Weighted norm inequalities for singular integral operators satisfying a variant of Hörmander's condition

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Abstract. In this paper we establish weighted norm inequalities for singular integral operators with kernel satisfying a variant of the classical Hörmander's condition.

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1. Introduction

In the classical Calderón-Zygmund theory, the Hörmander's condition

$$\int_{|x|>2|y|} |K(x-y) - K(x)| \, dx \leq C \tag{1.1}$$

plays a fundamental role and became the weakest restriction on the kernel in order to develop all the theory. Hörmander's condition was introduced in [7] and relaxed the original Dini property given in the work of Calderón and Zygmund ([3]). On the other hand, there has been also a great interest in operators which are not in the scope of the Calderón-Zygmund theory. The particular situation of singular integral operators which do not satisfy the Hörmander's condition has been extensively considered (say, among others, oscillatory and rough singular integral operators).

In [6], D.J. Grubb and C.N. Moore introduced a variant of the Hörmander's condition in order to study the L^p-boundedness of certain singular integral operators. In particular, these authors considered convolution operators bounded in $L^2(\mathbb{R}^n)$ with kernel K satisfying the so called variant of Hörmander's condition

$$\int_{|x|>2|y|} \left| K(x-y) - \sum_{j=1}^{m} B_j(x)\phi_j(y) \right| dx \leq C \tag{1.2}$$

with B_j and ϕ_j's appropriate functions (see Theorem 3.1). As an example we mention the kernel $K(x) = \sin x/x$, which verifies (1.2) (see Example 3.3) but

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it is not a Calderón-Zygmund kernel since its derivative does not decay quickly enough at infinity. Condition (1.2) makes it possible to develop a study similar to the classical for Calderón-Zygmund singular integral operators.

As it is well known, the classical Hörmander's condition (1.1) is too weak to get weighted inequalities by any known method. The usual hypothesis on the kernel K to obtain them is the Lipschitz condition

$$|K(x-y)-K(x)| \leq C\,\frac{|y|^{\alpha}}{|x|^{\alpha+n}}, \quad |x|>c|y|. \tag{1.3}$$

Weaker conditions than (1.3), but stronger than (1.1), have been also considered in [10] or [12].

The goal of this paper is the study of the weighted norm inequalities for operators satisfying an appropriate version of (1.3) in the scope of (1.2). Thus, we proceed with the same philosophy as Gruub and Moore ([6]) trying to develop the classical scheme in this setting (cf. [5, Chapter IV-Sect. 3]). We remark that the key of the arguments is the definition of an appropriate #-maximal operator.

The paper is organized as follows. In Section 2 we introduce the basic tool, a #-maximal type operator. We study its main properties and we establish the version of the classical Fefferman-Stein's weighted inequality with the Hardy-Littlewood maximal operator (Theorem 2.9). Section 3 is devoted to the proof of the main results on the weighted inequalities of the singular integral operators. Having introduced the class of singular integral operators of our interest, we first analyze the action on them of the #-maximal operator introduced, determinating the maximal operator that controls it (Theorem 3.6). Finally, we establish the L^p and weak-(1,1) weighted estimates for these operators (Theorem 3.7 and Theorem 3.8).

Throughout this paper, we denote by C a constant, not necessarily the same at each occurrence, which depends only on the parameters indicated.

2. The #-maximal type operator

We begin by introducing the definition of the #-maximal type operator associated to a family of bounded functions.

Definition 2.1. *Let $\Phi=\{\phi_1,\dots,\phi_m\}$ be a finite family of bounded functions in $\mathbb{R}^n$. For any $f\in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, we define the Φ-# maximal function $M^{\#}_{\Phi}f$ by*

$$M^{\#}_{\Phi}f(x)=\sup_{Q\ni x}\inf_{\{c_1,\dots,c_m\}}\left(\frac{1}{|Q|}\int_Q |f(y)-\sum_{j=1}^{m}c_j\phi_j(-(y-x_Q))|\,dy\right),$$

where the infimum is taken over all m-tuples $\{c_1,\dots,c_m\}$ of complex numbers, and the supremum is taken over all cubes Q with sides parallel to the coordinate axes that contain the point x and center denoted by x_Q.

Remark 2.2. We note that in the particular case of $m = 1$ and $\phi_1 \equiv 1$, $M_\Phi^\#$ is the classical sharp maximal operator $M^\#$ (see [5], [11] for details about this operator). So, in some sense, $M_\Phi^\#$ can be understood as a generalization of this well-known operator.

The basic properties of this operator needed for our study will require an additional property of the family Φ, say, a *reverse Hölder* condition which we first introduce in the general sense.

Definition 2.3. *Given a positive and locally integrable function g in $\mathbb{R}^n$, we say that g satisfies the reverse Hölder RH_∞ condition, in short, $g \in RH_\infty(\mathbb{R}^n)$, if for any cube Q centered at the origin we have*

$$0 < \sup_{x \in Q} g(x) \leq C \frac{1}{|Q|} \int_Q g(x)\, dx$$

with $C > 0$ being an absolute constant independent of Q.

Remark 2.4. It will be useful for later on to notice that if $g \in RH_\infty(\mathbb{R}^n)$ then also $g(-x) \in RH_\infty(\mathbb{R}^n)$.

The condition RH_∞ implies the well known Reverse Hölder condition $RH_{1+\varepsilon}$

$$\left(\frac{1}{|Q|} \int_Q g(x)^{1+\varepsilon}\, dx \right)^{\frac{1}{1+\varepsilon}} \leq C \left(\frac{1}{|Q|} \int_Q g(x)\, dx \right) \tag{2.4}$$

for all $\varepsilon > 0$. The Reverse Hölder condition $RH_{1+\varepsilon}$ characterizes the A_p classes of weights that we introduce next (cf. [5, p. 403]). We say that a positive and locally integrable function w belongs to A_p, $1 \leq p < \infty$, if there exists a constant C such that

$$\left(\frac{1}{|Q|} \int_Q w(y)\, dy \right) \left(\frac{1}{|Q|} \int_Q w(y)^{1-p'}\, dy \right)^{p-1} \leq C, \quad (1 < p < \infty)$$

$$\frac{1}{|Q|} \int_Q w(y)\, dy \leq C \inf_Q w, \quad (p = 1).$$

for any cube Q with sides parallel to the coordinate axes.

It follows that the A_p classes are increasing with respect to p and the widest one is defined as $A_\infty = \bigcup_{p \geq 1} A_p$. Moreover, there exists another characterization of the elements of A_∞ that will be useful later on. If $w \in A_\infty$ then there exist positive constants c and r such that, for any cube Q and any measurable set E contained in Q, denoting $w(A) = \int_A w$ for any subset $A \subset \mathbb{R}^n$, we have

$$\frac{w(E)}{w(Q)} \leq c \left(\frac{|E|}{|Q|} \right)^r. \tag{2.5}$$

For the basic theory of A_p weights, we refer the reader to the classical references [5] or [11].

Our interest will be concentrated in the projection of any function on the subspace generated by Φ. This projection, under a certain RH_∞-conditions on the family Φ, will be the optimal linear combination for the estimation of $M_\Phi^{\#}$. We detail all these comments.

By the projection of an L^1-function f onto a finite-dimensional subspace Y we refer to such an element, if it exists, $P(f)$ of Y verifying

$$\int f\bar{h}\,dx = \int P(f)\bar{h}\,dx$$

for every $h \in Y$. The uniqueness of this projection is immediate from its existence.

The most simple example is given for the subspace Y generated by that one functions constant on a fixed cube Q. Then the projection of any function f integrable on Q always exists and is given by its average f_Q on Q, which trivially satisfies $|f_Q| \le |f|_Q$. For the subspace generated by a finite family Φ of bounded functions, the existence of this projection and this type of estimate also holds under an appropriate RH_∞-condition on the family Φ. This result is stated in the following lemma, which became the key for the proof of the main result of [6].

Lemma 2.5. *Let $\Phi = \{\phi_1, \dots, \phi_m\}$ be a finite family of bounded functions in $\mathbb{R}^n$ satisfying that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{mn})$. Then, for any cube Q centered at the origin and any $f \in L^1(Q)$, there exists the projection $P_Q f$ of f onto the subspace of $L^1(Q)$ generated by the family Φ and satisfies*

$$\sup_{y\in Q} |P_Q f(y)| \le C_0 \frac{1}{|Q|} \int_Q |f(y)|\,dy, \tag{2.6}$$

where the constant C_0 depends only on n, m and the constant in the RH_∞-condition satisfied by the family Φ.

We specially remark that the RH_∞ condition imposed to the family Φ is the essential hypothesis to give both the existence of the projection and the boundedness (2.6).

Remark 2.6. Lemma 2.5 means that the projection operator $P_Q : L^1(Q) \to Y$ is bounded with norm $\|P_Q\|_{L^1(Q)\to L^1(Q)} \le C_0$.

Lemma 2.5 allows us to establish a close relationship between the operator projection and the operator $M_\Phi^{\#}$.

Lemma 2.7. *Let $\Phi = \{\phi_1, \dots, \phi_m\}$ be a finite family of bounded functions in $\mathbb{R}^n$ satisfying that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{nm})$. For any cube Q in $\mathbb{R}^n$ with center*

x_Q and any $f \in L^1(Q)$, let $P_Q f$ be the projection of f onto the subspace Y of $L^1(Q)$ generated by $\{\phi_1(-(\cdot - x_Q)), \dots, \phi_m(-(\cdot - x_Q))\}$.

Then

$$\frac{1}{|Q|}\int_Q |f(x) - P_Q f(x)|\,dx \le C \inf_{g\in Y} \frac{1}{|Q|}\int_Q |f(x) - g(x)|\,dx, \tag{2.7}$$

with $C > 0$ being an absolute constant depending only on the RH_∞-condition of the family Φ.

Proof: The main idea of this result follows from the concept of near-best L^1-approximation (cf. [9]). Since $P_Q g = g$ for any $g \in Y$, if I denotes the identity operator, we have

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f(x) - P_Q f(x)|\,dx &= \frac{1}{|Q|}\int_Q |f(x) - g(x) - P_Q(f-g)(x)|\,dx \\ &= \frac{1}{|Q|}\int_Q |[I - P_Q](f-g)(x)|\,dx \\ &\le \|I - P_Q\|_{L^1(Q)\to L^1(Q)} \frac{1}{|Q|}\int_Q |f(x) - g(x)|\,dx \\ &\le \Big(1 + \|P_Q\|_{L^1(Q)\to L^1(Q)}\Big) \frac{1}{|Q|}\int_Q |f(x) - g(x)|\,dx.\end{aligned}$$

Now, from Remark 2.4, Lemma 2.5 and Remark 2.6 it follows (2.7) and the lemma is proved. □

Remark 2.8. Lemma 2.7 implies that

$$\sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y) - P_Q f(y)|\,dy \sim M_\Phi^\# f(x),$$

where, as usual, the supremum is taken over all cubes Q with sides parallel to the coordinate axes that contain the point x.

Our first weighted estimate relates the operator $M_\Phi^\#$ and the non-centered maximal operator of Hardy-Littlewood

$$Mf(x) = \sup_{Q\ni x} \frac{1}{|Q|}\int_Q |f(y)|\,dy,$$

where the supremum is taken over all cubes Q with sides parallel to the coordinate axes that contain the point x.

Theorem 2.9. *Let $1 < p < \infty$, $w \in A_\infty$ and $\Phi = \{\phi_1, \dots, \phi_m\}$ be a finite family of bounded functions satisfying $|\det[\phi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{nm})$. Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}^n} Mf(x)^p\, w(x)\, dx \le C \int_{\mathbb{R}^n} M_\Phi^\# f(x)^p\, w(x)\, dx, \tag{2.8}$$

for every smooth function f such that the left hand side is finite.

This result became the version for these operators of the classical one due to C. Fefferman and E. Stein (see [8]). Its proof reduces to show that the RH_∞-condition on the family Φ is sufficient to establish a good-λ inequality between these two operators. This technique was introduced by Burkholder and Gundy in [2], and it was first used by Coifman and Fefferman in [4] for the study of the boundedness of Calderón-Zygmund singular integral operators. The use of this tool to relate the classical #-maximal and Hardy-Littlewood maximal operators was first considered by Bagby and Kurtz in [1].

Proof of Theorem 2.9: Following [11, Chapter XIII], we have to show that the operators M and $M_\Phi^\#$ verify a good-λ inequality with respect to the measure $w(x)\, dx$. For this purpose it is enough to check that the following conditions hold:

(i) M and $M_\Phi^\#$ are sublinear and positive,
(ii) $\{x \in \mathbb{R}^n : Mf(x) > \lambda\}$ is an open set of finite Lebesgue measure for each f in $C_0^\infty(\mathbb{R}^n)$ and each $\lambda > 0$,
(iii) if a cube Q contains a point x_0 where $Mf(x_0) \le \lambda$, then for each $0 < \eta < 1$ there exists a constant $\gamma > 0$ independent of λ, Q and f such that

$$w(\{y \in Q : Mf(y) > (C_0 + 1)\lambda, M_\Phi^\# f(y) \le \gamma\lambda\}) \le \eta w(Q), \tag{2.9}$$

where $C_0 > 0$ is the constant given in (2.6).

Conditions (i) and (ii) are readily seen from the definition of the operators M and $M_\Phi^\#$ and the basic properties of the Hardy-Littlewood maximal operator, respectively. So, it only remains to show (iii).

By simplicity, let us denote

$$E = \{y \in Q : Mf(y) > (C_0 + 1)\lambda, M_\Phi^\# f(y) \le \gamma\lambda\}.$$

First note that, since $w \in A_\infty$, there exist constants $c, r > 0$ such that

$$\frac{w(E)}{w(Q)} \le c \left(\frac{|E|}{|Q|}\right)^r,$$

and this reduces the problem to prove (2.9) with the Lebesgue measure instead of the measure $w(x)\, dx$.

Fix $f \in C_0^\infty(\mathbb{R}^n)$ and a cube Q that contains a point x_0 such that $Mf(x_0) \leq \lambda$. Then, by the non-centered definition of the operator M, the cube $\widetilde{Q}$ concentric with Q and side length two times that of Q satisfies

$$\frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} |f(y)|\,dy \leq \lambda. \tag{2.10}$$

We claim that, for all $x \in E$,

$$M(f\chi_{\widetilde{Q}})(x) > (C_0+1)\lambda. \tag{2.11}$$

Indeed, since $Mf(x) > (C_0+1)\lambda$, any cube that contains x and where the average of $|f|$ is bigger than $(C_0+1)\lambda$ it cannot contain x_0, so its diameter is less than the diameter of Q, and consequently it is contained in $\widetilde{Q}$.

Now, since $P_{\widetilde{Q}}f$ is the projection of f onto the subspace of $L^1(\widetilde{Q})$ generated by $\{\phi_1(-(\cdot - x_Q)), \ldots, \phi_m(-(\cdot - x_Q))\}$, from (2.11) it follows that

$$M((f - P_{\widetilde{Q}}f)\chi_{\widetilde{Q}})(x) > \lambda. \tag{2.12}$$

To see this, first recall that, for any cube R, by (2.6) and (2.10), we have

$$\begin{aligned}\frac{1}{|R|}\int_{\widetilde{Q}\cap R} |P_{\widetilde{Q}}f(y)|\,dy &\leq C_0\frac{|\widetilde{Q}\cap R|}{|R|}\Big(\frac{1}{|\widetilde{Q}|}\int_{\widetilde{Q}} |f(y)|\,dy\Big)\\ &\leq C_0\lambda\end{aligned} \tag{2.13}$$

with $C > 0$ independent of $\widetilde{Q}$, R, f and λ. On the other hand, by (2.11), we can choose a cube R such that

$$\frac{1}{|R|}\int_{\widetilde{Q}\cap R} |f(y)|\,dy > (C_0+1)\lambda. \tag{2.14}$$

From (2.13) and (2.14) we conclude

$$\begin{aligned}\frac{1}{|R|}\int_{\widetilde{Q}\cap R} |f(y) - P_{\widetilde{Q}}f(y)|\,dy &\geq \left|\frac{1}{|R|}\int_{\widetilde{Q}\cap R} |f(y)|\,dy - \frac{1}{|R|}\int_{\widetilde{Q}\cap R} |P_{\widetilde{Q}}f(y)|\,dy\right|\\ &> |(C_0+1)\lambda - C_0\lambda|\\ &= \lambda,\end{aligned}$$

which proves (2.12).

Finally, we have

$$\begin{aligned}|E| &= |\{y \in Q : Mf(y) > (C+1)\lambda, M^{\#}_{\Phi} f(y) \le \gamma\lambda\}| \\ &\le |\{y \in Q : M((f - P_{\widetilde{Q}} f)\chi_{\widetilde{Q}})(y) > \lambda, M^{\#}_{\Phi} f(y) \le \gamma\lambda\}| \\ &\le \frac{C}{\lambda} \int_{\widetilde{Q}} |f(y) - P_{\widetilde{Q}} f(y)|\, dy \\ &\le \frac{C}{\lambda} |\widetilde{Q}| M^{\#}_{\Phi} f(y) \\ &\le \frac{C}{\lambda} |Q| \gamma\lambda \\ &= C\gamma |Q|,\end{aligned}$$

where the first inequality follows from (2.12), the second one holds because M is of weak type-(1,1) and the third one by Remark 2.8 for any $y \in E$.

Thus, (iii) is verified by taking $\gamma = \eta/C$ and the proof of the theorem is finished. □

3. Weighted norm inequalities

In this section we establish the main results of the paper. We begin by giving the main theorem of [6] on the L^p-boundedness of the singular integral operators that satisfy the variant of the Hörmander's condition (1.2).

Theorem 3.1 [6]. *Let $K \in L^2(\mathbb{R}^n)$ satisfy*

(i) $\|\hat{K}\|_\infty \le C$,

(ii) *there exist functions $B_1, \dots, B_m$ and $\Phi = \{\phi_1, \dots, \phi_m\} \subset L^\infty(\mathbb{R}^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{nm})$, and*

(iii) *for all $|y| > 0$,*

$$\int_{|x|>2|y|} |K(x-y) - \sum_{j=1}^{m} B_j(x)\phi_j(y)|\, dx \le C.$$

For $f \in C_0^\infty(\mathbb{R}^n)$, $1 < p < \infty$, we define the singular integral operator

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy.$$

Then

$$\|Tf\|_p \le C\|f\|_p$$

with a constant C depending only on p, the dimension of the space and the constant in the RH_∞-condition for the ϕ_j's.

Example 3.2. The simplest example is given by $K(x) \in L^2(\mathbb{R}^n)$ being a Calderón-Zygmund kernel. Then, with $m = 1$, $B_1(x) = K(x)$ and $\phi_1 \equiv 1$, Theorem 3.1 is the Hörmander's version of the Calderón-Zygmund theorem ([7]).

Example 3.3. Consider the kernel

$$K(x) = \frac{1}{x} \sum_{j=1}^{m} c_j e^{i\lambda_j x}$$

with $\lambda_j \in \mathbb{R}$ for all j and $\sum_{j=1}^{m} c_j = 0$. This kernel satisfies that $\widehat{K}$ is a step function, so it includes the particular case $K(x) = \sin x/x$ mentioned in the Introduction. It verifies (iii) in Theorem 3.1 with $B_j(x) = c_j e^{i\lambda_j x}/x$ and $\phi_j(y) = e^{-i\lambda_j y}$. In [6] is proved that $|\det[\phi_j(y_i)]|^2$ satisfies the RH_∞-condition in $\mathbb{R}^m$.

As we pointed out in the Introduction, we will require a pointwise version of condition (iii) to be satisfied by the kernels of the operators considered. We precise it.

Let $K \in L^2(\mathbb{R}^n)$ verify for certain constant $C > 0$

(K1) $\|\widehat{K}\|_\infty \le C$,
(K2) $|K(x)| \le \frac{C}{|x|^n}$,
(K3) there exist functions $B_1, \dots, B_m \in L^1_{\text{loc}}(\mathbb{R}^n \setminus \{0\})$ and $\Phi = \{\phi_1, \dots, \phi_m\} \subset L^\infty(\mathbb{R}^n)$ such that $|\det[\phi_j(y_i)]|^2 \in RH_\infty(\mathbb{R}^{nm})$, and
(K4) for a fixed $\gamma > 0$ and for any $|x| > 2|y| > 0$,

$$\left| K(x-y) - \sum_{j=1}^{m} B_j(x)\phi_j(y) \right| \le C \frac{|y|^\gamma}{|x-y|^{n+\gamma}}.$$

For $f \in C_0^\infty(\mathbb{R}^n)$, we define the convolution operator associated to the kernel K by

$$Tf(x) = \int_{\mathbb{R}^n} K(x-y) f(y)\, dy. \tag{3.1}$$

Remark 3.4. It is immediate that any operator satisfying (K4) also verifies (iii) in Theorem 3.1.

Remark 3.5. The family of kernels given in Example 3.3 also satisfies (K1)–(K4).

As in the classical case, we will use the Φ-# maximal operator as a bridge to pass from weighted estimates of the operator to weighted estimates of the functions (cf. [5, Chapter II]). This is shown in the following result that reflect the action of $M_\Phi^\#$ on these operators.

Theorem 3.6. *Let T be a singular integral operator given by* (3.1) *with kernel K satisfying* (K1)–(K4)*. Then, for any $q > 1$,*

$$M_\Phi^\#(Tf)(x) \le CM_q f(x) \tag{3.2}$$

with an absolute constant $C > 0$ independent of f and q.

Proof: Fix $f \in C_0^\infty(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$. Let Q be an arbitrary cube containing x, with sides parallel to the coordinate axes and center denoted by x_Q. Taking $f_1 = f\chi_{Q^*}$ with $Q^* = 2\sqrt{n}Q$, the cube which has the same center as Q_j but with side length $2\sqrt{n}$ times as long, and $f_2 = f - f_1$, we define

$$b_j = \int_{\mathbb{R}^n} B_j(-(y - x_Q))f_2(y)\,dy, \quad 1 \le j \le m, \tag{3.3}$$

which by the choice of the B_j's are all finite. So, we can split

$$\begin{aligned} &\frac{1}{|Q|}\int_Q |Tf(y) - \sum_{j=1}^m b_j\phi_j(-(y - x_Q))|\,dy \\ &\le \frac{1}{|Q|}\int_Q |Tf_1(y)|\,dy \\ &\quad + \frac{1}{|Q|}\int_Q |Tf_2(y) - \sum_{j=1}^m b_j\phi_j(-(y - x_Q))|\,dy \\ &= I + J. \end{aligned} \tag{3.4}$$

The estimate of the first term is straightforward. Indeed, for any $q > 1$,

$$\begin{aligned} I &\le \left(\frac{1}{|Q|}\int_Q |Tf_1(y)|^q\,dy\right)^{1/q} \\ &\le C\left(\frac{1}{|Q|}\int_{Q^*} |f(y)|^q\,dy\right)^{1/q} \\ &\le CM_q f(x), \end{aligned} \tag{3.5}$$

where the second inequality follows from (K1), (K3) and (K4) since these hypotheses imply that T is of type-(q, q) by Remark 3.4 and Theorem 3.1.

On the other hand, taking into account (3.3) and (K4),

(3.6)

$$
\begin{aligned}
J &= \frac{1}{|Q|}\int_Q \left| \int_{\mathbb{R}^n} \left(K(y-s) - \sum_{j=1}^m B_j(-(s-x_Q))\phi_j(-(y-x_Q)) \right) f_2(s)\,ds \right| dy \\
&\le \frac{1}{|Q|}\int_Q \left(\int_{\mathbb{R}^n\setminus Q^*} \left| K(y-s) - \sum_{j=1}^m B_j(-(s-x_Q))\phi_j(-(y-x_Q)) \right| |f(s)|\,ds \right) dy \\
&\le C\frac{1}{|Q|}\int_Q \left(\int_{|s-x_Q|>2|y-x_Q|} \frac{|y-x_Q|^\gamma}{|s-x_Q|^{n+\gamma}} |f(s)|\,ds \right) dy \\
&\le C\frac{1}{|Q|}\int_Q \left(\sum_{l=1}^\infty \int_{2^{l-1}(2|y-x_Q|)<|s-x_Q|\le 2^l(2|y-x_Q|)} \frac{|y-x_Q|^\gamma}{|s-x_Q|^{n+\gamma}} |f(s)|\,ds \right) dy \\
&\le C\frac{1}{|Q|}\int_Q \left(\sum_{l=1}^\infty \frac{1}{2^{l\gamma}} \frac{1}{(2^l 2|y-x_Q|)^n} \int_{|s-x_Q|\le 2^l(2|y-x_Q|)} |f(s)|\,ds \right) dy \\
&\le CMf(x) \left(\sum_{l=1}^\infty \frac{1}{2^{l\gamma}} \right) \\
&\le CM_q f(x).
\end{aligned}
$$

Concluding, from (3.5) and (3.6) we get (3.2) and the theorem is proved. □

At this point we can state the weighted L^p-boundedness of these operators.

Theorem 3.7. *Let $1 < p < \infty$, $w \in A_p$ and T be a singular integral operator given by* (3.1) *with kernel K satisfying* (K1)-(K4). *Then there exists a constant C such that*

$$
\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx \le C \int_{\mathbb{R}^n} |f(x)|^p w(x)\,dx \tag{3.7}
$$

for every smooth function f with compact support.

Proof: The theorem will be proved if we can show that $M(Tf) \in L^p(w)$ for any $f \in C_0^\infty(\mathbb{R}^n)$. Indeed, taking this for granted, by Theorem 2.9, Theorem 3.6 and choosing $q > 1$ such that $w \in A_{p/q} \subset A_p$ (cf. [11, p. 236]), we have

$$
\begin{aligned}
\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\,dx &\le \int_{\mathbb{R}^n} M(Tf)(x)^p w(x)\,dx \\
&\le C \int_{\mathbb{R}^n} M_\Phi^\#(Tf)(x)^p w(x)\,dx \\
&\le C \int_{\mathbb{R}^n} M_q(f)(x)^p w(x)\,dx
\end{aligned}
$$

$$
\begin{aligned}
&= C \int_{\mathbb{R}^n} M(|f|^q)(x)^{p/q} w(x)\, dx \\
&\leq C \int_{\mathbb{R}^n} |f(x)|^p w(x)\, dx,
\end{aligned}
$$

where the last estimate follows by the classical result of Muckehoupt ([5, Chapter IV]).

So, to conclude the proof it only remains to prove the claim made. To show that $M(Tf) \in L^p(w)$ for any $f \in C_0^\infty(\mathbb{R}^n)$, since $w \in A_p$, again by the classical Muckehoupt's result it is enough to see that $Tf \in L^p(w)$. In order to make the proof as selfcontained as possible, we detail this estimate (cf. [5, Chapter IV]).

Fix any $f \in C_0^\infty(\mathbb{R}^n)$ with $\|f\|_\infty = 1$ and let $R > 0$ such that $f(y) = 0$ for $|y| \geq R$. Then, by (K2) and for any $x \in \mathbb{R}^n$ with $|x| \geq 2R$,

$$
\begin{aligned}
|Tf(x)| &\leq \int_{|y|<R} |K(x-y)||f(y)|\, dy \\
&\leq \int_{|y|<R} \frac{C}{|x-y|^n} |f(y)|\, dy \qquad (3.8)\\
&\leq \frac{C}{|x|^n}.
\end{aligned}
$$

Now, we split the integral to estimate in two parts

$$
\int_{\mathbb{R}^n} |Tf(x)|^p w(x)\, dx = \int_{|x|<2R} |Tf(x)|^p w(x)\, dx + \int_{|x|\geq 2R} |Tf(x)|^p w(x)\, dx.
$$

On one hand, if $\varepsilon > 0$ is given by the reverse Hölder condition (2.4) of w, then

$$
\begin{aligned}
(3.9) \quad &\int_{|x|<2R} |Tf(x)|^p w(x)\, dx \\
&\leq \left(\int_{\mathbb{R}^n} |Tf(x)|^{p(1+\frac{1}{\varepsilon})}\, dx \right)^{\frac{\varepsilon}{(1+\varepsilon)}} \left(\int_{|x|<2R} w(x)^{1+\varepsilon}\, dx \right)^{\frac{1}{(1+\varepsilon)}} < \infty,
\end{aligned}
$$

where the first term is finite since, by Remark 3.4 and Theorem 3.1, T is bounded in $L^{p(1+1/\varepsilon)}(\mathbb{R}^n)$.

On the other hand, recalling that $w(x)\, dx$ is a doubling measure and, as usual, denoting $w(B) = \int_B w(x)\, dx$ for any subset B, from (3.8) and by taking $1 < r < p$

such that $w \in A_r$, it follows (cf. [5, p. 412]) that

$$\begin{aligned}
\int_{|x|\geq 2R} |Tf(x)|^p w(x)\,dx &\leq C \int_{|x|\geq 2R} \frac{1}{|x|^{np}} w(x)\,dx \\
&= C \sum_{l=1}^{\infty} \int_{2^{l-1}2R<|x|\leq 2^l 2R} \frac{1}{|x|^{np}} w(x)\,dx \\
&\leq C \sum_{l=1}^{\infty} \frac{1}{(2^{l-1}2R)^{np}} w(2^l B(0,2R)) \qquad (3.10)\\
&\leq C \sum_{l=1}^{\infty} \frac{1}{(2^{l-1}2R)^{np}} 2^{lrn} w(B(0,2R)) \\
&\leq C \left(\sum_{l=1}^{\infty} \frac{1}{2^{ln(p-r)}} \right) w(B(0,2R)) < \infty.
\end{aligned}$$

Finally, from (3.9) and (3.10) we conclude that $Tf \in L^p(w)$ and the proof is complete. □

We can also prove that T is of weak type-(1,1) with respect to the A_1 weights. The proof of this result follows the classical scheme, but making use of the modification of the Calderón-Zygmund decomposition introduced in [6].

Theorem 3.8. *Let $w \in A_1$ and T be a singular integral operator given by* (3.1) *with kernel K satisfying* (K1)-(K4). *Then there exists a constant C such that*

$$w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) \leq \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x)\,dx$$

for every smooth function f with compact support.

Proof: We first recall that $w \in A_1$ means that there exists a constant C such that, for any cube Q,

$$\frac{w(Q)}{|Q|} \leq Cw(x) \quad \text{a.e. on } Q. \tag{3.11}$$

Fix $\lambda > 0$ and $f \in C_0^\infty(\mathbb{R}^n)$ that we can assume to be real. The Calderón-Zygmund decomposition of f at level λ provides a family $\{Q_j\}$ of non-overlapping cubes such that

(i) for $\Omega = \bigcup_j Q_j$ it follows that $|f(x)| \leq \lambda$ a.e. on $\mathbb{R}^n \setminus \Omega$',

(ii) for any cube Q_j

$$\lambda \leq \frac{1}{|Q_j|} \int_{Q_j} |f(x)|\,dx \leq 2^n \lambda,$$

(iii) if $Q_j^* = 2\sqrt{n}Q$ and $\Omega^* = \bigcup_j Q_j^*$, then

$$|\Omega^*| = |\bigcup_j Q_j^*| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(x)|w(x)\,dx.$$

We now introduce the variant of the classical decomposition of f in its "good" and "bad" part (cf. [6, p. 169]). For each Q_j, if y_j denotes its center, let $g_j(x)$ be the projection of the restriction of f to Q onto the span of $\{\phi_1(\cdot - y_j), \phi_2(\cdot - y_j), \cdots, \phi_m(\cdot - y_j)\}$.

Now set

$$g(x) = \begin{cases} f(x), & x \in \mathbb{R}^n \setminus \Omega, \\ g_j(x) & x \in Q_j, \end{cases}$$

and $b(x) = f(x) - g(x) = \sum_j b_j(x) = \sum_j f(x) - g_j(x)$.

Two facts are consequence of this construction. First, it readily follows that $|g(x)| \leq \lambda$ a.e. on $\mathbb{R}^n \setminus \Omega$. On the other hand, for any $x \in \Omega$ and by Lemma 2.5 and (ii), $|g(x)| \leq C|f|_{Q_j} \leq C\lambda$. Both inequalities can be fusioned in the general

$$|g(x)| \leq C\lambda \quad \text{a.e.} \tag{3.12}$$

With respect to the "bad" part b, we first note that for any $1 \leq i \leq m$ and any j

$$\int_{Q_j} \phi_i(x - y_j) b_j(x)\,dx = 0. \tag{3.13}$$

The basic Lemma 2.5 provides a fundamental weighted estimate of g based in the A_1 condition (3.11) of the weight. Indeed,

$$\begin{aligned} \int_{\mathbb{R}^n} |g(x)|w(x)\,dx &\leq \int_{\mathbb{R}^n \setminus \Omega} |f(x)|w(x)\,dx + \sum_j w(Q_j)\frac{1}{|Q_j|}\int_{Q_j} |f(x)|\,dx \\ &\leq C\int_{\mathbb{R}^n} |f(x)|w(x)\,dx. \end{aligned} \tag{3.14}$$

Having fixed the decomposition $f = g + b$ of the function, we now proceed as in the classical case (cf. [5, p. 413]), hence we will simplify the computations detailing those steps that differ from it.

The decomposition of f reduces the problem to estimate

$$\begin{aligned} w(\{x \in \mathbb{R}^n : |Tf(x)| > \lambda\}) &\leq w(\{x \in \mathbb{R}^n : |Tg(x)| > \lambda\}) \\ &\quad + w(\{x \in \mathbb{R}^n \setminus \Omega^* : |Tb(x)| > \lambda\}) \\ &\quad + w(\Omega^*) \\ &= I + II + III. \end{aligned}$$

The third term is readily estimated from the doubling property of the weight w (see [5, p. 396]), (ii) and (3.11)

$$
\begin{aligned}
III &\le C \sum_j w(Q_j) \\
&\le C \sum_j \frac{w(Q)}{|Q|} \frac{1}{\lambda} \int_{Q_j} |f(x)|\, dx \\
&\le C \sum_j \frac{1}{\lambda} \int_{Q_j} |f(x)| w(x)\, dx \\
&\le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x)\, dx.
\end{aligned}
\tag{3.15}
$$

For the first term, choosing any $p > 1$ and taking into account that $w \in A_1 \subset A_p$, by Theorem 3.7 we have

$$
\begin{aligned}
I &\le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |Tg(x)|^p w(x)\, dx \\
&\le \frac{C}{\lambda^p} \int_{\mathbb{R}^n} |g(x)|^p w(x)\, dx \\
&\le \frac{C}{\lambda} \int_{\mathbb{R}^n} |g(x)| w(x)\, dx \\
&\le \frac{C}{\lambda} \int_{\mathbb{R}^n} |f(x)| w(x)\, dx,
\end{aligned}
\tag{3.16}
$$

where the third inequality follows by (3.12) and the last one from (3.14).

Finally, for the second term we recall (3.13), the hypothesis (K4) and (3.11), and we get

(3.17)

$$
\begin{aligned}
II &\le \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \Omega^*} |Tb(x)| w(x)\, dx \\
&\le \frac{C}{\lambda} \int_{\mathbb{R}^n \setminus \Omega^*} \sum_j \int_{Q_j} \left| \left(K(x-y) - \sum_{r=1}^m B_r(x-y_j)\phi_r(y-y_j) \right) \right| |b_j(y)|\, dy w(x)\, dx \\
&\le \frac{C}{\lambda} \sum_j \int_{x \notin Q^*} \int_{Q_j} \left| \left(K(x-y) - \sum_{r=1}^m B_r(x-y_j)\phi_r(y-y_j) \right) \right| |b_j(y)|\, dy w(x)\, dx \\
&\le \frac{C}{\lambda} \sum_j \int_{Q_j} \left(\int_{x \notin Q^*} \left| \left(K(x-y) - \sum_{r=1}^m B_r(x-y_j)\phi_r(y-y_j) \right) \right| w(x)\, dx \right) |b_j(y)|\, dy
\end{aligned}
$$

$$\leq \frac{C}{\lambda}\sum_j \int_{Q_j} |b_j(y)| Mw(y)\, dy$$
$$\leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |b(y)| w(y)\, dy$$
$$\leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f(y)| w(y)\, dy,$$

where the last inequality holds since $b = f - g$ and (3.14).

From (3.15), (3.16) and (3.17) we deduce (3.7) and the theorem is proved. □

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