Remarks on extremally disconnected semitopological groups

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Abstract. Answering recent question of A.V. Arhangel'skii we construct in ZFC an extremally disconnected semitopological group with continuous inverse having no open Abelian subgroups.

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All topological spaces under consideration are supposed to be Hausdorff. A topological space X is called *extremally disconnected* if the closure of every open subset of X is open. A topological space X without isolated points is called *maximal* if X has an isolated point in any stronger topology. Every maximal space is extremally disconnected. A group G provided with a topology τ is called maximal if (G,τ) is maximal as a topological space.

A group G provided with a topology is called *left* (*right*) topological if all mappings $x \mapsto gx$, $g \in G$ ($x \mapsto xg$, $g \in G$) are continuous. A group G with a topology τ is called *semitopological* if (G,τ) is left and right topological.

In [1] A.V. Arhangel'skii established some properties of extremally disconnected semitopological groups and posed three problems.

Problem 1. *Is there in ZFC an example of a non-discrete extremally disconnected topological group?*

This is a reminiscence of old (and still unsolved) problem from [2]. It is worth of mentioning that for some types of extremally disconnected topological groups the answer to Problem 1 is negative. For example, if there exists a maximal topological group, then there exists a P-point in ω^*, the reminder of the Stone-Čech compactification of the discrete space ω ([4, Theorem 7.3]). For further results in this direction see [5, Theorem 5.1], [7, Theorem 2.5] and [9].

Problem 2. *Is there in ZFC an example of a non-discrete extremally disconnected semitopological group with continuous inverse?*

Several kinds of such examples follow from [5] and [8]. We describe three of them.

By [5, Theorem 1.2], every infinite group G of cardinality α admits a maximal left invariant topology τ of dispersion character α. Remind that the dispersion character of a topological space (X, τ) is the cardinal $\Delta(\tau) = \min\{|U| : U \in \tau, U \neq \emptyset\}$. If G is Abelian then (G, τ) is semitopological. If G is Boolean ($g^2 = e$ for every $g \in G$, e is the identity of G) then (G, τ) is a semitopological group with continuous inverse. Note that a maximal left topological group need not be regular. However, every countable group admits a maximal regular left invariant topology ([5, Corollary 2.7]). It is still unknown ([5, Problem 2.10]) whether there is in ZFC an example of a regular maximal left topological group of uncountable dispersion character.

By [5, Theorem 1.3 and 4.5], every infinite group G admits an extremally disconnected left invariant topology τ such that (G, τ) is zero-dimensional (i.e. every point of G has base of neighborhoods consisting of clopen subsets) and left totally bounded (i.e. for every neighborhood U of the identity, there exists a finite subset F with $G = FU$). By the above argument, there is a zero-dimensional example to Problem 2 of arbitrary dispersion character.

Let τ, τ' be left invariant topologies on a group G. We say that (G, τ') is an open refinement of (G, τ) if $\tau \subseteq \tau'$ and every nonempty open subset from (G, τ') contains a nonempty open subset from (G, τ). By [8], every left topological group (G, τ) has an extremally disconnected open refinement. If (G, τ) is regular, then there exists a zero-dimensional extremally disconnected open refinement (G, τ'). Now suppose that, for every element $g \in G$, there exists a neighborhood U of identity in τ such $gx = xg$ for each $x \in U$. Then every open refinement (G, τ') of (G, τ) is a semitopological group. In addition, if the subset $\{g : g^2 = e\}$ is open in $(G,)$, then the mapping $x \mapsto x^{-1}$ is continuous in (G, τ').

Problem 3. *Let G be an extremally disconnected semitopological group with continuous inverse. Does there exist an open and closed Abelian subgroup of G?*

There are two reasons for considering this problem. By Malykhin's theorem [1, Theorem 2], every extremally disconnected topological group has a clopen Boolean subgroup. By [1, Theorem 3], for every non-discrete extremally disconnected semitopological group with continuous inverse, there exists a neighborhood U of the identity e such that $g^2 = e$ for every $g \in U$. In one special case this problem has been mentioned in [6]: does every maximal semitopological group with continuous inverse contain an open Boolean subgroup?

The following two theorems give us a negative answer to Problem 3.

Theorem 1. *For every infinite cardinal α, there exists a semitopological group (G, τ) with continuous inverse and following properties: $\Delta(\tau) = \alpha$, (G, τ) has no open Abelian subgroups, (G, τ) is extremally disconnected and zero-dimensional.*

Theorem 2. *For every infinite cardinal α, there exists a maximal semitopological group (G, τ) with continuous inverse such that $\Delta(\tau) = \alpha$ and (G, τ) has no open Abelian subgroups.*

To prove these theorems we need some definitions, constructions and results from [3], [5].

Given a discrete space X, we take the points of βX, the Stone-Čech compactification of X, to be the ultrafilters on X, with the points of X identified with the principal ultrafilters. The topology of βX can be defined by stating that the sets of the form $\{p \in \beta X : A \in p\}$, where A is a subset of X, form a base for the open sets. We note that the sets of this form are clopen and that, for any $p \in \beta X$ and any $A \subseteq X$, $A \in p$ if and only if $p \in \overline{A}$, where $\overline{A}$ is a closure of A in βX. If A is a subset of X we shall use A^* to denote $\overline{A} \setminus A$, in particular X^* is a set of all free ultrafilters on X. For every filter φ on X denote $\overline{\varphi} = \{p \in \beta X : \varphi \subseteq p\}$, $\varphi^* = \overline{\varphi} \cap G^*$.

Let G be a discrete group. There are two natural ways for extension of multiplication from G to βG. We follow [3, Chapter 4]. Given any $p, q \in \beta G$ and $A \subseteq G$, put

$$A \in pq \text{ if and only if } \{g \in G : g^{-1}A \in q\} \in p.$$

Take any member $P \in p$ and, for every $x \in P$, choose some element $Q_x \in q$. Then $\bigcup_{X \in P} xQ_x \in pq$ and the family of subsets of this form is a base of the ultrafilter pq. This multiplication on βG is associative, so βG is a semigroup and G^* is a subsemigroup of βG.

Every closed subsemigroup of βG has an idempotent p, $p^2 = p$ ([3, Theorem 2.5]). Given any idempotent $p \in G^*$, the family of subsets $\{P \cup \{e\} : P \in p\}$ is a filter of neighborhoods of e for the uniquely determined maximal left invariant topology on G ([5, §1]). A group G provided with this topology is denoted by $G(p)$. We need also another type of topologies determined by idempotents. Fix $p \in G^*$ with $p^2 = p$ and, for every subset $A \subseteq G$, put $\mathrm{cl}(A, p) = \{x \in G : A \in xp\}$. Then the family $\{\mathrm{cl}(A, p) : A \in p\}$ is a base of neighborhoods of e for the uniquely determined zero-dimensional extremally disconnected left invariant topology on G ([5, §1]). A group G provided with this topology is denoted by $G[p]$.

Let X be an infinite set of cardinality α. For every permutation f of X, put $\operatorname{supp} f = \{x \in X : f(x) \neq x\}$. Consider the group $S(X)$ of all permutations of X with finite supports. For every nonempty subset $Y \subseteq X$, identify $S(Y)$ with the subgroup of all permutations $f \in S(X)$ such that $f(x) \in Y$, $x \in Y$ and $f(x) = x$, $x \in X \setminus Y$. The identity permutation is denoted by e.

Let $\mathbf{F} = \{Y \subseteq X : X \setminus Y \text{ is finite}\}$ be a filter of all cofinite subsets of X. Denote by φ_0 the filter on $S(X)$ with base $\{S(Y) : Y \in \mathbf{F}\}$. Note that φ_0^* is a subsemigroup of $\beta S(X)$ and

$(*)$ for every $f \in S(X)$, there exists $F \in \varphi$ such that $fg = gf$ for every $g \in F$.

Put $S_2(X) = \{g \in S(X) : g^2 = e\}$ and denote by φ_2 the filter on $S(X)$ with the base $\{F \cap S_2(X) : F \in \varphi_0\}$. By $(*)$, φ_2^* is a subsemigroup of $\beta S(X)$.

Call a subset $A \subseteq S(X)$ sparse if there exists $x \in X$ such that $|\{g(x) : g \in A\}| = \alpha$. A filter φ' on $S(X)$ is called sparse if every member of φ' is sparse.

Clearly, φ_2 is a sparse filter. By Zorn's Lemma, for every sparse filter φ', there exists a sparse ultrafilter p with $\varphi' \subseteq p$. Hence, the subset SP of all sparse ultrafilters from φ_2^* is nonempty. Clearly, SP is closed in $\beta S(X)$. For every sparse subset $A \subseteq S(X)$ and every $f \in S(X)$, the subset fA is sparse. It follows that SP is a subsemigroup of $\beta S(X)$. We shall use the following claim

(∗∗) for every sparse subset $A \subseteq S(X)$, there exist $h, g \in A$ such that $hg \neq gh$.

To prove (∗∗), choose $x \in X$ such that the subset $\{f(x) : f \in A\}$ is infinite. Fix any $h \in A$ with $h(x) \neq x$ and pick $g \in A$ such that $g(x) \notin \operatorname{supp} h$. Since $hg(x) = g(x)$ and $h(x) \neq x$, then $hg(x) \neq gh(x)$ so $hg \neq gh$.

Proof of Theorem 2: Put $G = S(X)$ and choose any idempotent $p \in SP$. Consider the maximal left topological group $(G, \tau) = G(p)$. Since p is sparse then $\Delta(\tau) = \alpha$. Since $\varphi_2 \subseteq p$ then, by (∗), (G, τ) is right topological with continuous inverse. By (∗∗), (G, τ) has no open Abelian subgroups. □

Proof of Theorem 1: Put $G = S(X)$ and choose any idempotent $p \in SP$. Consider the extremally disconnected zero-dimensional left topological group $(G, \tau) = G[p]$. Clearly, $\Delta(\tau) = \alpha$. Denote by τ_e the filter of neighborhoods of e in τ. For every ultrafilter q on G, $\tau_e \subseteq q$ if and only if $qp = p$ ([5, §2]). By (∗) and the definition of product of ultrafilters, $\varphi_2 \subseteq \tau_e$. By the above paragraph, (G, τ) is right topological with continuous inverse and (G, τ) has no open Abelian subgroups. □

We conclude the paper with four remarks.

1. Using arguments from [5, §2], we can add the following statement to Theorem 2: there exists a countable zero-dimensional maximal semitopological group with continuous inverse and without open Abelian subgroups.

2. A topological space S is called strongly extremally disconnected if, for every open nonclosed subset U of S, there exists $x \in \operatorname{cl} U \setminus U$ such that $\{x\} \cup U$ is a neighborhood of x. Let (G, τ) be a left topological group and let an ultrafilter q converge to the identity in τ. By [8, Theorem 4.12], the strongest left invariant topology τ_q on G in which q converges to e is strongly extremally disconnected. Put $G = S(X)$ and denote by τ the left invariant topology on G such that φ_2 is a filter of neighborhoods of e. Choose any ultrafilter $q \in SP$. Then (G, τ_q) is a particular example to Problem 3.

3. The group $S(X)$ has been used in [5, Example 6.2] to prove the following statement. Let B be a non-discrete extremally disconnected topological Abelian group. Then there exists an extremally disconnected topological group G with distinct left and right uniformities such that B is topologically isomorphic to some open subgroup of G.

4. A group G with a topology τ is called paratopological if the multiplication $(x, y) \mapsto xy$ is jointly continuous in G. By [8], every maximal paratopological group is a topological group. Let G be an extremally disconnected paratopological group. Is G a topological group?

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