

Products of k -spaces, and questions

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Abstract. As is well-known, every product of a locally compact space with a k -space is a k -space. But, the product of a separable metric space with a k -space need not be a k -space. In this paper, we consider conditions for products to be k -spaces, and pose some related questions.

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Definitions and preliminaries

We assume that all spaces are *regular* and T_1 , and all maps are continuous surjections.

Let X be a space. For a (not necessarily open or closed) cover \mathcal{P} of X , X is *determined by* a cover \mathcal{P} , if $U \subset X$ is open in X if and only if $U \cap P$ is relatively open in P for every $P \in \mathcal{P}$. Here, we can replace “open” by “closed”. (Following [3], we shall use “ X is determined by \mathcal{P} ” instead of the usual “ X has the *weak topology* with respect to \mathcal{P} ”.) Obviously, every space is determined by its open cover. As is well-known, a space is a k -space (resp. *sequential space*) if it is determined by the cover of *all* compact (resp. compact metric) subsets. We recall that every k -space (resp. sequential space) is characterized as a quotient space of a locally compact space (resp. metric space [2]). Every sequential space is a k -space, and the converse holds if each point is a G_δ -set ([7]).

We recall elementary facts which will be used later on. These are routinely shown, but Fact (4) is due to [15].

Facts: (1) Let X be a space determined by a cover \mathcal{P} , and let \mathcal{C} be a cover of X . If each element of \mathcal{P} is contained in some element of \mathcal{C} , then X is also determined by \mathcal{C} .

(2) Let X be a space determined by a cover $\{X_\alpha : \alpha\}$. If each X_α is determined by a cover \mathcal{P}_α , then X is determined by the cover $\bigcup\{\mathcal{P}_\alpha : \alpha\}$.

(3) (i) Let $f : X \rightarrow Y$ be a quotient map. If X is determined by a cover \mathcal{C} , then Y is determined by the cover $\{f(C) : C \in \mathcal{C}\}$.

(ii) For a cover \mathcal{P} of a space Y , Y is determined by the cover \mathcal{P} if and only if the obvious map $f : \Sigma \mathcal{P} \rightarrow Y$ is quotient (where $\Sigma \mathcal{P}$ is the topological sum of \mathcal{P}).

(4) Let $f : X \rightarrow Y$ be a closed map. If Y is determined by a cover \mathcal{P} , then X is determined by the cover $\{f^{-1}(P) : P \in \mathcal{P}\}$.

A space X is *Fréchet* if for any $A \subset X$ and any $x \in \overline{A}$, there exist points $x_n \in A$ such that $\{x_n : n \in \mathbb{N}\}$ converges to x . Also, a space X is *strongly Fréchet* [12]; or *countably bi-sequential* [7], if for every decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then there exist points $x_n \in A_n$ ($n \in \mathbb{N}$) such that $\{x_n : n \in \mathbb{N}\}$ converges to the point x .

Let $f : X \rightarrow Y$ be a map. Then f is *bi-quotient* [6] (resp. *countably bi-quotient* [12]) if, whenever $y \in Y$ and \mathcal{U} is a cover (resp. countable cover) of $f^{-1}(y)$ by open subsets of X , then finitely many $f(U)$, with $U \in \mathcal{U}$, cover a nbd of y in Y . Also, f is *hereditarily quotient* (or *pseudo-open*) if $f|f^{-1}(S) : f^{-1}(S) \rightarrow S$ is quotient for every $S \subset Y$ (equivalently, for any nbd U of $f^{-1}(y)$ in X , $\text{int } f(U)$ is a nbd of y in Y (see [7])).

Obviously, we have the following implications: open (or perfect) map \rightarrow bi-quotient map \rightarrow countably bi-quotient map \rightarrow hereditarily quotient map \leftarrow closed map. Also, hereditarily quotient map \rightarrow quotient map.

We recall the following characterizations by means of these maps. For these, and intrinsic definitions of related spaces, see [7]. Here, a space is an *M-space* if it is an inverse image of a metric space under a quasi-perfect map.

Characterizations: (1) A space X is bi-sequential (resp. countably bi-sequential; Fréchet; sequential) $\Leftrightarrow X$ is a bi-quotient (resp. countably bi-quotient; hereditarily quotient; quotient) image of a metric space.

(2) A space X is bi- k (resp. countably bi- k ; singly bi- k ; k) $\Leftrightarrow X$ is a bi-quotient (resp. countably bi-quotient; hereditarily quotient; quotient) image of a paracompact M -space.

(3) A space X is bi-quasi- k (resp. countably bi-quasi- k ; singly bi-quasi- k ; quasi- k) $\Leftrightarrow X$ is a bi-quotient (resp. countably bi-quotient; hereditarily quotient; quotient) image of an M -space.

We recall that a decreasing sequence $\{A_n : n \in \mathbb{N}\}$ of sets is a *k-sequence* (resp. *q-sequence*) [7] if $K = \bigcap\{A_n : n \in \mathbb{N}\}$ is compact (resp. countably compact), and any open set $U \supset K$ contains some A_n . Recall that a space X is of *pointwise countable type* (resp. *q-space*) if each point has nbds $\{V_n : n \in \mathbb{N}\}$ which is a *k-sequence* (resp. *q-sequence*). Obviously, every first countable space is of pointwise countable type.

Recall that a space X is of pointwise countable type (resp. *q-space*) if and only if X an open image of a paracompact M -space (resp. M -space); see [7]. Thus, we can replace “paracompact M -space (M -space)” by “space of pointwise countable type (resp. *q-space*)” in Characterizations.

As weaker concepts than “strongly Fréchet spaces”, let us recall Tanaka spaces and strongly sequential spaces defined by F. Mynard.

A space X is a *Tanaka space* (or *Tanaka topology*) [10] if it satisfies the following condition (C) in [16].

(C) Let $\{A_n : n \in \mathbb{N}\}$ be a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$. Then there exist $x_n \in A_n$ such that $\{x_n : n \in \mathbb{N}\}$ converges to some point $y \in X$.

Obviously, every sequentially compact space is precisely a countably compact Tanaka space. We note that every Tanaka space need not be sequential (not even a k -space).

A space X is *strongly sequential* [9] if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then the point x belongs to the (idempotent) *sequential closure* of the set A of limit points of convergent sequences $\{x_n : n \in \mathbb{N}\}$ with $x_n \in A_n$. Namely, a space X is strongly sequential if and only if it is a sequential space such that if $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n}$ for any $n \in \mathbb{N}$, then the point x belongs to the (usual) closure of the above set A .

A space X is *inner-closed* A [8] if, whenever $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence of subsets of X with $x \in \overline{A_n - \{x\}}$ for any $n \in \mathbb{N}$, then there exist $F_n \subset A_n$ which are closed in X such that $\bigcup\{F_n : n \in \mathbb{N}\}$ is not closed in X . Among sequential spaces, we can assume that the F_n are singletons.

Let $S = \{\infty\} \cup \{p_n : n \in \mathbb{N}\} \cup \{p_{nm} : n, m \in \mathbb{N}\}$ be an infinite countable space such that each p_{nm} is isolated in S , $K = \{p_n : n \in \mathbb{N}\}$ converges to $\infty \notin K$, and each $L_n = \{p_{nm} : m \in \mathbb{N}\}$ converges to $p_n \notin L_n$. The space S is called the *Arens' space* S_2 , if for every finite $F_n \subset L_n$ ($n \in \mathbb{N}$), $\bigcup\{F_n : n \in \mathbb{N}\}$ is closed in S . The quotient space $S_2/(K \cup \{\infty\})$ is the *sequential fan* S_ω ; that is, S_ω is the space obtained from the topological sum of countably many convergent sequences by identifying all the limit points. The sequential spaces S_2 , S_ω , and their modifications have played important roles in the theory of products of k -spaces; see [21], [22], [24], and [25], for example.

The following diagrams, etc., hold in view of Characterizations, or these are easily shown, but the first implication in Diagram (6) is shown in [7].

Diagrams: (1) First countable space \rightarrow bi-sequential space \rightarrow countably bi-sequential \rightarrow Fréchet space \rightarrow sequential space \rightarrow k -space.

(2) Countably bi-sequential space (= strongly Fréchet space) \rightarrow strongly sequential space \rightarrow sequential space.

(3) Countably bi-sequential space \rightarrow countably bi- k -space \rightarrow countably bi-quasi- k -space \leftarrow bi-quasi- k -space \leftarrow bi- k -space \leftarrow bi-sequential space.

(4) Compact space \rightarrow paracompact M -space \rightarrow space of pointwise countable type \rightarrow bi- k -space \rightarrow countably bi- k -space \rightarrow singly bi- k -space \rightarrow k -space \rightarrow quasi- k -space.

(5) Countably compact space \rightarrow M -space \rightarrow q -space \rightarrow bi-quasi- k -space \rightarrow countably bi-quasi- k -space \rightarrow singly bi-quasi- k -space \rightarrow quasi- k -space.

(6) Countably bi-quasi- k -space \rightarrow inner-closed A -space \rightarrow space which contains no closed copy of S_ω , and no $S_2 \leftarrow$ Tanaka space.

For a space, let us consider the following properties. Here, a cover \mathcal{P} of a space X is called a k -network for X if, for any compact subset K of X and any open set $V \supset K$, $K \subset \bigcup \mathcal{P}' \subset V$ for some finite $\mathcal{P}' \subset \mathcal{P}$. Every countably bi- k -space having property (P4) has a point-countable base, and every quotient Lindelöf image of a metric space has property (P4); see [3]. Here, a map is *Lindelöf* if every inverse-image of a point is Lindelöf. As for products of k -spaces having certain point-countable k -networks, see [24].

- (P1) Fréchet space.
- (P2) Space in which every point is a G_δ -set.
- (P3) Hereditarily normal space.
- (P4) Space with a point-countable k -network.
- (P5) Closed image of a countably bi- k -space.
- (P6) Closed image of an M -space.

Let us recall the following results. For (1) & (2); (3); (4); (5); and (6), see [20]; [4]; [25]; [16]; and [7] respectively.

Results: (1) For a space X having (P1), X is strongly Fréchet $\Leftrightarrow X$ contains no (closed) copy of S_ω .

(2) For a sequential space X having (P2) or (P3), X is strongly Fréchet $\Leftrightarrow X$ contains no (closed) copy of S_ω , and no S_2 .

(3) For a k -space X having property (P4), X is first countable $\Leftrightarrow X$ contains no (closed) copy of S_ω , and no S_2 .

(4) For a sequential space X having properties (P5) (resp. (P6)), X is a countably bi- k -space (resp. q -space) $\Leftrightarrow X$ contains no (closed) copy of S_ω .

(5) For a space X having (P1) or (P2), X is strongly Fréchet $\Leftrightarrow X$ is a Tanaka space.

(6) For a sequential space X which is a quotient Lindelöf image of a paracompact space S , if S is bi-sequential (resp. countably bi-sequential; bi- k ; countably bi- k ; bi-quasi- k ; countably bi-quasi- k), then so is X respectively $\Leftrightarrow X$ is inner-closed A (equivalently, Tanaka space).

Results and questions

In [16], it is shown that, for a first countable space X , if $X \times Y$ is sequential, then X is locally countably compact, or Y is a sequential Tanaka space, and that the converse holds under some conditions on Y . F. Mynard [10] and [9] obtained Theorems 1 and 2 below respectively. Theorem 1 implies that every sequential countably bi-quasi- k -space is strongly sequential by Diagram (6).

Theorem 1. *For a space X , the following are equivalent.*

- (a) X is strongly sequential.

- (b) X is sequential inner-closed A .
- (c) X is a sequential Tanaka space.

Theorem 2. *Let X be first countable, and Y be sequential. Then $X \times Y$ is sequential if and only if X is locally countably compact, or Y is a strongly sequential space.*

The following lemma holds in view of the proof of Lemma 6 in [19].

Lemma 3. *Let X be a space determined by a point-countable cover \mathcal{C} . Then, for a q -sequence $\{A_n : n \in \mathbb{N}\}$ in X , some A_n is contained in a finite union of elements of \mathcal{C} .*

Proposition 4. *Let X be a space determined by a point-countable cover \mathcal{C} . Then each point of X has a nbd which is contained in a finite union of elements of \mathcal{C} if the following (a), (b), or (c) holds.*

- (a) X is countably bi-quasi- k , and \mathcal{C} is closed.
- (b) X is sequential and inner-closed A (equivalently, strongly sequential).
- (c) X is inner-closed A , and \mathcal{C} is countable, and closed (or increasing).

PROOF: Case (a): Suppose that some point x of X has no nbds which are contained in a finite union of elements of \mathcal{C} . Let $\{C \in \mathcal{C} : x \in C\} = \{C_n : n \in \mathbb{N}\}$, and let $B_n = \bigcup\{C_m : m \leq n\}$ for each $n \in \mathbb{N}$. Then, $x \in \overline{X - B_n}$ for each $n \in \mathbb{N}$. Since X is countably bi-quasi- k , there exists a q -sequence $\{A_n : n \in \mathbb{N}\}$ such that $x \in \overline{(X - B_n) \cap A_n}$ for all $n \in \mathbb{N}$ ([7]). But, by Lemma 3, some A_m is contained in a union of finitely many closed sets F_n in \mathcal{C} . Let $V = X - \bigcup\{F_n : x \notin F_n\}$, then V is a nbd of x , so $x \in \overline{(V - B_n) \cap A_n}$ for all $n \in \mathbb{N}$. But, some $(V - B_n) \cap A_n$ must be empty. This is a contradiction. Thus each point has a nbd which is contained in an element of the cover \mathcal{C} .

Case (b): Suppose that some point x of X has no nbds which are contained in a finite union of elements of \mathcal{C} . As is known, since X is sequential, if $x \in \overline{A}$, then $x \in \overline{B}$ for some countable $B \subset A$; see [7], for example. Then, there exists a sequence $\{B_n : n \in \mathbb{N}\}$ of countable subsets such that $B_1 = \{x\}$, $x \in \overline{B_n}$, and $B_n \cap C_i(B_j) = \emptyset$ whenever $i < n$ and $j < n$, here $\{C_i(B_j) : i \in \mathbb{N}\} = \{C \in \mathcal{C} : C \cap B_j \neq \emptyset\}$. Let $A_n = \bigcup\{B_k : k \geq n\}$. Then $\{A_n : n \in \mathbb{N}\}$ is a decreasing sequence such that $x \in \overline{A_n}$, but any $C \in \mathcal{C}$ meets only finitely many A_n . Since X is inner-closed A , there exist $F_n \subset A_n$ which are closed in X , but $A = \bigcup\{F_n : n \in \mathbb{N}\}$ is not closed in X . Since X is determined by \mathcal{C} and A is not closed in X , some $C \in \mathcal{C}$ meets infinitely many closed sets F_n , so infinitely many A_n . This is a contradiction. Thus, each point has a nbd which is contained in a finite union of elements of \mathcal{C} .

Case (c): We can assume that $\mathcal{C} = \{C_n : n \in \mathbb{N}\}$ is increasing. Suppose that some point x of X has no nbds which are contained in some C_n . Then $x \in \overline{X - C_n}$

for each $n \in \mathbb{N}$. But, X is inner-closed A , so some C_m meets infinitely many $X - C_n$, a contradiction. \square

As is well-known, every product of a closed map with the identity map need not be quotient. Also, every product of a quasi-perfect map f with the identity map need not be closed ([13]).

Lemma 5. (1) *Let $f_i : X_i \rightarrow Y_i$ ($i = 1, 2$) be quasi-perfect maps. If X_1 is a k -space, and $Y_1 \times Y_2$ is sequential, then $f_1 \times f_2$ is quasi-perfect ([13]).*

(2) *Every product of bi-quotient maps is bi-quotient ([6]).*

(3) *Let $f : X \rightarrow Y$ be a countably bi-quotient map. If Z is first countable, then $f \times \text{id}_Z$ is countably bi-quotient ([7]).*

We have the following sufficient conditions for products to be k -spaces. The result for case (c) or (e) is shown in [25]. For case (d) (resp. (e)), when X is countably bi-quasi- k or strongly sequential (resp. inner-closed A), X is locally compact by means of Proposition 4.

Theorem 6. *One of the following (a)–(e) implies that $X \times Y$ is a k -space.*

- (a) *X is strongly sequential, and Y is a k -space which is bi-quasi- k .*
- (b) *X is bi- k , and Y is a k -space which is countably bi-quasi- k .*
- (c) *X is sequential, and Y is a k -space which is locally countably compact.*
- (d) *X and Y are singly bi-quasi- k -spaces determined by a point-countable closed cover of locally compact subsets.*
- (e) *X and Y are spaces determined by a countable, and closed (or increasing) cover of locally compact subsets.*

PROOF: Case (a): Let Y be an image of an M -space S under a bi-quotient map f by Characterization (3). Let S be an inverse image of a metric space T under a quasi-perfect map g . Then $X \times T$ is a sequential space by Theorem 2. Hence, $X \times S$ is an inverse image of the sequential space $X \times T$ under a quasi-perfect map $\text{id}_X \times g$ by Lemma 5(1). While, $X \times T$ is a k -space, so it is determined by the cover of all compact subsets. Thus, by Fact (1), $X \times T$ is determined by a cover $\{C \times T : C \text{ is compact in } X\}$. Thus, by Fact (4), $X \times S$ is determined by a cover $\{C \times S : C \text{ is compact in } X\}$. But, $X \times Y$ is an image of $X \times S$ under a quotient map $\text{id}_X \times f$ by Lemma 5(2). Thus, $X \times Y$ is determined by a cover $\{C \times Y : C \text{ is compact in } X\}$ by Fact (3). But, as is well-known, each element $C \times Y$ is a k -space, for C is compact and Y is a k -space. Thus, $X \times Y$ is a k -space, for it is determined by the cover of all compact subsets by Facts (1) & (2).

Case (b): Let X be the image of a paracompact M -space S under a bi-quotient map f by Characterization (2). Let Y be the image of an M -space S' under a countably bi-quotient map g by Characterization (3). Let S be the inverse image of a metric space T under a perfect map p . Let S' be the inverse image of a metric space T' under a quasi-perfect map q . Then, $T \times S'$ is the inverse image of a metric

space $T \times T'$ under a quasi-perfect map $\text{id}_T \times q$. Then, $T \times S'$ is determined by a cover $\{C \times S' : C \text{ is a compact in } T\}$. While, $T \times Y$ is the image of $T \times S'$ under a quotient map $\text{id}_T \times g$ by Lemma 5(3). Thus, $T \times Y$ is determined by a cover $\{C \times Y : C \text{ is a compact in } T\}$. But, $S \times Y$ is the inverse image of $T \times Y$ under a perfect map $p \times \text{id}_Y$, so $S \times Y$ is determined by a cover $\{p^{-1}(C) \times Y : C \text{ is a compact in } T\}$ (by means of Fact (4)). Since $X \times Y$ is the image of $S \times Y$ under a quotient map $f \times \text{id}_Y$, $X \times Y$ is determined by a cover $\{f(p^{-1}(C)) \times Y : C \text{ is a compact in } T\}$. But, each element $f(p^{-1}(C)) \times Y$ is a k -space since $f(p^{-1}(C))$ is compact in X . Thus, $X \times Y$ is also a k -space.

Case (d): Let X (resp. Y) be determined by a point-countable closed cover \mathcal{C} (resp. \mathcal{K}) of locally compact subsets. We will show that $X \times Y$ is determined by a cover $\mathcal{L} = \{C \times K : C \in \mathcal{C}, K \in \mathcal{K}\}$, then $X \times Y$ is a k -space, because each element $C \times K$ is locally compact (hence a k -space). So, for $F \subset X \times Y$, suppose that $F \cap (C \times K)$ is closed in $C \times K$ for each $C \times K \in \mathcal{L}$. To show F is closed in $X \times Y$, let $A = X - F$, and let $(x, y) \in A$. Let $\{C \in \mathcal{C} : x \in C\} = \{C_n : n \in \mathbb{N}\}$, and $\{K \in \mathcal{K} : y \in K\} = \{K_n : n \in \mathbb{N}\}$. Here, we can assume that $C_n \subset C_{n+1}$ and $K_n \subset K_{n+1}$ since C_n and K_n are closed in X and Y respectively. Since $A \cap (C_n \times K_n)$ is open in $C_n \times K_n$ for each $n \in \mathbb{N}$, by induction, it is routine to show that there exist nbds U_n of x in C_n , and nbds V_n of y in K_n such that $(\overline{U_n} \times \overline{V_n}) \subset (U_{n+1} \times V_{n+1}) \subset (\overline{U_{n+1}} \times \overline{V_{n+1}}) \subset (C_{n+1} \times K_{n+1}) \cap A$, and all $\overline{U_n}, \overline{V_n}$ are compact sets. Let $U = \bigcup\{U_n : n \in \mathbb{N}\}$, and $V = \bigcup\{V_n : n \in \mathbb{N}\}$. Then $U \times V \subset A$. Also, U is a nbd of x in X . Indeed, suppose U is not a nbd of x in X . Then $x \in \overline{X - U}$. Since X is singly bi-quasi- k , there exists a q -sequence $\{A_n : n \in \mathbb{N}\}$ such that $x \in \overline{((X - U) \cap A_n)}$ for all $n \in \mathbb{N}$ ([7]). But, by Lemma 3, some A_m is contained in a finite union of elements of \mathcal{C} . Then, $x \in \overline{((X - U) \cap C)}$ for some $C \in \mathcal{C}$. Then, $x \in C$, so we can assume $C = C_k$ for some $k \in \mathbb{N}$. But, $U_k \cap (X - U) \supset U_k \cap ((X - U) \cap C_k) \neq \emptyset$. This is a contradiction to $U_k \cap (X - U) = \emptyset$. Hence U is a nbd of x in X . Similarly, V is a nbd of y in Y . Then A is open in $X \times Y$, thus F is closed in $X \times Y$. This shows that $X \times Y$ is a k -space. (For case (e), we can assume that the countable closed cover \mathcal{C} is increasing by Fact (1), then $X \times Y$ is a k -space as in the first half of the proof). □

Corollary 7. *The following (a) or (b) implies that $X \times Y$ is a k -space.*

- (a) *X is sequential countably bi-quasi- k , and Y is a k -space which is bi-quasi- k .*
- (b) *X is bi- k , and Y is countably bi- k ([17]).*

Lemma 8. *Let X and Y be sequential spaces. Then $X \times Y$ is sequential if and only if it is a k -space ([14]).*

Corollary 9. *Each of the following items (a), (b), or (c) implies that $X \times Y$ is a sequential space.*

- (a) *X is strongly sequential, and Y is sequential bi-quasi- k .*

- (b) X is sequential countably bi-quasi- k , and Y is sequential bi-quasi- k .
- (c) X is sequential, and Y is sequential locally countably compact ([1]).

Corollary 10. Let $f_i : X_i \rightarrow Y_i$ ($i = 1, 2$) be maps such that X_i are locally compact (resp. sequential locally compact). Then each of the following items (a)–(e) implies that $Y_1 \times Y_2$ is a k -space (resp. sequential space).

- (a) f_i are quotient maps, and X_i are Lindelöf.
- (b) f_i are quotient Lindelöf maps, X_i are paracompact, and Y_i are singly bi-quasi- k .
- (c) f_i are hereditarily quotient Lindelöf maps, X_i are paracompact.
- (d) f_i are closed maps, X_i are paracompact, and Y_i are locally Lindelöf.
- (e) f_i are closed Lindelöf maps.

PROOF: For case (a), X_i are determined by a countable cover of compact subsets, then so are Y_i by Fact (3). Hence $Y_1 \times Y_2$ is a k -space by Theorem 6. For case (b), X_i are determined by a locally finite cover of compact subsets, Y_i are determined by a point-countable cover of compact subsets by Fact (3). But, Y_i are singly bi-quasi- k , then $Y_1 \times Y_2$ is a k -space by Theorem 6. Case (c) implies case (b), because every hereditarily quotient image of a locally compact space is singly bi- k ([7]), hence singly bi-quasi- k . For case (d), it is routine that Y_i are determined by a hereditarily closure-preserving cover of compact subsets. Then, since Y_i are locally Lindelöf, each point of Y_i has a nbd which is determined by a countable cover of compact subsets. Thus, $Y_1 \times Y_2$ is a locally k -space by Theorem 6. Hence, $Y_1 \times Y_2$ is a k -space by means of Fact (2). The result for case (e) is due to [18]. The parenthetic part holds by means of Lemma 8. \square

Remark 11. (1) As is well-known, there exist a separable metric space X (or, closed image X of a metric locally compact space), and a closed image Y of a separable metric locally compact space such that $X \times Y$ is not a k -space; see [2], [18], for example.

(2) ($2^{\omega_0} < 2^{\omega_1}$). There exist countable, strongly Fréchet spaces X and Y such that $X \times Y$ is not a k -space ([11]).

(3) There exists a paracompact space X which is a quotient compact image of a metric locally compact space such that X^2 is not a k -space ([19]).

We note that every quotient Lindelöf image of a paracompact locally compact space is precisely a space determined by a point-countable cover of compact subsets by Fact (3). In view of Theorem 6, Corollary 10 and Remark 11(3), the author has a question whether every product of quotient Lindelöf images of paracompact locally compact spaces is a k -space if the images are Lindelöf. We recall the following general question. (1), (2) was posed in [21], [22] respectively.

Question 12. (1) Let X and Y be quotient Lindelöf images of paracompact locally compact spaces. What is a necessarily and sufficient condition for $X \times Y$ to be a k -space?

(2) Let X and Y be closed images of paracompact bi- k -spaces. What is a necessarily and sufficient condition for $X \times Y$ to be a k -space ?

Let us review partial answers to Question 12. First, we recall some related matters. For a cardinal number α , a space is a k_α -space if it is determined by a cover \mathcal{C} of compact subsets with $|\mathcal{C}| \leq \alpha$. A space X is *locally* $< k_\alpha$ if each point $x \in X$ has a nbd whose closure is a k_{β_x} -space, $\beta_x < \alpha$. Every locally compact space is locally k_ω (i.e., locally $< k_{\omega_1}$), and so is every space determined by a countable closed cover of locally compact subsets ([24]).

For products of k -spaces, we recall the following Hypotheses (H) and (H*); see [21] and [22] (or [24]). (A pair of spaces X and Y is said to have *Tanaka's condition* in [5], if (a'), (b), or (c) in (H) holds, where (a') X and Y are first countable.)

(H): Let X and Y be k -spaces. Then $X \times Y$ is a k -space if and only if (a), (b) or (c) below holds. (The "if" part of (H) is valid).

- (a) X and Y are bi- k .
- (b) X or Y is locally compact.
- (c) X and Y are locally k_ω .

(H*): Same as (H), but change (c) to (c'): One of X and Y is locally k_ω , and another is locally $< k_\mathfrak{c}$, $\mathfrak{c} = 2^\omega$.

Let F be the collection of all functions from \mathbb{N} to \mathbb{N} . The set-theoretic axiom $\text{BF}(\omega_2)$ means that if whenever $A \subset F$ with $|A| < \omega_2$, there exists $g \in F$ such that $f \leq g$ for all $f \in A$, here $f \leq g$ means $\{n \in \mathbb{N} : f(n) > g(n)\}$ is finite. (CH) implies $\text{BF}(\omega_2)$ is false, and Martin's axiom (MA) + \neg CH implies $\text{BF}(\omega_2)$.

Then, for example, we have the following partial answers to Question 12. (1) holds by means of [24, Theorem 1.1] and [23, Theorem 2.3], and (2), (3), and (4) are due to [22, Theorem 1.1].

Theorem 13. (1) Let X and Y be Fréchet spaces which are quotient Lindelöf images of metric spaces. Then Hypothesis (H) holds.

(2) Let X and Y be sequential spaces which are closed Lindelöf images of paracompact bi- k -spaces. Then Hypothesis (H) holds.

(3) $\text{BF}(\omega_2)$ is false if and only if the assertion (*) below is valid. When $X = Y$, (*) is valid without any set-theoretic axiom.

(*): Let X and Y be sequential spaces which are closed images of paracompact bi- k -spaces. Then Hypothesis (H) holds.

(4) Let X and Y be sequential spaces which are closed images of paracompact bi- k -spaces. Then the "only if" part of Hypothesis (H*) holds. Also, under (MA), Hypothesis (H*) holds if all compact sets in X and Y are metric (in particular, X and Y are closed images of metric spaces).

However, Hypothesis (H) does not suggest an answer to Question 12. Indeed, under $\text{BF}(\omega_2)$, there exist spaces X and Y which are quotient finite-to-one (or

closed) images of metric locally compact spaces such that $X \times Y$ is a k -space, but none of the properties (a), (b), and (c) holds ([5] or [24]).

Now, the author does not know whether every product of countably compact k -spaces is a k -space, more generally he has the following question in view of Theorem 6. When X is sequential, this is affirmative by Corollary 7.

Question 14. Let X be a k -space which is bi-quasi- k (or countably bi-quasi- k), and let Y be a k -space which is bi-quasi- k . Is $X \times Y$ a k -space ?

Lemma 15. Let X be a bi- k -space, and let Y be sequential. If $X \times Y$ is a k -space, then X is locally countably compact, or Y is a Tanaka space ([25]).

The following holds by means of Lemma 15, and Theorems 1 & 6.

Theorem 16. Let X be a bi- k -space, and let Y be sequential. Then $X \times Y$ is a k -space if and only if X is locally countably compact, or Y is a Tanaka space.

Question 17. In the previous theorem, is it possible to replace “bi- k -space” by “ k -space which is a bi-quasi- k -space (or M -space)” ?

The “if” part of Theorem 16, under Y being a k -space which is bi-quasi- k , remains true by Theorems 1 & 6. Thus, Question 17 is reduced to the question whether the replacement in Lemma 15 remains valid.

The following holds by means of Theorem 16, Corollary 7, and Results.

Corollary 18. Let X be a bi- k -space. Let Y be a sequential space having one of the properties (P1)–(P6) in the previous section. Then the following (a), (b), and (c) are equivalent ([25]).

- (a) $X \times Y$ is a k -space.
- (b) X is locally countably compact, or Y is a Tanaka space.
- (c) X is locally countably compact, or Y contains no (closed) copy of S_ω , and no S_2 .

Question 19. Let X be a bi- k -space, and let Y be a sequential space. Is it true that $X \times Y$ is a k -space if and only if X is locally countably compact, or Y contains no (closed) copy of S_ω , and no S_2 ?

The “only if” part holds by Theorem 16. Question 19 is reduced to the question whether every sequential is a Tanaka space if it contains no (closed) copy of S_ω , and no S_2 .

Question 19 is affirmative if Y is a quotient Lindelöf image of a metric space by Corollary 18. But, the author does not know whether Question 19 is also affirmative if Y is a sequential space which is a quotient Lindelöf image of a paracompact, and M -space (or bi- k -space) in view of Results.

REFERENCES

- [1] Boehme T.K., *Linear s -spaces*, Proc. Symp. Convergent structures, Univ. Oklahoma, 1965.
- [2] Franklin S.P., *Spaces in which sequences suffice*, Fund. Math. **LVII** (1965), 107–113.
- [3] Gruenhagen G., Michael E., Tanaka Y., *Spaces determined by point-countable covers*, Pacific J. Math. **113** (1984), 303–332.
- [4] Lin S., *A note on the Arens' space and sequential fan*, Topology Appl. **81** (1997), 185–196.
- [5] Lin S., Liu C., *On spaces with point-countable cs -networks*, Topology Appl. **74** (1996), 51–60.
- [6] Michael E., *Bi-quotient maps and cartesian products of quotient maps*, Ann. Inst. Fourier, Grenoble **18** (1968), 287–302.
- [7] Michael E., *A quintuple quotient quest*, Gen. Topology Appl. **2** (1972), 91–138.
- [8] Michael E., Olson R.C., Siwiec F., *A -spaces and countably bi-quotient maps*, Dissertationes Math. **133** (1976), 4–43.
- [9] Mynard F., *Strongly sequential spaces*, Comment. Math. Univ. Carolinae **41** (2000), 143–153.
- [10] Mynard F., *More on strongly sequential spaces*, Comment. Math. Univ. Carolinae **43** (2002), 525–530.
- [11] Olson R.C., *Bi-quotient maps, countably bi-sequential spaces, and related topics*, Gen. Topology Appl. **4** (1974), 1–28.
- [12] Siwiec F., *Sequence-converging and countably bi-quotient mappings*, Gen. Topology Appl. **1** (1971), 143–154.
- [13] Tanaka Y., *On products of quasi-perfect maps*, Proc. Japan Acad. **46** (1970), 1070–1073.
- [14] Tanaka Y., *On quasi- k -spaces*, Proc. Japan Acad. **46** (1970), 1074–1079.
- [15] Tanaka Y., *On sequential spaces*, Science Reports of Tokyo Kyoiku Daigaku Sect. A, **11** (1971), 68–72.
- [16] Tanaka Y., *Products of sequential spaces*, Proc. Amer. Math. Soc. **54** (1976), 371–375.
- [17] Tanaka Y., *A characterization for the products of k -and- \aleph_0 -spaces and related results*, Proc. Amer. Math. Soc. **59** (1976), 149–155.
- [18] Tanaka Y., *On the k -ness for the products of closed images of metric spaces*, Gen. Topology Appl. **9** (1978), 175–183.
- [19] Tanaka Y., *Point-countable k -systems and products of k -spaces*, Pacific J. Math. **101** (1982), 199–208.
- [20] Tanaka Y., *Metrizability of certain quotient spaces*, Fund. Math. **119** (1983), 157–168.
- [21] Tanaka Y., *On the products of k -spaces questions*, Questions Answers Gen. Topology **1** (1983), 36–50.
- [22] Tanaka Y., *k -spaces and the products of closed images*, Questions Answers Gen. Topology **1** (1983), 88–99.
- [23] Tanaka Y., *Point-countable covers and k -networks*, Topology Proc. **12** (1987), 327–349.
- [24] Tanaka Y., *Products of k -spaces having point-countable k -networks*, Topology Proc. **22** (1997), 305–329.
- [25] Tanaka Y., Shimizu Y., *Products of k -spaces, and special countable spaces*, to appear in Tsukuba J. Math.

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