On the intrinsic geometry of a unit vector field

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Dedicated to Professor Oldřich Kowalski on the occasion of his 65th birthday

Abstract. We study the geometrical properties of a unit vector field on a Riemannian 2-manifold, considering the field as a local imbedding of the manifold into its tangent sphere bundle with the Sasaki metric. For the case of constant curvature K, we give a description of the totally geodesic unit vector fields for $K = 0$ and $K = 1$ and prove a non-existence result for $K \neq 0, 1$. We also found a family ξ_ω of vector fields on the hyperbolic 2-plane L^2 of curvature $-c^2$ which generate foliations on T_1L^2 with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$.

Keywords: Sasaki metric, vector field, sectional curvature, totally geodesic submanifolds

Classification: Primary 54C40, 14E20; Secondary 46E25, 20C20

Introduction

A unit vector field ξ on a Riemannian manifold M is called *holonomic* if ξ is a field of normals of some family of regular hypersurfaces in M and *non-holonomic* otherwise. The geometry of non-holonomic unit vector fields has been developed by A. Voss at the end of the 19-th century. The foundations of this theory can be found in [1]. Recently, the geometry of a unit vector field has been considered from another point of view. Namely, let T_1M be the unit tangent sphere bundle of M endowed with the Sasaki metric ([9]). If ξ is a unit vector field on M, then one may consider ξ as a mapping $\xi : M \to T_1M$ so that the image $\xi(M)$ is a submanifold in T_1M with the metric induced from T_1M. So, one may apply the methods from the study of the geometry of submanifolds to determine geometrical characteristics of a unit vector field. For example, the unit vector field ξ is said to be *minimal* if $\xi(M)$ is of minimal volume with respect to the induced metric ([6]). A number of examples of locally minimal unit vector fields has been found (see [2], [3], [7]). On the other hand, using the geometry of submanifolds, we may find the Riemannian, Ricci or scalar curvature of a unit vector field using the second fundamental form of the submanifold $\xi(M) \in T_1M$ found in [11]. In this paper we apply this approach to the simplest case when the base space is 2-dimensional and hence the submanifold $\xi(M) \in T_1M$ is a hypersurface.

2. The results

Let ξ be a given unit vector field on a 2-dimensional Riemannian manifold (M, g). Denote by e_0 a unit vector field such that $\nabla_{e_0}\xi = 0$. Denote by e_1 a unit vector field, orthogonal to e_0, such that

$$\nabla_{e_1}\xi = \lambda\eta,$$

where η is a unit vector field, orthogonal to ξ. The function λ is a *signed* singular value of a linear operator $\nabla\xi : TM \to \xi^{\perp}$ (acting as $(\nabla\xi)X = \nabla_X\xi$). Set

$$\nabla_{\xi}\,\xi = k\,\eta, \quad \nabla_{\eta}\,\eta = \kappa\,\xi.$$

The functions k and κ are the *signed* geodesic curvatures of the integral curves of the fields ξ and η respectively. We prove that $\lambda^2 = k^2 + \kappa^2$.

Denote the *signed* geodesic curvatures of the integral curves of the fields e_0 and e_1 as μ and σ respectively. Then

$$\nabla_{e_0}\,e_0 = \mu\,e_1, \quad \nabla_{e_1}\,e_1 = \sigma\,e_0.$$

The orientations of the frames (ξ, η) and (e_0, e_1) are independent. Set $s = 1$ if the orientations are coherent and $s = 0$ otherwise.

The following result (Lemma 3.2) is a basic tool for the study.

Let M be a 2-dimensional Riemannian manifold of Gaussian curvature K. The second fundamental form Ω of the submanifold $\xi(M) \subset T_1M$ is given by

$$\Omega = \begin{bmatrix} -\mu\,\frac{\lambda}{\sqrt{1+\lambda^2}} & (-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2} \\ (-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2} & e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) \end{bmatrix}.$$

Using the formula for the sectional curvature of T_1M^n, we find an expression for the Gaussian curvature of $\xi(M^2)$ (Lemma 3.4).

The Gaussian curvature K_ξ of a hypersurface $\xi(M) \in T_1M$ is given by

$$K_\xi = \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K) + \frac{1}{2}\mu e_1\left(\frac{1}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2,$$

where K is the Gaussian curvature of M.

As applications of these lemmas, we prove the following theorems.

Theorem 1. *Let M^2 be a Riemannian manifold of constant Gaussian curvature K. A unit vector field ξ generating a totally geodesic submanifold in T_1M^2 exists if and only if $K=0$ or $K=1$. Moreover,*

(a) *if $K=0$, then ξ is either a parallel vector field or moving along a family of parallel geodesics with constant angle speed. Geometrically, $\xi(M^2)$ is either M^2 imbedded isometrically into $M^2\times S^1$ as a factor or a (helical) flat submanifold in $M^2\times S^1$;*

(b) *if $K=1$, then ξ is a vector field on a standard sphere S^2 which is parallel along the meridians and moving along the parallels with a unit angle speed. Geometrically, $\xi(M^2)$ is a part of totally geodesic RP^2 locally isometric to the sphere S^2 of radius 2 in $T_1S^2 \overset{isom}{\approx} RP^3$.*

Theorem 2. *Let M^2 be a 2-dimensional Riemannian manifold of Gaussian curvature K. Suppose that ξ is a unit geodesic vector field on M^2. Then the submanifold $\xi(M^2)\subset T_1M^2$ has non-positive extrinsic curvature.*

Theorem 3. *Let M^2 be a space of constant Gaussian curvature K. Suppose that ξ is a unit geodesic vector field on M^2. Then $\xi(M^2)$ has constant Gaussian curvature in one of the following cases:*

(a) *$K=-c^2<0$ and ξ is a normal vector field for the family of horocycles on the hyperbolic 2-plane L^2 of curvature $-c^2$. In this case, $K_\xi=-c^2$ and therefore $\xi(M^2)$ is locally isometric the base space;*

(b) *$K=0$ and ξ is a parallel vector field on M^2. In this case $K_\xi=0$ and $\xi(M^2)$ is also locally isometric to the base space;*

(c) *$K=1$ and ξ is any (local) geodesic vector field on the standard sphere S^2. In this case, $K_\xi=0$.*

The case (a) of Theorem 3 has an interesting generalization of the following kind.

Theorem 4. *Let L^2 be a hyperbolic 2-plane of constant curvature $-c^2$. Then T_1L^2 admits a hyperfoliation with leaves of constant intrinsic curvature $-c^2$ and of constant extrinsic curvature $-\frac{c^2}{4}$. The leaves are generated by unit vector fields making a constant angle with a pencil of parallel geodesics on L^2.*

2. Basic definitions and preliminary results

Let (M,g) be an $(n+1)$-dimensional Riemannian manifold with metric g. Let ∇ denote the Levi-Civita connection on M. Then $\nabla_X\xi$ is always orthogonal to ξ and hence, $(\nabla\xi)X \overset{def}{=} \nabla_X\xi : T_pM\to\xi_p^\perp$ is a linear operator at each $p\in M$. We define an adjoint operator $(\nabla\xi)^*X:\xi_p^\perp\to T_pM$ by

$$\langle(\nabla\xi)^*X,Y\rangle_g=\langle X,\nabla_Y\xi\rangle_g\,.$$

Then there is an orthonormal frame $e_0, e_1, \ldots, e_n$ in T_pM and an orthonormal frame $f_1, \ldots, f_n$ in $\xi_p^\perp$ such that

$$(1)\qquad (\nabla\xi)e_0 = 0, \quad (\nabla\xi)e_\alpha = \lambda_\alpha f_\alpha, \quad (\nabla\xi)^* f_\alpha = \lambda_\alpha e_\alpha, \qquad \alpha = 1, \ldots, n,$$

where $\lambda_1, \lambda_2, \ldots \lambda_n$ are real-valued functions.

Definition 2.1. The orthonormal frames satisfying (1) are called *singular frames* for the linear operator $(\nabla\xi)$ and the real valued functions $\lambda_1, \lambda_2, \ldots \lambda_n$ are called the (signed) *singular values* of the operator $\nabla\xi$ with respect to the singular frame.

Remark that the sign of the singular value is defined up to the directions of the vectors of the singular frame.

For each $\tilde X \in T_{(p,\xi)}TM$ there is a decomposition

$$\tilde X = X_1^h + X_2^v,$$

where $(\cdot)^h$ and $(\cdot)^v$ are the horizontal and vertical lifts of vectors X_1 and X_2 from T_pM to $T_{(p,\xi)}TM$. The Sasaki metric is defined by the scalar product of the form

$$\langle\langle \tilde X, \tilde Y\rangle\rangle = \langle X_1, Y_1\rangle + \langle X_2, Y_2\rangle,$$

where $\langle\cdot\,,\cdot\rangle$ means the scalar product with respect to metric g.

The following lemma has been proved in [11].

Lemma 2.1. *At each point $(p,\xi) \in \xi(M) \subset TM$ the vectors*

$$(2)\qquad \begin{cases} \tilde e_0 = e_0^h, \\ \tilde e_\alpha = \dfrac{1}{\sqrt{1+\lambda_\alpha^2}}(e_\alpha^h + \lambda_\alpha f_\alpha^v), \qquad \alpha = 1, \ldots, n, \end{cases}$$

form an orthonormal frame in the tangent space of $\xi(M)$ and the vectors

$$(3)\qquad \tilde n_{\sigma|} = \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big(-\lambda_\sigma e_\sigma^h + f_\sigma^v\,\big), \qquad \sigma = 1, \ldots, n,$$

form an orthonormal frame in the normal space of $\xi(M)$.

Let $R(X,Y)\xi = [\nabla_X, \nabla_Y]\,\xi - \nabla_{[X,Y]}\,\xi$ be the curvature tensor of M. Introduce the following notation

$$(4)\qquad r(X,Y)\xi = \nabla_X\nabla_Y\xi - \nabla_{\nabla_XY}\xi.$$

Then, evidently,

$$R(X,Y)\xi = r(X,Y)\xi - r(Y,X)\xi.$$

The following lemma has also been proved in [11].

Lemma 2.2. *The components of second fundamental form of $\xi(M) \subset T_1M$ with respect to the frame* (3) *are given by*

$$\begin{aligned}
\tilde\Omega_{\sigma|00} &= \frac{1}{\sqrt{1+\lambda_\sigma^2}}\big\langle r(e_0,e_0)\xi, f_\sigma\big\rangle,\\
\tilde\Omega_{\sigma|\alpha 0} &= \frac{1}{2}\frac{1}{\sqrt{(1+\lambda_\sigma^2)(1+\lambda_\alpha^2)}}\Big[\big\langle r(e_\alpha,e_0)\xi + r(e_0,e_\alpha)\xi, f_\sigma\big\rangle\\
&\qquad\qquad + \lambda_\sigma\lambda_\alpha\big\langle R(e_\sigma,e_0)\xi, f_\alpha\big\rangle\Big],\\
\tilde\Omega_{\sigma|\alpha\beta} &= \frac{1}{2}\frac{1}{\sqrt{(1+\lambda_\sigma^2)(1+\lambda_\alpha^2)(1+\lambda_\beta^2)}}\Big[\big\langle r(e_\alpha,e_\beta)\xi + r(e_\beta,e_\alpha)\xi, f_\sigma\big\rangle\\
&\quad + \lambda_\alpha\lambda_\sigma\big\langle R(e_\sigma,e_\beta)\xi, f_\alpha\big\rangle + \lambda_\beta\lambda_\sigma\big\langle R(e_\sigma,e_\alpha)\xi, f_\beta\big\rangle\Big],
\end{aligned}$$

where $\{e_0, e_1, \dots, e_n; f_1, \dots, f_n\}$ is a singular frame of $(\nabla\xi)$ and $\lambda_1, \dots, \lambda_n$ are the corresponding singular values.

Let $\tilde\nabla$ and ∇ be the Levi-Civita connections of the Sasaki metric of TM and the metric of M respectively. The Kowalski formulas [8] give the covariant derivatives of combinations of lifts of vector fields.

Lemma 2.3 (O. Kowalski)**.** *Let X and Y be vector fields on M. Then at each point $(p,\xi) \in TM$ we have*

$$\begin{aligned}
\tilde\nabla_{X^h}Y^h &= (\nabla_XY)^h - \frac{1}{2}\left(R(X,Y)\xi\right)^v,\\
\tilde\nabla_{X^h}Y^v &= \frac{1}{2}\left(R(\xi,Y)X\right)^h + (\nabla_XY)^v,\\
\tilde\nabla_{X^v}Y^h &= \frac{1}{2}\left(R(\xi,X)Y\right)^h,\\
\tilde\nabla_{X^v}Y^v &= 0,
\end{aligned}$$

where R is the Riemannian curvature tensor of (M,g).

This basic result allows to find the curvature tensor of TM (see [8]) and the curvature tensor of T_1M (see [4]). As a corollary, it is not too hard to find an expression for the *sectional curvature* of T_1M. It is well-known that ξ^v is a unit normal for T_1M as a hypersurface in TM. Thus, $\tilde X = X_1^h + X_2^v$ is tangent to T_1M if and only if $\langle X_2, \xi\rangle = 0$.

Let $\tilde X = X_1^h + X_2^v$ and $\tilde Y = Y_1^h + Y_2^v$, where $X_2, Y_2 \in \xi^\perp$, form an orthonormal base of a 2-plane $\tilde\pi \subset T_{(p,\xi)}T_1M$. Then we have ([5]):

$$
\begin{aligned}
\tilde K(\tilde\pi) = {} & \langle R(X_1,Y_1)Y_1,X_1\rangle - \frac{3}{4}\|R(X_1,Y_1)\xi\|^2 \\
& + \frac{1}{4}\|R(\xi,Y_2)X_1 + R(\xi,X_2)Y_1\|^2 + \|X_2\|^2\|Y_2\|^2 - \langle X_2,Y_2\rangle^2 \\
& + 3\langle R(X_1,Y_1)Y_2,X_2\rangle - \langle R(\xi,X_2)X_1, R(\xi,Y_2)Y_1\rangle \\
& + \langle(\nabla_{X_1}R)(\xi,Y_2)Y_1,X_1\rangle + \langle(\nabla_{Y_1}R)(\xi,X_2)X_1,Y_1\rangle.
\end{aligned} \tag{5}
$$

Combining the results of Lemma 2.1, Lemma 2.2 and (5), we can write an expression for the sectional curvature of $\xi(M)$.

Lemma 2.4. *Let $\tilde X$ and $\tilde Y$ be an orthonormal vectors which span a 2-plane $\tilde\pi$ tangent to $\xi(M) \subset T_1M$. Denote by $K_\xi(\tilde\pi)$ the sectional curvature $\xi(M)$ with respect to the metric induced by Sasaki metric of T_1M. Then*

$$
K_\xi(\tilde\pi) = \tilde K(\tilde\pi) + \sum_\sigma \left(\Omega_{\sigma|}(\tilde X,\tilde X)\Omega_{\sigma|}(\tilde Y,\tilde Y) - \Omega^2_{\sigma|}(\tilde X,\tilde Y)\right), \tag{6}
$$

where $\tilde K(\tilde\pi)$ is the sectional curvature of T_1M given by (5), $\Omega_{|\sigma}$ are the components of the second fundamental form of $\xi(M)$ given by Lemma 2.2 and the vectors are given with respect to the frame (2).

3. The 2-dimensional case

Let M be a 2-dimensional Riemannian manifold. The following proposition gives useful information about the relation between the singular values of the $(\nabla\xi)$-operator, geometric characteristics of the integral curves of singular frame and the Gaussian curvature of the manifold.

Lemma 3.1. *Let ξ be a given smooth unit vector field on M^2. Denote by e_0 a unit vector field on M^2 such that $\nabla_{e_0}\xi = 0$. Let η and e_1 be the unit vector fields on M^2 such that (ξ,η) and (e_0,e_1) form two orthonormal frames on M^2. Denote by λ a signed singular value of the operator $(\nabla\xi)$. Then we have*

$$\nabla_{e_1}\xi = \lambda\eta,$$

and the following relations hold:

(a) *if $k = \langle\nabla_\xi\xi,\eta\rangle$ is the signed geodesic curvature of a ξ-curve and $\kappa = \langle\nabla_\eta\eta,\xi\rangle$ is the signed geodesic curvature of an η-curve, then*

$$\lambda^2 = k^2 + \kappa^2;$$

(b) *if K is the Gaussian curvature of M^2, then*

$$(-1)^s K = e_0(\lambda) - \lambda\sigma,$$

where $\sigma = \langle \nabla_{e_1} e_1, e_0 \rangle$ is the signed geodesic curvature of an e_1-curve and

$$s = \begin{cases} 1 & \text{if the frames } (\xi,\eta) \text{ and } (e_0,e_1) \text{ have the same orientation,} \\ 0 & \text{if the frames } (\xi,\eta) \text{ and } (e_0,e_1) \text{ have an opposite orientation.} \end{cases}$$

PROOF: (a) If (ξ, η) is an orthonormal frame on M^2, then

$$\begin{aligned} \nabla_\xi \xi &= k\,\eta, & \nabla_\xi \eta &= -k\,\xi, \\ \nabla_\eta \xi &= -\kappa\,\eta, & \nabla_\eta \eta &= \kappa\,\xi. \end{aligned} \tag{7}$$

Geometrically, the functions k and κ are the signed geodesic curvatures of ξ- and η-curves respectively.

In a similar way we get

$$\begin{aligned} \nabla_{e_0} e_0 &= \mu\, e_1, & \nabla_{e_0} e_1 &= -\mu\, e_0, \\ \nabla_{e_1} e_0 &= -\sigma\, e_1, & \nabla_{e_1} e_1 &= \sigma\, e_0, \end{aligned} \tag{8}$$

where μ and σ are the signed geodesic curvatures of the e_0- and e_1-curves respectively.

Let ω be an angle function between ξ and e_0. Then we have two possible decompositions:

$$\mathrm{Or}(+) \begin{cases} e_0 = \cos\omega\,\xi + \sin\omega\,\eta, \\ e_1 = -\sin\omega\,\xi + \cos\omega\,\eta, \end{cases} \qquad \mathrm{Or}(-) \begin{cases} e_0 = \cos\omega\,\xi + \sin\omega\,\eta, \\ e_1 = \sin\omega\,\xi - \cos\omega\,\eta. \end{cases}$$

In the case Or(+) we have

$$\begin{aligned} \nabla_{e_0} \xi &= (k\cos\omega - \kappa\sin\omega)\,\eta, \\ \nabla_{e_1} \xi &= -(k\sin\omega + \kappa\cos\omega)\,\eta, \end{aligned}$$

and due to the choice of e_0 and e_1 we see that

$$\begin{cases} k\cos\omega - \kappa\sin\omega &= 0, \\ k\sin\omega + \kappa\cos\omega &= -\lambda. \end{cases}$$

So, for the case of Or(+), $k = -\lambda\sin\omega$, $\kappa = -\lambda\cos\omega$.

In a similar way, for the case of Or(−), $k = \lambda\sin\omega$, $\kappa = \lambda\cos\omega$. In both cases

$$\lambda^2 = k^2 + \kappa^2.$$

(b) Due to the choice of the frames,

$$\begin{aligned}\big\langle R(e_0,e_1)\xi,\eta\big\rangle &= \big\langle \nabla_{e_0}\nabla_{e_1}\xi - \nabla_{e_1}\nabla_{e_0}\xi - \nabla_{\nabla_{e_0}e_1-\nabla_{e_1}e_0}\xi,\eta\big\rangle\\ &= \big\langle \nabla_{e_0}(\lambda\,\eta) - \nabla_{-\mu\,e_0+\sigma\,e_1}\,\xi,\eta\big\rangle = e_0(\lambda)-\lambda\sigma.\end{aligned}$$

On the other hand,

$$\big\langle R(e_0,e_1)\xi,\eta\big\rangle = \begin{cases} -K \text{ for the case of } \mathrm{Or}(+),\\ +K \text{ for the case of } \mathrm{Or}(-).\end{cases} \tag{9}$$

Set $s=1$ for the case $\mathrm{Or}(+)$ and $s=0$ for the case $\mathrm{Or}(-)$. Combining the results, we get $(-1)^sK = e_0(\lambda)-\lambda\sigma$, which completes the proof. □

The result of Lemma 2.2 can also be simplified in the following way.

Lemma 3.2. *Let M be a 2-dimensional Riemannian manifold of Gaussian curvature K. In terms of Lemma 3.1 the second fundamental form of the submanifold $\xi(M)\subset T_1M$ can be presented in two equivalent forms:*

(i)

$$\Omega = \begin{bmatrix} -\mu\,\frac{\lambda}{\sqrt{1+\lambda^2}} & (-1)^{s+1}\frac{K}{2}+\frac{e_0(\lambda)}{1+\lambda^2}\\ (-1)^{s+1}\frac{K}{2}+\frac{e_0(\lambda)}{1+\lambda^2} & e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)\end{bmatrix},$$

(ii)

$$\Omega = \begin{bmatrix} -\mu\,\frac{\lambda}{\sqrt{1+\lambda^2}} & \frac12\left(\sigma\,\lambda+\frac{1-\lambda^2}{1+\lambda^2}e_0(\lambda)\right)\\ \frac12\left(\sigma\,\lambda+\frac{1-\lambda^2}{1+\lambda^2}e_0(\lambda)\right) & e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right)\end{bmatrix}.$$

Proof: At each point $(p,\xi)\in\xi(M)$ the vectors

$$\begin{cases}\tilde e_0 = e_0^h,\\ \tilde e_1 = \frac{1}{\sqrt{1+\lambda^2}}(e_1^h+\lambda\eta^v)\end{cases}$$

form an orthonormal frame in the tangent space of $\xi(M)$ and

$$\tilde n = \frac{1}{\sqrt{1+\lambda^2}}\big(-\lambda e_1^h+\eta^v\ \big),$$

is a unit normal for $\xi(M)\subset T_1M$.

Thus we see that in a 2-dimensional case the components of Ω take the form

$$\Omega_{00} = \frac{1}{\sqrt{1+\lambda^2}}\big\langle r(e_0,e_0)\xi,\eta\big\rangle,\qquad \Omega_{11} = \frac{1}{(1+\lambda^2)^{3/2}}\big\langle r(e_1,e_1)\xi,\eta\big\rangle,$$

$$\Omega_{01} = \frac12\frac{1}{1+\lambda^2}\Big[\big\langle r(e_1,e_0)\xi+r(e_0,e_1)\xi,\eta\big\rangle+\lambda^2\big\langle R(e_1,e_0)\xi,\eta\big\rangle\Big].$$

Keeping in mind (4), (8) and (9), we see that

$$\begin{aligned} &\big\langle r(e_0,e_0)\xi,\eta\big\rangle = -\mu\lambda, && \big\langle r(e_0,e_1)\xi,\eta\big\rangle = e_0(\lambda),\\ &\big\langle r(e_1,e_0)\xi,\eta\big\rangle = \sigma\lambda, && \big\langle r(e_1,e_1)\xi,\eta\big\rangle = e_1(\lambda),\\ &\big\langle R(e_0,e_1)\xi,\eta\big\rangle = (-1)^s K. \end{aligned}$$

So we have

$$\Omega_{00} = -\mu\,\frac{\lambda}{\sqrt{1+\lambda^2}}, \qquad \Omega_{11} = \frac{e_1(\lambda)}{(1+\lambda^2)^{3/2}} = e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right),$$

$$\Omega_{01} = \frac{1}{2(1+\lambda^2)}(e_0(\lambda)+\lambda\sigma-\lambda^2(-1)^sK) = \begin{cases} (-1)^{s+1}\frac{K}{2}+\frac{e_0(\lambda)}{1+\lambda^2} \\ \frac{1}{2}\left(\sigma\,\lambda+\frac{1-\lambda^2}{1+\lambda^2}e_0(\lambda)\right) \end{cases},$$

where Lemma 3.1(b) has been applied in two ways. □

3.1 Totally geodesic vector fields

The main goal of this section is to prove Theorem 1. The proof will be divided into a series of separate propositions.

Proposition 3.1. *Let M^2 be a Riemannian manifold. Let D be a domain in M^2 endowed with a semi-geodesic coordinate system such that $ds^2 = du^2 + f^2\,dv^2$, where $f(u,v)$ is some non-vanishing function. Denote by (e_0,e_1) an orthonormal frame in D and specify $e_0 = \partial_u,\ e_1 = f^{-1}\partial_v$. If ξ is a unit vector field in D parallel along u-geodesics, then ξ can be written given as*

$$\xi = \cos\omega\, e_0 + \sin\omega\, e_1,$$

where $\omega = \omega(v)$ is an angle function and

(a) *a singular frame for ξ may be chosen as $\big\{e_0,\ e_1, \eta = -\sin\omega\, e_0 + \cos\omega\, e_1\big\}$;*
(b) *a singular value for ξ in this case is $\lambda = e_1(\omega) - \sigma$, where σ is a signed geodesic curvature of the e_1-curves.*

Proof: Indeed, if ξ is parallel along u-geodesics, then evidently the angle function ω between ξ and the u-curves does not depend on u. So this function has the form $\omega = \omega(v)$ and $\xi = \cos\omega\, e_0 + \sin\omega\, e_1$. Moreover, since

$$\begin{aligned} &\nabla_{e_0}e_0 = 0, && \nabla_{e_0}e_1 = 0,\\ &\nabla_{e_1}e_0 = \frac{f_u}{f}e_1, && \nabla_{e_1}e_1 = -\frac{f_u}{f}e_0, \end{aligned}$$

we see that $\sigma = -\frac{f_u}{f}$ and $\nabla_{e_1}\xi = (e_1(\omega)-\sigma)\,\eta$, where $\eta = -\sin\omega\, e_0 + \cos\omega\, e_1$. Therefore, $\lambda = e_1(\omega) - \sigma$ and the proof is complete. □

Proposition 3.2. *Let M^2 be a Riemannian manifold of constant negative curvature $K = -r^{-2} < 0$. Then there is no totally geodesic unit vector field on M^2.*

Proof: Suppose ξ is a totally geodesic unit vector field on M^2. Set $\Omega \equiv 0$ in Lemma 3.2. Then $\lambda\mu \equiv 0$. If $\lambda \equiv 0$ in some domain $D \subset M^2$, then ξ is parallel in this domain and hence M^2 is flat in D, which contradicts the hypothesis. Suppose that $\mu \equiv 0$ at least in some domain $D \subset M^2$. This means that e_0-curves are geodesics in D and the field ξ is parallel along them. Choose a family of e_0-curves and the orthogonal trajectories as a local coordinate net in D. Then the first fundamental form of M^2 takes the form

$$ds^2 = du^2 + f^2\, dv^2,$$

where $f(u,v)$ is some function. Since M^2 is of constant curvature $K = -\frac{1}{r^2}$, the function f satisfies the equation

$$f_{uu} - \frac{1}{r^2} f = 0.$$

The general solution of this equation is

$$f(u,v) = A(v)\cosh(u/r) + B(v)\sinh(u/r).$$

There are two possible cases:

(i) $A^2(v) \equiv B^2(v)$ over the whole domain D;
(ii) $A^2(v) \neq B^2(v)$ in some subdomain $D' \subset D$.

Case (i). In this case, in dependence of the signs of $A(v)$ and $B(v)$,

$$f(u,v) = A(v)e^{u/r} \quad \text{or} \quad f(u,v) = A(v)e^{-u/r}.$$

Consider the first case (the second case can be reduced to the first one after the parameter change $u \mapsto -u$). Making an evident v-parameter change, we reduce the metric to the form

$$ds^2 = du^2 + r^2 e^{2\,u/r}\, dv^2.$$

Applying Proposition 3.1 for $f = re^{u/r}$, we get $\lambda = \frac{1}{r}(\omega'\, e^{-u/r} + 1)$. Setting $\Omega_{11} \equiv 0$, we see that $e_1(\lambda) \equiv 0$. Hence $\omega'' = 0$, i.e., $\omega = av + b$. Therefore,

$$\lambda = \frac{1}{r}\left(a\, e^{-u/r} + 1\right).$$

Considering $\Omega_{01} \equiv 0$ (with $s = 1$ because of Or(+)-case), we get

$$-\frac{1}{2r^2} + \frac{\frac{1}{r}e_0(a\, e^{-u/r} + 1)}{1 + \frac{1}{r^2}(e^{-u/r}a + 1)^2} = -\frac{(\frac{1}{r^2} + 1)(ae^{-u/r} + 1)^2 - a^2 e^{-2u/r}}{2\, r^2[1 + \frac{1}{r^2}(ae^{-u/r} + 1)^2]} \not\equiv 0,$$

and hence, this case is not possible.

Case (ii). Choose a subdomain $D' \subset D$ such that $A^2(v) < B^2(v)$ or $A^2(v) > B^2(v)$ over D'. Then the function f may be presented respectively in two forms:

(a) $f(u,v) = \sqrt{B^2 - A^2}\,\sinh(u/r + \theta)$ or
(b) $f(u,v) = \sqrt{A^2 - B^2}\cosh(u/r + \theta)$,

where $\theta(v)$ is some function.

Consider the case (a). After a v-parameter change, the metric in D' takes the form

$$ds^2 = du^2 + r^2 \sinh^2(u/r + \theta)\, dv^2.$$

Applying Proposition 3.1 for $f = r\sinh(u/r + \theta)$, we get

$$\lambda = \frac{\omega'}{r\,\sinh(u/r + \theta)} + \frac{1}{r}\coth(u/r + \theta).$$

Considering $\Omega_{11} \equiv 0$, we have $e_1(\lambda) \equiv 0$ which implies the identity

$$\omega''\,\sinh(u/r + \theta) - \omega'\theta'\,\cosh(u/r + \theta) - \theta' \equiv 0.$$

From this we get $\omega'' = 0$, $\theta' = 0$ and hence $\begin{cases}\theta = \text{const},\\ \omega = av + b\end{cases}$ $(a, b = \text{const})$. After a parameter change we reduce the metric to the form

$$ds^2 = du^2 + r^2 \sinh^2(u/r)\, dv^2.$$

Applying Proposition 3.1 for $f = r\sinh(u/r)$, we get $\lambda = \frac{a + \cosh(u/r)}{r\sinh(u/r)}$. The substitution into Ω_{01} gives

$$-\frac{1}{2}\frac{(\frac{1}{r^2} + 1)[a + \cosh(u/r)]^2 - a^2 + 1}{r^2\sinh^2(u/r) + [a + \cosh(u/r)]^2} \not\equiv 0,$$

which completes the proof for the case (a).

The case (b) consideration gives $\omega = av + b$, $\lambda = \frac{a + \sinh(u/r)}{r\cosh(u/r)}$ and $\Omega_{01} = -\frac{1}{2}\frac{(\frac{1}{r^2}+1)[a+\sinh(u/r)]^2 - a^2 - 1}{r^2\cosh^2(u/r) + [a+\sinh(u/r)]^2} \not\equiv 0$, which completes the proof. □

Proposition 3.3. *Let M^2 be a Riemannian manifold of constant positive curvature $K = r^{-2} > 0$. Then a totally geodesic unit vector field ξ on M^2 exists if $r = 1$ and ξ is parallel along the meridians of M^2 locally isometric to S^2 and moves along the parallels with a unit angle speed. Geometrically, $\xi(M^2)$*

is a part of totally geodesic RP^2 locally isometric to sphere S^2 of radius 2 in $T_1S^2 \overset{isom}{\approx} RP^3$.

PROOF: Suppose ξ is a totally geodesic unit vector field on M^2. The same arguments as in Proposition 3.2 lead to the case $\mu \equiv 0$ at least in some domain $D \subset M^2$. So, choose again a family of e_0-curves and the orthogonal trajectories as a local coordinate net in D. Then the first fundamental form of M^2 can be expressed as $ds^2 = du^2 + f^2\,dv^2$, where $f(u,v)$ is some function. Since M^2 is of constant curvature $K = r^{-2}$, the function f satisfies the equation

$$f_{uu} + \frac{1}{r^2} f = 0.$$

The general solution of this equation $f(u,v) = A(v)\cos(u/r) + B(v)\sin(u/r)$ may be presented in two forms:

(a) $f(u,v) = \sqrt{A^2+B^2}\,\sin(u/r+\theta)$ or

(b) $f(u,v) = \sqrt{A^2+B^2}\cos(u/r+\theta)$,

where $\theta(v)$ is some function.

Consider first, the case (a). After v-parameter change, the metric in D takes the form

$$ds^2 = du^2 + r^2\sin^2(u/r+\theta)\,dv^2.$$

Applying Proposition 3.1 for $f = r\sin(u/r+\theta)$, we get

$$\lambda = \frac{\omega'}{r\,\sin(u/r+\theta)} + \frac{1}{r}\cot(u/r+\theta).$$

Setting $\Omega_{11} \equiv 0$, we find $e_1(\lambda) \equiv 0$ which implies the identity

$$\omega''\,\sin(u/r+\theta) - \omega'\theta'\,\cos(u/r+\theta) + \theta' \equiv 0.$$

From this $\omega'' = 0$, $\theta' = 0$ and we have again $\begin{cases}\theta = \text{const}, \\ \omega = av+b\end{cases}$ $a, b = \text{const}$. After a suitable u-parameter change, we reduce the metric to the form

$$ds^2 = du^2 + r^2\sin^2(u/r)\,dv^2$$

Applying Proposition 3.1 for $f = r\sin(u/r)$, we get $\lambda = \frac{a+\cos(u/r)}{r\,\sin(u/r)}$. Substitution into Ω_{01} gives

$$\frac{1}{2}\frac{(\frac{1}{r^2}-1)[a+\cos(u/r)]^2 + a^2 - 1}{r^2\sin^2(u/r) + [a+\cos(u/r)]^2} \equiv 0,$$

which is possible only if $r = 1$ and $|a| = 1$. So, we obtain to the standard sphere metric

$$ds^2 = du^2 + \sin^2 u\, dv^2$$

and (after the $\pm v + b \to v$ parameter change) the unit vector field

$$\xi = \left\{ \cos v, \frac{\sin v}{\sin u} \right\}.$$

This vector field is parallel along the meridians of S^2 and moves helically along the parallels of S^2 with unit angle speed.

For the case (b) one can find $\omega = av + b$, $\lambda = \frac{a - \sin(u/r)}{r \cos(u/r)}$ and

$$\Omega_{01} = \frac{1}{2} \frac{(\frac{1}{r^2} - 1)[a - \sin(u/r)]^2 + a^2 - 1}{r^2 \cos^2(u/r) + [a - \sin(u/r)]^2} \equiv 0,$$

which gives $r = 1$ and $|a| = 1$ as a result. Thus, we have a metric

$$ds^2 = du^2 + \cos^2 u\, dv^2$$

and a vector field $\xi = \left\{ \cos v, \frac{\sin v}{\cos u} \right\}$. It is easy to see that the results of cases (a) and (b) are geometrically equivalent.

Introduce the local coordinates (u, v, ω) on $T_1 S^2$, where ω is the angle between arbitrary unit vector ξ and the coordinate vector field $X_1 = \{1, 0\}$. The first fundamental form of $T_1 S^2$ with respect to these coordinates is [10]

$$d\tilde{s}^2 = du^2 + dv^2 + 2 \cos u\, dv\, d\omega + d\omega^2.$$

The local parameterization of the submanifold $\xi(S^2)$, generated by the given field, is $\omega = v$ and the induced metric on $\xi(S^2)$ is

$$d\tilde{s}^2 = du^2 + 2(1 + \cos u)\, dv^2 = du^2 + 4 \cos^2 u/2\, dv^2.$$

Thus, $\xi(S^2)$ is locally isometric to sphere S^2 of radius 2. Since $T_1 S^2 \overset{isom}{\approx} RP^3$ and there are no other totally geodesic submanifolds in RP^3 except RP^2, we see that $\xi(S^2)$ is a part of RP^2. So the proof is complete. □

Proposition 3.4. *Let M^2 be a Riemannian manifold of constant zero curvature $K = 0$. Then a totally geodesic unit vector field ξ on M^2 is either parallel or moves along the family of parallel geodesics with constant angle speed. Geometrically,*

$\xi(M^2)$ is either E^2 imbedded isometrically into $E^2 \times S^1$ as a factor or a helical flat submanifold in $E^2 \times S^1$.

PROOF: Suppose ξ is a totally geodesic unit vector field on M^2. Set $\Omega \equiv 0$ in Lemma 3.2. Then $\lambda\mu \equiv 0$. If $\lambda \equiv 0$ over some domain $D \subset M^2$, then ξ is *parallel in this domain.*

Suppose $\lambda \not\equiv 0$ in a domain $D \subset M^2$. Then $\mu \equiv 0$ on at least a subdomain $D' \subset D$. This means that the e_0-curves are geodesics in D' and the field ξ is parallel along them. Choose a family of e_0-curves and the orthogonal trajectories as a local coordinate net in D'. Then the first fundamental form of M^2 takes the form $ds^2 = du^2 + f^2\, dv^2$ and since M^2 is of zero curvature, f satisfies the equation

$$f_{uu} = 0.$$

A general solution of this equation is $f(u,v) = A(v)u + B(v)$. There are two possible cases:

(a) $A(v) \neq 0$ in some subdomain $D'' \subset D'$;

(b) $A(v) \equiv 0$ over the whole domain D'.

Case (a). The function f may be presented over D'' in the form

$$f(u,v) = A(v)(u+\theta),$$

where $\theta(v) = B(v)/A(v)$. After a v-parameter change, the metric in D'' takes the form $ds^2 = du^2 + (u+\theta)^2\, dv^2$. Applying Proposition 3.1 for $f = u + \theta$, we get $\lambda = \frac{\omega'+1}{u+\theta}$. Setting $\Omega_{11} \equiv 0$, we obtain the identity

$$\omega''(u+\theta) - (\omega'+1)\theta' \equiv 0.$$

From this we get $\left\{\begin{smallmatrix} \omega''=0 \\ \omega'=-1 \end{smallmatrix}\right.$ or $\left\{\begin{smallmatrix} \omega''=0 \\ \theta'=0 \end{smallmatrix}\right.$. In the first case, $\lambda = 0$ and the field ξ is parallel again. In the second case $\left\{\begin{smallmatrix} \theta=\text{const}, \\ \omega=av+b \end{smallmatrix}\right.$ $a, b = \text{const}$.

Making a parameter change, we reduce the metric to the form

$$ds^2 = du^2 + u^2\, dv^2.$$

Applying Proposition 3.1 with $f(u,v) = u$, we get $\lambda = \frac{a+1}{u}$. The substitution into Ω_{01} gives the condition

$$-\frac{a+1}{u^2 + (a+1)^2} = 0$$

which is possible only if $a = -1$. But this means that again $\lambda = 0$ and hence ξ is a parallel vector field.

Case (b). After a v-parameter change, the metric takes the form

$$ds^2 = du^2 + dv^2.$$

Applying Proposition 3.1 for $f \equiv 1$, we get $\lambda = \omega'$. Setting $\Omega_{11} \equiv 0$, we find $\omega'' \equiv 0$. This means that $\omega = av + b$ and ξ is either parallel along the u-lines ($a = 0$) or moves along the u-lines helically with constant angle speed.

Let (u, v, ω) be the standard coordinates in $E^2 \times S^1$. Then the first fundamental form of $E^2 \times S^1$ is

$$d\tilde{s}^2 = du^2 + dv^2 + d\omega^2.$$

If $a = 0$, then with respect to these coordinates the local parameterization of $\xi(E^2)$ is $\omega = \text{const}$ and $\xi(E^2)$ is nothing else but E^2 isometrically imbedded into $E^2 \times S^1$. If $a \neq 0$, then the local parameterization of $\xi(E^2)$ is $\omega = av + b$ and the induced metric is

$$d\tilde{s}^2 = du^2 + (1 + a^2)\, dv^2$$

which is flat. The imbedding is helical in the sense that this submanifold meets each flat element of the cylinder $p : E^2 \times S^1 \to S^1$ under constant angle $\varphi = \arccos \frac{1}{\sqrt{1+a^2}}$. So the proof is complete. $\square$

3.2 The curvature

The main goal of this section is to obtain an explicit formula for the Gaussian curvature of $\xi(M^2)$ and apply it to some specific cases. The first step is the following lemma.

Lemma 3.3. *Let ξ be a unit vector field on a 2-dimensional Riemannian manifold of Gaussian curvature K. In terms of Lemma 3.1, the sectional curvature $K_{T_1M}(\xi)$ of T_1M along 2-planes tangent to $\xi(M)$ is given by*

$$K_{T_1M}(\xi) = \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K).$$

Proof: Let $\tilde{\pi}$ be a 2-plane tangent to $\xi(M)$. Then $\tilde{X} = e_0^h$ and $\tilde{Y} = \frac{1}{\sqrt{1+\lambda^2}}(e_1^h + \lambda\eta^v)$ form an orthonormal basis of $\tilde{\pi}$. So we may apply (5) setting $X_1 = e_0$, $X_2 = 0$, $Y_1 = \frac{1}{\sqrt{1+\lambda^2}}e_1$, $Y_2 = \frac{\lambda}{\sqrt{1+\lambda^2}}\eta$.

We get

$$\langle R(X_1, Y_1)Y_1, X_1\rangle = \frac{1}{1+\lambda^2}\langle R(e_0, e_1)e_1, e_0\rangle = \frac{1}{1+\lambda^2}\, K,$$

$$\|R(X_1, Y_1)\xi\|^2 = \frac{1}{1+\lambda^2}\|R(e_0, e_1)\xi\|^2 = \frac{1}{1+\lambda^2}\, K^2,$$

$$\|R(\xi, Y_2)X_1\|^2 = \frac{\lambda^2}{1+\lambda^2}\|R(\xi, \eta)e_0\|^2 = \frac{\lambda^2}{1+\lambda^2}\, K^2,$$

$$\langle(\nabla_{X_1}R)(\xi, Y_2)Y_1, X_1\rangle = \frac{\lambda}{1+\lambda^2}\langle(\nabla_{e_0}R)(\xi, \eta)e_1, e_0\rangle = -(-1)^s\frac{\lambda}{1+\lambda^2}\, e_0(K),$$

where K is the Gaussian curvature of M. Applying directly (5) we obtain

$$\begin{aligned} K_{T_1M}(\xi) &= \frac{1}{1+\lambda^2}\left(K - \frac{3}{4}K^2 + \frac{\lambda^2K^2}{4} + (-1)^{s+1}\lambda e_0(K)\right) \\ &= \frac{1}{1+\lambda^2}\left(K(1-K) + \frac{(1+\lambda^2)K^2}{4} + (-1)^{s+1}\lambda e_0(K)\right) \\ &= \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K). \end{aligned}$$

□

Now we have the following.

Lemma 3.4. *Let ξ be a unit vector field on a 2-dimensional Riemannian manifold M. In terms of Lemma 3.1, the Gaussian curvature K_ξ of the hypersurface $\xi(M) \in T_1M$ is given by*

$$\begin{aligned} K_\xi = \frac{K^2}{4} + \frac{K(1-K)}{1+\lambda^2} + (-1)^{s+1}\frac{\lambda}{1+\lambda^2}e_0(K) \\ + \frac{1}{2}\mu e_1\left(\frac{1}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2, \end{aligned}$$

where K is the Gaussian curvature of M.

Proof: In our case, one can easily reduce the formula (6) to the form

$$K_\xi = K_{T_1M}(\xi) + \det\Omega.$$

Applying Lemma 3.2, we see that

$$\begin{aligned} \det\Omega &= -\mu\frac{\lambda}{\sqrt{1+\lambda^2}}e_1\left(\frac{\lambda}{\sqrt{1+\lambda^2}}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2 \\ &= -\frac{1}{2}\mu e_1\left(\frac{\lambda^2}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2 \\ &= \frac{1}{2}\mu e_1\left(\frac{1}{1+\lambda^2}\right) - \left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2. \end{aligned}$$

Combining this result with Lemma 3.3, we get what was claimed. □

As an application of Lemma 3.4 we prove Theorems 2, 3 and 4.

Proof of Theorem 2: By definition, the extrinsic curvature of a submanifold is the difference between the sectional curvature of the submanifold and the sectional curvature of ambient space along the planes, tangent to the submanifold. In our case, this is $\det\Omega$. If ξ is a geodesic vector field, then we may choose $e_0 = \xi$ and then $\mu = k = 0$. Therefore, for the extrinsic curvature we get

$$-\left((-1)^{s+1}\frac{K}{2} + \frac{e_0(\lambda)}{1+\lambda^2}\right)^2 \le 0.$$

□

Proof of Theorem 3: Since ξ is geodesic, we may set $e_0 = \xi$, $e_1 = \eta$, $s = 1$. Taking into account (7) and (8), we see that $\lambda = -\kappa = -\sigma$. Lemma 3.1(b) gives $-K = -e_0(\sigma) + \sigma^2$. So the result of Lemma 3.4 takes the form

$$\begin{aligned} K_\xi &= \frac{K^2}{4} + \frac{K(1-K)}{1+\sigma^2} - \left(\frac{K}{2} - \frac{e_0(\sigma)}{1+\sigma^2}\right)^2 \\ &= \frac{K^2}{4} + \frac{K(1-K)}{1+\sigma^2} - \left(\frac{K}{2} - \frac{K+\sigma^2}{1+\sigma^2}\right)^2 \\ &= \frac{K(1-K)}{1+\sigma^2} + \frac{K(K+\sigma^2)}{1+\sigma^2} - \left(\frac{K+\sigma^2}{1+\sigma^2}\right)^2 \\ &= K - \left(\frac{K+\sigma^2}{1+\sigma^2}\right)^2. \end{aligned}$$

Suppose that K_ξ is constant. Then the following cases should be considered:

(a) $\sigma = \text{const} \neq 0$. This means that the orthogonal trajectories of the field ξ consist of curves of constant curvature. With respect to this natural coordinate system, the metric of M^2 takes the form $ds^2 = du^2 + f^2\,dv^2$. Set $\sigma = -c$. Then the function f should satisfy the equation

$$\frac{f_u}{f} = c$$

the general solution of which is $f(u,v) = A(v)e^{cu}$. After a v-parameter change we obtain a metric of the form

$$ds^2 = du^2 + e^{2cu}\,dv^2.$$

So, the manifold M^2 is locally isometric to the hyperbolic 2-plane L^2 of curvature $-c^2$ and the field ξ is a geodesic field of (internal or external) normals to the family of horocycles.

(b) $\sigma = 0$. Then evidently ξ is a parallel vector field and therefore the manifold M^2 is locally Euclidean which implies K_ξ=0.

(c) σ is not constant. Then K_ξ is constant if $K = 1$ only. So, M^2 is contained in a standard sphere S^2 and the curvature of $\xi(S^2)$ does not depend on σ. Thus, the field ξ is any (local) geodesic vector field. Evidently, $K_\xi = 0$ for this case. □

Proof of Theorem 4: Consider L^2 with metric $ds^2 = du^2 + e^{2cu}\,dv^2$ and a family of vector fields

$$\xi_\omega = \cos\omega X_1 + \sin\omega X_2 \quad (\omega = \text{const}),$$

where $X_1 = \{1, 0\}$, $X_2 = \{0, e^{-cu}\}$ are the unit vector fields.

Since $\nabla_{X_1}\xi_\omega = 0$, we may set $e_0 = X_1$, $e_1 = X_2$ and therefore we have $\sigma = -c$, $\lambda = c$. Then, setting $K = -c^2$ and $\lambda = c$ in Lemma 3.4, we get

$$K_\xi = -c^2.$$

The extrinsic curvature of $\xi(L^2)$ is also constant since

$$\det\Omega = -\frac{1}{4}c^2.$$

Now fix a point P_∞ at infinity boundary of L^2 and draw a pencil of parallel geodesics from P_∞ through each point of L^2. Define a family of submanifolds $\xi_\omega(L^2)$ for this pencil. Evidently, through each point $(p,\zeta) \in T_1L^2$ there passes only one submanifold of this family. Thus, a family of submanifolds ξ_ω form a hyperfoliation on T_1L^2 of constant intrinsic curvature $-c^2$ and constant extrinsic curvature $-\frac{c^2}{4}$.

Geometrically, $\xi_\omega(L^2)$ is a family of coordinate hypersurfaces $\omega = \text{const}$ in T_1L^2. Indeed, let (u, v, ω) form a natural local coordinate system on T_1L^2. Then the metric of T_1L^2 has the form

$$ds^2 = du^2 + 2\,e^{2cu}dv^2 + 2\,dvd\,\omega + d\,\omega^2.$$

With respect to these coordinates, the coordinate hypersurface $\omega = \text{const}$ is nothing else but $\xi_\omega(L^2)$ and the induced metric is

$$ds^2 = du^2 + 2\,e^{2cu}dv^2.$$

Evidently, its Gaussian curvature is constant and equal to $-c^2$. □

Acknowledgment. The author expresses his thanks to E. Boeckx for valuable remarks and the referee for careful reading and corrections.

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(*Received* August 14, 2001, *revised* February 19, 2002)