {}_A C \to {}_A F$. As before, we obtain a $p$-injective left $A$-module $F$ which is injective by hypothesis. Thus (3) implies (1) by [3, Theorem 24.20]. □

Recall that a left $A$-module $M$ is YJ-injective if, for any $o \neq a \in A$, there exists a positive integer $n$ with $a^n \neq 0$ such that every left $A$-homomorphism of $Aa^n$ into $M$ extends to $A$ (cf. [15], [24], [30]). $A$ is called left YJ-injective if ${}_A A$ is YJ-injective. Similarly, YJ-injectivity is defined on the right side. In [15], it is shown that YJ-injectivity generalizes $p$-injectivity even for rings (quasi-injectivity generalizes injectivity but the two concepts coincide for rings). Also left YJ-injective rings generalize right IF-rings. If $A$ is left YJ-injective, then every non-zero-divisor is invertible in $A$. Consequently, $A$ coincides with $Q^l_{cl}(A)$ and $Q^r_{cl}(A)$, the classical left and right quotient rings of $A$. It is well-known that $A$ is left non-singular iff $A$ has a maximal left quotient ring, noted $Q^l_{\max}(A)$, which is a left self-injective regular ring. If $A$ is a reduced ring, $Q^l_{\max}(A)$, is not necessarily strongly regular. However, if $A$ is reduced and admits a classical left quotient ring $Q^l_{cl}(A)$, then by [22, Proposition 1.5], $Q^l_{cl}(A)$ is a reduced ring (this is the case when $A$ is left non-singular, left duo). In that case, if $Q^l_{cl}(A)$ is left or right YJ-injective, then it is strongly regular. Consequently, a reduced left YJ-injective ring is strongly regular.

A well-known theorem of Y. Utumi asserts that if $A$ is left and right non-singular, then $Q^l_{\max}(A)$ and $Q^r_{\max}(A)$ coincide iff every complement one-sided ideal of $A$ is an annihilator. (The terms "annihilator" and "complement" should be permuted in [4, p. 181]).

Left (or right) Johns rings are studied in [4].

**Question 3.** Is a left Johns, left YJ-injective ring quasi-Frobeniusean?

(We know that a left $p$-injective left Johns ring is quasi-Frobeniusean).

Now let $J$, $Y$, $Z$ denote respectively the Jacobson radical, the right singular ideal and the left singular ideal of the ring $A$.

**Proposition 10.** *Let $A$ be a ring whose simple right modules are flat. If every maximal left ideal of $A$ is either injective or an ideal of $A$, then either $A$ is a left self-injective regular left $V$-ring or $A$ is strongly regular. Consequently, $A$ must be a regular left $V$-ring.*

Proof: First suppose that every maximal left ideal of $A$ is an ideal of $A$. For any maximal left ideal $N$ of $A$, $N$ is a maximal right ideal of $A$ and by hypothesis, $A/N_A$ is flat. Then ${}_AA/N$ is injective by [21, Lemma 1] which implies that $A$ is a left $V$-ring, whence $A$ is strongly regular (cf [17, Proposition 3]). Now suppose that there exists a maximal left ideal $M$ of $A$ which is not an ideal of $A$. Then ${}_AM$ is injective which implies $A = M \oplus U$, where $U$ is a minimal projective left ideal of $A$. Let $V$ be an arbitrary minimal projective left ideal of $A$. Write $V = Av$, $0 \neq v \in A$. If $MV = 0$, then $MAv = 0$ which implies that $MA = M$ (because $MA \neq A$). This contradicts the hypothesis that $M$ is not an ideal of $A$. Therefore $MV \neq 0$ and $Mw \neq 0$ for some $0 \neq w \in V$. Now $Mw = V$ and we have an epimorphism $p : M \to V$ defined by $p(m) = mw$ for all $m \in M$. Then $M/\ker p \approx V$ which yields $M \approx \ker p \oplus M/\ker p$ (in as much as ${}_AV$ is projective). Since ${}_AM$ is injective, then so is $M/\ker p$, proving that ${}_AV$ is injective. In particular, ${}_AU$ is injective which implies that $A$ is left self-injective. Now for any maximal left ideal $L$ of $A$, if ${}_AL$ is injective, then ${}_AA/L$ is injective as just seen. If $L$ is an ideal of $A$, then $A/L_A$ is flat which implies that ${}_AA/L$ is injective ([21, Lemma 1]). In any case, ${}_AA/L$ is injective, proving that $A$ is a left $V$-ring, whence $J = 0$ (cf. [18, Lemma 1]). Since $A$ is left self-injective, $Z = J = 0$

and hence $A$ is von Neumann regular. We conclude that $A$ must be a regular, left $V$-ring. □

We now turn to a characterization of commutative quasi-Frobeniusean rings.

**Theorem 11.** *The following conditions are equivalent for a commutative ring $A$:*

(1) *$A$ is quasi-Frobeniusean;*
(2) *$A$ is a* YJ*-injective ring with maximum condition on annihilators and* Soc$(A)$*, the socle of $A$, is finitely generated.*

Proof: It is clear that (1) implies (2).

Assume (2). Since $A$ is a commutative YJ-injective, then $A$ coincides with its classical quotient ring. Since $A$ satisfies the maximum condition on annihilators, then $A/J$ is Artinian ([4, Theorem 16.31]) and also $Y$ is nilpotent. Now $A$ being YJ-injective implies that $J = Y$ ([24, p. 103]), whence $A$ is semi-primary. Therefore $A$ has an essential socle. Given Soc$(A)$ finitely generated, we then have a finitely embedded ring $A$ satisfying the maximum condition on annihilators which yields $A$ Artinian ([4, p. 164]). Since $A$ is YJ-injective, every minimal ideal of $A$ must be an annihilator. In that case, $A$ is quasi-Frobeniusean by a theorem of H.H. Storrer. Thus (2) implies (1). □

**Remark 8.** A right YJ-injective ring whose simple right modules are either $p$-injective or projective is fully right idempotent (this is because $Y = J = Y \cap J = 0$).

Theorem 11 motivates the next question on YJ-injectivity.

**Question 4.** Is a commutative YJ-injective ring with maximum condition on annihilators quasi-Frobeniusean?

We add a remark on flatness and $p$-injectivity.

**Remark 9.** We know that $A$ is von Neumann regular if every cyclic singular left $A$-module is flat (Math. J. Okayama Univ. 20 (1978), 123–129 (Theorem 5)). If every singular left $A$-module is injective, $A$ needs not be von Neumann regular ([4, p. 92]). Consequently, this is also the case when every cyclic singular left $A$-module is $p$-injective. However, $A$ is von Neumann regular if $A$ is also left $p$-injective. We may also recall the following: If $I$ is a $p$-injective left ideal of $A$, then ${}_AA/I$ is flat.

In 1974, we introduced the concept of $p$-injective modules to study von Neumann regular rings, $V$-rings and associated rings ([17]). In 1985, this is generalized to YJ-injective modules ([24]). In 1998, Xue Weimin showed that even for rings, YJ-injectivity effectively generalizes $p$-injectivity [15]. Finally, Zhang-Wu [30] proved that if every left $A$-module is YJ-injective, then $A$ is von Neumann regular (which answers a question raised in [24]).

K.R. Goodearl's volume on von Neumann regular rings [5] has motivated numerous papers in that area during the last twenty years. It is now a classic for people interested in VNR rings (cf. [4]).

In view of [17, Proposition 1] and our Theorem 1 here, we raise the last question.

**Question 5.** Is $A$ von Neumann regular if $A$ is a semi-prime ring such that every principal one-sided ideal is the annihilator of an element of $A$?

Note that semi-prime rings whose principal one-sided ideals are annihilators need not be von Neumann regular (cf. K. Beidar–R. Wisbauer, Comm. Algebra 23 (1995), 841–861 (Example 4.8), which answered in the negative a question raised in 1981).

In recent years, Xue Weimin and Zhang Jule solved several problems raised in my papers. Among the still unanswered questions, we recall the following:

**U.Q.1.** Is $A$ von Neumann regular if $A$ satisfies any one of the following conditions: (a) $A$ is left semi-hereditary and every maximal left ideal of $A$ is $p$-injective; (b) $A$ is a left $p$-injective left $V$-ring; (c) every principal left ideal of $A$ is a projective left annihilator; (d) $A$ is left semi-hereditary and every simple left $A$-module is flat; (e) $A$ is a semi-prime left self-injective ring whose maximal essential left ideals are two-sided.

**U.Q.2.** Is $A$ strongly regular if $A$ is a reduced ring whose principal left ideals are complement left ideals?

**U.Q.3.** Is $A$ fully left idempotent if every simple left $A$-module is YJ-injective? (The answer is "yes" if "YJ-injective" is replaced by "$p$-injective").

**U.Q.4.** Is $A$ Artinian if $A$ is a prime left self-injective ring whose maximal essential left ideals are two-sided?

**U.Q.5.** Is $A$ left pseudo-Frobeniusean if $A$ is a left Kasch ring containing an injective maximal left ideal?

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