Filling boxes densely and disjointly

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Dedicated to my teacher Professor Gerhard Preuss on the occasion of his 62nd birthday

Abstract. We effectively construct in the Hilbert cube $\mathbb{H} = [0,1]^\omega$ two sets $V, W \subset \mathbb{H}$ with the following properties:

(a) $V \cap W = \emptyset$,
(b) $V \cup W$ is discrete-dense, i.e. dense in $[0,1]_D{}^\omega$, where $[0,1]_D$ denotes the unit interval equipped with the discrete topology,
(c) V, W are open in $\mathbb{H}$. In fact, $V = \bigcup_{\mathbb{N}} V_i$, $W = \bigcup_{\mathbb{N}} W_i$, where $V_i = \bigcup_0^{2^{i-1}-1} V_{ij}$, $W_i = \bigcup_0^{2^{i-1}-1} W_{ij}$. V_{ij}, W_{ij} are basic open sets and $(0,0,0,\dots) \in V_{ij}$, $(1,1,1,\dots) \in W_{ij}$,
(d) $V_i \cup W_i$, $i \in \mathbb{N}$ is point symmetric about $(1/2, 1/2, 1/2, \dots)$.

Instead of $[0,1]$ we could have taken any T_4-space or a digital interval, where the resolution (number of points) increases with i.

Keywords: Hilbert cube, discrete-dense, disjoint, disconnected, covering, constructive, computation, digital interval, T_4-space

Classification: Primary 54-04; Secondary 05-04, 54B10

Introduction

This is a paper in computational general topology. It originates in problems of submaximal spaces and the attempt to construct dense connected subspaces of product spaces. Our $V \cup W$ is not connected, despite fulfilling strong conditions. A similar, non-constructive, instance was discovered in [Wat90], using essentially the compactness of $[0,1]$. In order to proceed in a strictly constructive manner, we will develop a language with a simple grammar. Translating words of this language into $\mathbb{H}$ yields V_i and W_i. Since on the one hand we need examples as basis for the induction process and on the other hand our imagination is poorly developed in higher dimensions, the symbolic mathematical software Maple 6.01© was used to create and check higher-dimensional cases, mainly utilizing its set data structure. Pictures were created by means of Maple 6.01© as well.This numeric investigation into set-theoretic topology leads to some, albeit simple, general theorems at the end of this article.

Definition 1. Let $E \subset \mathbb{N}$ be finite and $\mathbb{H} = [0,1]^\omega$.

(a) $p_E : \mathbb{H} \to [0,1]^E$ is the projection of $\mathbb{H}$ onto the finite subproduct $[0,1]^E$ of $\mathbb{H}$. For $p_{\{i\}}$, $i \in \mathbb{N}$, we write p_i.

(b) $A \subseteq \mathbb{H}$ is called discrete-dense, if $p_E[A] = [0,1]^E$ for all finite $E \subset \mathbb{N}$.

(c) Let $A \subseteq \mathbb{H}$. The carrier $c(A)$ of A is defined by $c(A) = \{i \mid i \in \mathbb{N} \wedge p_i[A] \neq [0,1]\}$.

Remark 2.

(a) In other words, $A \subseteq \mathbb{H}$ is discrete-dense, if A covers all finite faces of $\mathbb{H}$ or equivalently A is dense in $[0,1]_D{}^\omega$, where $[0,1]_D$ is the unit interval equipped with the discrete topology.

(b) What is the idea behind the construction of V_i and W_i? We start by defining W_0 as follows: $c(W_0) = \{0\}, p_0[W_0] = \{1\}$. Hence $W_0 = \{1\} \times \prod_{\geq 1}[0,1]$. Similarly $V_0 = \{0\} \times \prod_{\geq 1}[0,1]$ (see Fig. 1). In the following pictures we draw only factors indexed by the carrier. V_0, W_0 do not cover $\mathbb{H}$, nor are they open. This latter problem we will address later. In the next step we have to increase the first factor of W_0, V_0 and shrink the second to keep disjointness:

$$\begin{aligned} W_1 &= [1/2,1] \times \{1\} \times \textstyle\prod_{\geq 2}[0,1] \\ V_1 &= [0,1/2] \times \{0\} \times \textstyle\prod_{\geq 2}[0,1] \end{aligned}$$

(see Figure 2). So, $V_1 \cup W_1$ covers the first coordinate. $V_2 \cup W_2$ is designed to cover the first two coordinates (i.e. the square). We are expanding W_0 and W_1 halfway to the nearest opposite member V_0 and V_1:

$$\begin{aligned} W_2 &= [3/4,1] \times [0,1] \times \{1\} \times \textstyle\prod_{\geq 3}[0,1] \cup [1/4,1] \times [1/2,1] \times \{1\} \times \textstyle\prod_{\geq 3}[0,1] \\ V_2 &= [0,1/4] \times [0,1] \times \{0\} \times \textstyle\prod_{\geq 3}[0,1] \cup [0,3/4] \times [0,1/2] \times \{0\} \times \textstyle\prod_{\geq 3}[0,1] \end{aligned}$$

(see Figure 3, note that W_2 lies in the top face of the cube and V_2 at the bottom). The next step takes place in a cube. We have to expand W_2 going halfway in the direction to V_0, V_1, V_2. At the top level opposite to W_2 there is V_0, V_1. Applying the same procedure as before we arrive at the sets:

$$\begin{aligned} W_3 = {} & [5/8,1] \times [0,1] \times [1/2,1] \times \{1\} \times \textstyle\prod_{\geq 4}[0,1] \;\cup \\ & [1/8,1] \times [1/4,1] \times [1/2,1] \times \{1\} \times \textstyle\prod_{\geq 4}[0,1] \;\cup \\ & [7/8,1] \times [0,1] \times [0,1] \times \{1\} \times \textstyle\prod_{\geq 4}[0,1] \;\cup \\ & [3/8,1] \times [3/4,1] \times [0,1] \times \{1\} \times \textstyle\prod_{\geq 4}[0,1] \;. \end{aligned}$$

V_3 is obtained by applying the symmetry transformation $s(x) := 1 - x$ to the factors, i.e. $s[[a,b]] = [1-b, 1-a]$, e.g. $s[[3/8,1]] = [0,5/8]$. (Compare with Lemma 15.)

(c) The next definition provides the tool to construct W_{ij} and V_{ij}.

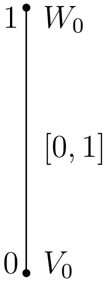

Fig. 1

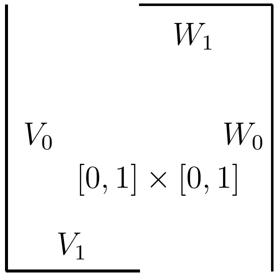

Fig. 2

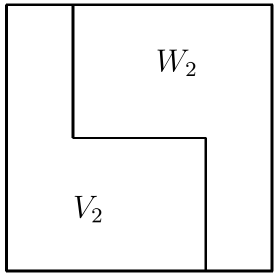

Fig.3

Definition 3. Let the alphabet $\{\downarrow, \uparrow, \varepsilon, \oplus, \ominus\}$ be given. A word in the language L is any finite sequence of uparrows $\uparrow$ and downarrows $\downarrow$ or a single ε, $\oplus$ or $\ominus$.

Definition 4. Let $w \neq \varepsilon, \oplus, \ominus$ be a word in L with length n, $n \in \mathbb{N}$. We are defining the lth derived word, $l \in \mathbb{N}$, of w. If $w = a_1 a_2 a_3 \dots a_n$, then $d^0(w) = w$ and

$$d^l(w) := \begin{cases} a_{l+1} a_{l+2} \dots a_n & \text{if } l < n \text{ and } a_l = a_n, \\ \varepsilon & \text{if } l < n \text{ and } a_l \neq a_n, \\ \oplus & \text{if } l = n \text{ and } a_n = \downarrow, \\ \ominus & \text{if } l = n \text{ and } a_n = \uparrow, \\ \varepsilon & \text{if } l > n. \end{cases}$$

Example 5. Let $w = \downarrow\uparrow\downarrow\uparrow\downarrow = d^0(w)$. Then

$$\begin{cases} d^1(w) = \uparrow\downarrow\uparrow\downarrow, & d^2(w) = \varepsilon, \\ d^3(w) = \uparrow\downarrow, & d^4(w) = \varepsilon, \\ d^5(w) = \oplus, & d^6(w) = d^7(w) = \dots = \varepsilon. \end{cases}$$

Definition 6. Let $0 < x < y < 1$. The meaning of $\uparrow$ and $\downarrow$ is to be a map from $<$ into $[0,1] \times [0,1]$. (The relation $<$ is a subset of $[0,1] \times [0,1]$.) In detail: $\begin{matrix} (x,y)\downarrow = (x, \frac{x+y}{2}) \\ (x,y)\uparrow = (\frac{x+y}{2}, y) \end{matrix}$. Additionally we need two initial symbols: $\begin{matrix} \bullet\downarrow = (0,1/2) \\ \bullet\uparrow = (1/2,1) \end{matrix}$.

Example 7. Let $w = \downarrow\uparrow\downarrow\uparrow\downarrow$. Then $\bullet w = \bullet \downarrow\uparrow\downarrow\uparrow\downarrow = (0, 1/2) \uparrow\downarrow\uparrow\downarrow = (1/4, 1/2) \downarrow\uparrow\downarrow = (1/4, 3/8) \uparrow\downarrow = (5/16, 3/8) \downarrow = (5/16, 11/32)$.

Definition 8. Let $w = a_1 a_2 \dots a_n$ be a word in L and $\bullet w = (x, y)$. Define the closed interval

$$\bullet w \bullet = \begin{cases} [y, 1] & \text{if } a_n = \downarrow, \\ [0, x] & \text{if } a_n = \uparrow, \\ [0, 1] & \text{if } a_n = \varepsilon \text{ (necessarily } n = 1), \\ \{1\} & \text{if } a_n = \oplus \text{ (necessarily } n = 1), \\ \{0\} & \text{if } a_n = \ominus \text{ (necessarily } n = 1). \end{cases}$$

Example 9. Let $w = \downarrow\uparrow\downarrow\uparrow\downarrow$. Then $\bullet w \bullet = [11/32, 1]$.

Definition 10.
(a) Given a binary number $b = b_1 b_2 \dots b_n$ then b_1 is the highest value bit and b_n is the lowest.

(b) Let $w = a_1 a_2 \dots a_n$ be a word in $L \setminus \{\oplus, \ominus, \varepsilon\}$. Define a binary number $b_1 b_2 \dots b_n = b_w$ by

$$b_i := \begin{cases} 1 & \text{if } a_i = \uparrow, \\ 0 & \text{if } a_i = \downarrow. \end{cases}$$

(c) Let $b_1 b_2 \dots b_n = b$ be a binary number. Define a word $a_1 a_2 \dots a_n = w_b$ in L by

$$a_i := \begin{cases} \uparrow & \text{if } b_i = 1, \\ \downarrow & \text{if } b_i = 0. \end{cases}$$

Lemma 11. *Let $v = a_1 a_2 \dots a_m$, $w = a_1 a_2 \dots a_m b_{m+1} \dots b_n$, $n \geq m$ be words in $L \setminus \{\oplus, \ominus, \varepsilon\}$ (w is an extension of v). Let $\bullet v = (r, s)$, $\bullet w = (x, y)$. Then $r \leq x \leq y \leq s$.*

Proof: By Definition 6, r can increase only and s can decrease only. □

Lemma 12. *Let $w = a_1 a_2 \dots a_n$, $w' = a'_1 a'_2 \dots a'_n$ be words in $L \setminus \{\oplus, \ominus, \varepsilon\}$. Assume $\bullet w = (x, y)$, $\bullet w' = (x', y')$. Then*

(a) $[b_w \leq b_{w'} \Leftrightarrow \bullet w \bullet \supseteq \bullet w' \bullet]$ *if* $a_n = a'_n = \downarrow$;
(b) $[b_w \leq b_{w'} \Leftrightarrow \bullet w \bullet \subseteq \bullet w' \bullet]$ *if* $a_n = a'_n = \uparrow$;
(c) *if* $a_1 = \downarrow$ *and* $a'_1 = \uparrow$ *then* $[0, x] \cap [y', 1] = \emptyset$.

Proof: If $\bullet w = (x, y)$, $\bullet w' = (x', y')$ and $a_1 = \downarrow$, $a'_1 = \uparrow$, then $x < y \leq 1/2 \leq x' < y'$. Hence $[0, x] \cap [y', 1] = \emptyset$ and $[y, 1] \supseteq [y', 1]$ and $[0, x] \subseteq [0, x']$. Now let l be the last index where w and w' coincide, $a_1 a_2 \dots a_l = a'_1 a'_2 \dots a'_l$. Then $a_{l+1} = \downarrow$ and $a'_{l+1} = \uparrow$. Let $\bullet a_1 a_2 \dots a_l = (r, s)$. Then $x < y \leq \frac{r+s}{2} \leq x' < y'$. □

Remark 13.
(a) Lemma 12 implies that $\bullet w \bullet$ is uniquely determined by w.
(b) $[0, x] \cap [y', 1] = \emptyset$ remains true, even if w and w' have different length (see Lemma 11) or if $a_1 a_2 \dots a_l = a'_1 a'_2 \dots a'_l$ and $a_{l+1} = \downarrow$ and $a'_{l+1} = \uparrow$.

Definition 14. Let w be a word in L. The 1-complement $-w$ is defined by

$$-w = \begin{Bmatrix} \varepsilon & \text{if } w = \varepsilon \\ \ominus & \text{if } w = \oplus \\ \oplus & \text{if } w = \ominus \\ r(a_1) r(a_2) \dots r(a_n) & \text{if } w = a_1 a_2 \dots a_n \end{Bmatrix}, \text{ where } r(a_i) = \begin{cases} \uparrow \text{ if } a_i = \downarrow, \\ \downarrow \text{ if } a_i = \uparrow. \end{cases}$$

Lemma 15. *Let w be a word in $L \setminus \{\oplus, \ominus, \varepsilon\}$ and $\bullet w = (x, y)$, $\bullet w \bullet = [0, x]$ or $\bullet w \bullet = [y, 1]$. Then $\bullet - w = [1-y, 1-x]$, $\bullet - w \bullet = [1-x, 1]$ or $\bullet - w \bullet = [0, 1-y]$, respectively.*

Proof: It is sufficient to show $\bullet - w = (1-y, 1-x)$. Let $w_n = a_1 a_2 \dots a_n$. We will proceed by induction on n: If $w_1 = \uparrow$, then $\bullet w_1 = (1/2, 1)$, $-w_1 = \downarrow$ and $\bullet - w_1 = (0, 1/2)$. Let $w_{n+1} = w_n \uparrow$ be given and $\bullet w_n = (x, y)$. Hence $\bullet w_n \uparrow = (\frac{x+y}{2}, y)$. By induction hypothesis $\bullet - w_n = (1-y, 1-x)$. Now $-(w_n \uparrow) = (-w_n) \downarrow$ and $\bullet(-w_n) \downarrow = (1-y, 1-x) \downarrow = (1-y, \frac{1-y+1-x}{2}) = (1-y, 1-\frac{x+y}{2})$. The cases $w_1 = \downarrow$, $w_{n+1} = w_n \downarrow$ are alike. □

Lemma 16. *Let w be a word in $L \setminus \{\oplus, \ominus, \varepsilon\}$. Then $\bullet w \downarrow \bullet \cup \bullet w \uparrow \bullet = [0, 1]$.*

Proof: Let $\bullet w = (x, y)$. Then $\bullet w \downarrow \bullet = [\frac{x+y}{2}, 1]$ and $\bullet w \uparrow \bullet = [0, \frac{x+y}{2}]$. □

Definition 17. Let $\mathcal{B}_n = \{00\dots00, 00\dots01, 00\dots10, \dots, 11\dots11\}$ be the set of all n-bit binary numbers. Let $c_j = b_{j1} b_{j2} \dots b_{jn} \in \mathcal{B}_n$, $0 \le j < 2^n$ (so $c_j = j$). Set (see Definition 4 and 10) $\begin{matrix} W_{nj} = \prod_{i=0}^{\infty} \bullet(d^i(w_{c_j}))\bullet \text{ if } j \text{ is even} \\ V_{nj} = \prod_{i=0}^{\infty} \bullet(d^i(w_{c_j}))\bullet \text{ if } j \text{ is odd} \end{matrix}$. Further, set $\begin{matrix} W_n = \bigcup_{j \text{ even}}^{<2^n} W_{nj} \\ V_n = \bigcup_{j \text{ odd}}^{<2^n} V_{nj} \end{matrix}$.

11110	↑↑↑↑↓	ε	ε	ε	ε	⊕
11100	↑↑↑↓↓	ε	ε	ε	↓	⊕
11010	↑↑↓↑↓	ε	ε	↑↓	ε	⊕
11000	↑↑↓↓↓	ε	ε	↓↓	↓	⊕
10110	↑↓↑↑↓	ε	↑↑↓	ε	ε	⊕
10100	↑↓↑↓↓	ε	↑↓↓	ε	↓	⊕
10010	↑↓↓↑↓	ε	↓↑↓	↑↓	ε	⊕
10000	↑↓↓↓↓	ε	↓↓↓	↓↓	↓	⊕
01110	↓↑↑↑↓	↑↑↑↓	ε	ε	ε	⊕
01100	↓↑↑↓↓	↑↑↓↓	ε	ε	↓	⊕
01010	↓↑↓↑↓	↑↓↑↓	ε	↑↓	ε	⊕
01000	↓↑↓↓↓	↑↓↓↓	ε	↓↓	↓	⊕
00110	↓↓↑↑↓	↓↑↑↓	↑↑↓	ε	ε	⊕
00100	↓↓↑↓↓	↓↑↓↓	↑↓↓	ε	↓	⊕
00010	↓↓↓↑↓	↓↓↑↓	↓↑↓	↑↓	ε	⊕
00000	↓↓↓↓↓	↓↓↓↓	↓↓↓	↓↓	↓	⊕
b_w	$d^0(w)$	$d^1(w)$	$d^2(w)$	$d^3(w)$	$d^4(w)$	$d^5(w)$

Fig. 4: dim = 5

Theorem 18. *Let $E = \{0, 1, \dots, n-1\}$. Then $p_E[V_n \cup W_n] = \prod_E[0,1]$ and $W_n \cap V_m = \emptyset$ for all $m \leq n$.*

Proof: We proceed by induction on n and j. We need the following notation: $c1(c0)$ is the binary number c followed by $1(0)$, $1c(0c)$ is the binary number c preceded by $1(0)$. $c_{m/2}$ is the binary number c_m $(= m)$ divided by 2. Let $W_{n+1} = \bigcup_{j \text{ even}}^{<2^{n+1}} W_{(n+1)j}$. $W_1 = [1/2, 1] \times \{1\} \times \prod_{\geq 2}[0,1]$, $V_1 = [0, 1/2] \times \{0\} \times \prod_{\geq 2}[0,1]$ (see Fig. 2) cover the first coordinate and W_1 is disjoint to V_0, V_1. Assume that $W_n \cup V_n$ covers (the product of) the first n coordinates. Take a point $(x_1, x_2, \dots, x_{n+1}) \in \prod_0^n[0,1]$. By symmetry and induction hypothesis we may assume that there is W_{nj} such that $(x_2, \dots, x_{n+1}) \in p_{\{0,1,\dots,n-1\}}[W_{nj}]$ (so j is even). We show now by induction on j that there is $W_{(n+1)k}$ or $V_{(n+1)l}$ with $(x_1, x_2, \dots, x_{n+1}) \in W_{(n+1)k}$ or $(x_1, x_2, \dots, x_{n+1}) \in V_{(n+1)l}$. Let $c_0 = 00\dots0 \in \mathcal{B}_n$. If $(x_2, \dots x_{n+1}) \in \prod_0^{n-1} \bullet(d^i(w_{c_0}))\bullet$ and $x_1 \notin \bullet(w_{0c_0})\bullet$, then $x_1 \in \bullet(w_{c_0 1})\bullet$ by Lemma 16 and $(x_1, x_2, \dots x_{n+1}) \in \prod_0^n \bullet(d^i(w_{c_0 1}))\bullet$ (the reader might wish to follow the line of proof by looking at Fig. 4). Assume we have shown for all $j < m$; j, m even, that $(x_2, \dots, x_{n+1}) \in p_{\{0,1,\dots,n-1\}}[W_{nj}]$ implies $(x_1, x_2, \dots, x_{n+1}) \in p_{\{0,1,\dots,n\}}[W_{(n+1)k} \cup V_{(n+1)l}]$ for some k, l. Let $(x_2, \dots, x_{n+1}) \in p_{\{0,1,\dots,n-1\}}[W_{nm}]$. Take $c_m \in \mathcal{B}_n$, hence $p_{\{0,1,\dots,n-1\}}[W_{nm}] = \prod_0^{n-1} \bullet(d^i(w_{c_m}))\bullet$. If $x_1 \in \bullet(w_{0c_m})\bullet$ we are finished, because then $(x_1, x_2, \dots, x_{n+1}) \in \bullet(w_{0c_m}) \bullet \times p_{\{0,1,\dots,n-1\}}[W_{nm}] = p_{\{0,1,\dots,n\}}[W_{(n+1)m}] = \prod_0^n \bullet(d^i(w_{0c_m}))\bullet$. If $x_1 \notin \bullet(w_{0c_m})\bullet$, then $x_1 \in \bullet(w_{0c_{m/2}1})\bullet$. Note $c_m = b_1 b_2 \dots b_n$, $b_i \in \{0,1\}$, $b_n = 0$. $\bullet(d^i(w_{c_m}))\bullet$ is either a proper subset of $[0,1]$ or equal to $[0,1]$. Since $x_2 \in \bullet(d^0(w_{c_m}))\bullet = \bullet(w_{c_m})\bullet$ and Lemma 12 we have $x_2 \in \bullet(w_{c_j})\bullet \supseteq \bullet(w_{c_m})\bullet$ for all $c_j \leq c_m$, $c_j \in \mathcal{B}_n$ even. The idea is to construct a set $V_{(n+1)l}$ with $(x_1, x_2, \dots, x_{n+1}) \in p_{\{0,1,\dots,n\}}[V_{(n+1)l}]$ assuming that for all even $j < m$ we have $(x_2, \dots x_n) \notin p_{\{0,1,\dots,n-1\}}[W_{nj}]$. Let $q : \{1, 2, \dots, n\} \times \mathcal{B}_n \to \{0,1\}$ be the function which picks the i-th digit in c_m. (e.g. $t = 10$ renders $q(1,t) = 1$, $q(2,t) = 0$) If $\bullet d^i(w_{c_m})\bullet = [0,1]$ we know $q(i, c_m) = 1$ by Definition 4. Let c_{t_i} differ from c_m in exactly the i-th digit, where $i \in \{u \mid 1 \leq u \leq n \wedge q(u, c_m) = 1\}$. Of course $c_{t_i} < c_m$ and $\bullet d^i(w_{c_{t_i}})\bullet = \bullet w_{b_{i+1}\dots b_n}\bullet$, where $c_{t_i} = b_1 b_2 \dots b_{i-1} 0 b_{i+1} \dots b_n$. Now $(x_2, \dots, x_{n+1}) \notin p_{\{0,1,\dots n-1\}}[W_{nt_i}]$ and since c_{t_i}, c_m differ in one digit only it implies $x_{i+2} \notin \bullet w_{b_{i+1}\dots b_n}\bullet$, hence $x_{i+2} \in \bullet w_{b_{i+1}\dots b_{n-1}1}\bullet = \bullet d^i(w_{c_{m/2}1})\bullet$. Hence $(x_1, x_2, \dots, x_{n+1}) \in \prod_0^n \bullet d^i(w_{0c_{m/2}1})\bullet = V_{(n+1)0c_{m/2}1}$. We are now turning to the quest for disjointness. Assume $W_{nl} \cap V_{mk} = \emptyset$ for all $m, n < t$; $0 \leq l < 2^n$, l even; $0 \leq k < 2^n$, k odd.

1. Then $W_{tl} \cap V_{tk} = \emptyset$, because $p_t[W_{tl}] = \{1\}$ and $p_t[V_{tk}] = \{0\}$.
2. By symmetry we may limit ourselves to the case W_{tl}, V_{mk}.
3. If c_l starts with a 0 and c_k starts with a 1 we are finished, because after deleting the first coordinate disjointness follows from the induction

hypothesis.

4. If c_l starts with a 1 and c_k starts with a 0 we may apply Remark 13 to get disjointness in the first coordinate. Therefore c_l, c_k both commence with 0 or 1.
   (a) c_l, c_k coincide for the length of c_k. Then $\bullet w_{c_k} = (x, y)$, $\bullet w_{c_k}\bullet = [0, x]$ and $\bullet w_{c_l} = (x', y')$, where $x \le x' < y'$. $\bullet w_{c_l}\bullet = [y', 1]$ is disjoint from $[0, x]$. (We only need the first coordinate of W_{tl}, V_{mk}.)
   (b) Let c_l, c_k coincide below position i and let $q(i, c_l) = 0$, $q(i, c_k) = 1$. Then disjointness follows from the induction hypothesis, because the next derived word does not translate into $[0, 1]$.
   (c) Let c_l, c_k coincide below position i and let $q(i, c_l) = 1$, $q(i, c_k) = 0$. In this case we may not apply the induction hypothesis, because $p_i[V_{mk}] = p_i[W_{tl}] = [0, 1]$, but we can apply again Remark 13(b) to get disjointness in the first coordinate.

□

Remark 19.

(a) We succeeded in filling the Hilbert space $\mathbb{H}$ densely and disjointly. But our sets W_n, V_n are closed. How can we achieve openess? The distance of W_n and V_m, $m < n$ in the hypercube $[0, 1]^n$ is at least 2^{-n}. We choose a positive $\epsilon < \frac{1}{2}$ and replace all intervals $[y, 1]$ appearing in W_n by $(y - \epsilon 2^{-n}, 1]$. A symmetric change is applied to V_n: $[0, x]$ is replaced by $[0, x + \epsilon 2^{-n})$. The remaining problem are the sets $\{1\}$ and $\{0\}$ which force W_n to be disjoint from V_n. We choose a small $\delta > 0$ and replace $\{1\}$ by $(1, 1 + \delta]$ and $\{0\}$ by $[-\delta, 0)$. As a consequence our construction takes place in the space $[-\delta, 1 + \delta]^\omega$ using intervals $(y - \epsilon 2^{-n}, 1 + \delta]$ and $[-\delta, x + \epsilon 2^{-n})$, which, of course, does no harm.

(b) Fig. 5 and Fig. 6 give an indication how the sets W_{nj}, $j < 2^{n-1}$ look in the 8-dimensional hypercube (we skip odd indices j). They are to be understood in the following way: Each picture consists of 128 slices each consisting of 8 factors. The factors represent the length of the closed interval $[y, 1]$. The cartesian product of the 8 factors in one slice yields one set W_{8j}.

(c) Fig. 5 and Fig. 6 were created by the following Maple 6.01© session:

```
> restart;
> N:=8;
N := 8
> h:=proc(r,t)
> x:=0;
> y:=1;
> if t>1 and r[nops(r)-t+2]=1 then x:=0 else
> for s from nops(r)-t+1 by -1 to 1 do
> if r[s]=0 then y:=(x+y)/2 else x:=(x+y)/2 fi:
> od;
```

```
> fi;
> end;
>
Warning,`x` is implicitly declared local to procedure `h`
Warning,`y` is implicitly declared local to procedure `h`
Warning,`s` is implicitly declared local to procedure `h`
h := proc(r, t)
local x, y, s;
x := 0;
y := 1;
if 1 < t and r[nops(r) - t + 2] = 1 then x := 0
else for s from nops(r) - t + 1 by -1 to 1 do
if r[s] = 0 then y := 1/2*x + 1/2*y
else x := 1/2*x + 1/2*y
end if
end do
end if
end proc
>
> a:= array(0..2^(N-1)-1,1..N);
a := array(0 .. 127, 1 .. 8, [])
> for i from 0 by 2 to 2^N - 1 do
> if i<2^(N-1) then z:=i+2^(N-1):
> c:=convert(z,base,2):
> c[nops(c)]:=0:
> else
> c:=convert(i,base,2):
> fi:
> for j from 1 by 1 to N do
> a[i/2,j]:= h(c,j);
> od:
> od:
> b:=map(x->1-x,a):
> M:=convert(b,matrix):
> plots[matrixplot](M,heights=histogram,orientation=
[-62,35],axes=framed,color=white);
> plots[matrixplot](M,heights=histogram,orientation=
[105,35],axes=framed,color=white);
```

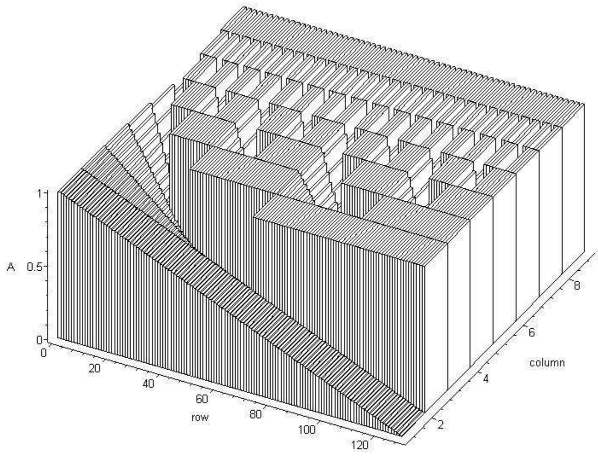

Fig. 5: Front view, see Remark 19(b)

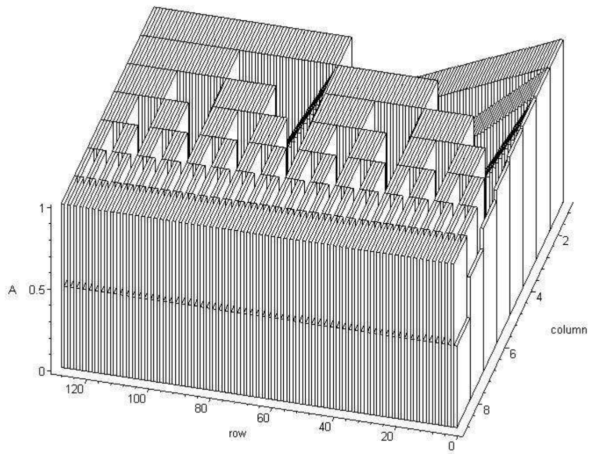

Fig. 6: Rear view, see Remark 19(b)

Remark 20.
(a) Are there more general spaces X than $[0,1]$ on which our algorithm can run? The basic step takes two open sets O_0, O_1 with disjoint closures and selects two open sets $O_{1/4}$, $O_{3/4}$ satisfying $O_0 \subset O_{1/4}$, $O_1 \subset O_{3/4}$ and $O_{1/4} \cup O_{3/4} = X$, $cl(O_0) \cap cl(O_{3/4}) = \emptyset$, $cl(O_1) \cap cl(O_{1/4}) = \emptyset$. Such constructions can be carried out in any T_4-space. In fact, we have the stronger Lemma 22. Recall that a space is

called functionally T_2 if its topology is finer than a completely regular T_1 topology.

(b) The other line of generalization looks at the information we need to pursue the construction. At least we need to have the defining end points of all intervals. For the first step the space $I_2 = \{0, 1, 2, 3, 4\}$ suffices with open points $\{0\}$, $\{2\}$, $\{4\}$ and closed points $\{1\}$, $\{3\}$. The next iteration already needs $I_3 = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ where even numbers are open and odd numbers closed. These digital intervals I_n with increasing resolution can be used to verify Theorem 18 on a computer up to a fixed dimension n.

(c) Digital intervals are Alexandroff spaces (each point has a minimal open neighborhood). The next Lemma 21 reconciles Remark 20(b) with [Wat90], who states that discrete-dense subspaces of products of connected Alexandroff spaces are connected.

Lemma 21. *Let $(X, \mathcal{X})$ be a connected Alexandroff space and $(O_i)_\mathbb{N}$ be an increasing sequence of non-empty open sets such that $cl(O_i) \subseteq O_{i+1}$. Then $\bigcup_\mathbb{N} O_i = X$.*

Proof: $\bigcup_\mathbb{N} O_i = \bigcup_\mathbb{N} cl(O_i)$ is closed and open. □

Lemma 22. *Let $(X, \mathcal{X})$ be a functionally T_2 space. Then X^ω can be filled densely and disjointly as $\mathbb{H}$.*

Proof: Lemma 22 is true (even trivial) if X is disconnected. Let X be connected. Take two points $a, b \in X$ and a continuous map $f : X \to [0, 1]$ with $f(a) = 0$ and $f(b) = 1$. f is surjective. Define $A(i, c_j) =: f^{-1}[\bullet(d^i(w_{c_j}))\bullet]$ if j is odd and $B(i, c_j) =: f^{-1}[\bullet(d^i(w_{c_j}))\bullet]$ if j is even (see Definition 17). □

Note added in proof: After my talk at the Free University of Berlin Vladimir Kadets communicated the following elegant method to show the existence of disjoint, discrete-dense open sets: Define $\phi : \mathbb{H} \to [0, 1]$ by $\phi(x) := \sum_1^\infty \frac{x_i}{2^i}$ for $x = (x_i) \in \mathbb{H} = [0, 1]^\omega$. Then $\phi^{-1}[[0, 1/2)]$ and $\phi^{-1}[(1/2, 1]]$ are as required. How do they look? His, St. Watson's [Wat90] and my sets are different.

References

[Sch98] Schröder J., *On sub-, pseudo- and quasimaximal spaces*, Comment. Math. Univ. Carolinae **39.1** (1998), 198–206.

[Wat90] Watson St., *Powers of the Sierpinski space*, Topology Appl. **35** (1990), 299–302.

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(*Received* November 22, 2001, *revised* July 1, 2002)