Uniform approximation of continuous functions on compact sets by biharmonic functions

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Abstract. We give a characterization of functions that are uniformly approximable on a compact subset K of $\mathbb{R}^n$ by biharmonic functions in neighborhoods of K.

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1. Introduction

Debiard and Gaveau proved the following [3]:

Theorem. *Let K be a compact subset of $\mathbb{R}^n$ and f a real function on K. Then the following statements are equivalent:*

1. *There exists a sequence (f_n) of harmonic functions on neighborhoods of K which converges uniformly to f in K.*
2. *f is continuous on K and finely harmonic on K', the fine interior of K.*

This result has been later extended to closed subsets of $\mathbb{R}^n$ by Gauthier and Ladouceur [7].

If we denote by $\mathcal{H}(K)$ the space of real functions that are restrictions to K of harmonic functions on neighborhoods of K, then the equivalence between conditions 1 and 2 in the above theorem means that the closure $\overline{\mathcal{H}(K)}$ of $\mathcal{H}(K)$ in $\mathcal{C}(K)$ under the uniform norm is the space of continuous functions on K that are finely harmonic in K'.

Our main purpose in this work is to extend the above theorem to functions that are uniformly approximable on a compact set K in $\mathbb{R}^n$ by restrictions to K of biharmonic functions on neighborhoods of K. More precisely, let $\mathcal{B}H(K)$ be the set of restrictions to K of biharmonic functions in neighborhoods of K endowed with the norm

$$\|f\| = \sup_{x\in K} |f(x)| + \sup_{x\in K} |\Delta f(x)|.$$

We shall prove that the completion of $\mathcal{B}H(K)$ under the norm $\|\cdot\|$ is exactly the space of continuous functions on K which are finely biharmonic in K' and whose fine Laplacian in K' can be extended continuously to K.

We recall here that the fine topology on $\mathbb{R}^n$ is the coarsest one, making all superharmonic functions in $\mathbb{R}^n$ continuous. We will use the word fine (finely) to distinguish between the notions related to the initial (euclidean) topology from those related to the fine topology. The fine topology on $\mathbb{R}^n$ has been extensively studied by Fuglede in many papers, where he shows in particular that it has nice properties such as local connectedness which allowed him to develop a beautiful (fine) potential theory on finely open sets (see [5]).

The word function always means, unless otherwise stated, a function with values in $\overline{\mathbb{R}}$. The order on the set of pairs of functions on a set M is the usual order:

$$(f,g) \le (h,k) \iff f \le h \ \text{ and } \ g \le k.$$

We also write $(h,k) \ge (f,g)$ instead of $(f,g) \le (h,k)$. If $(f,g) \ge (0,0)$, we simply write $(f,g) \ge 0$. If A is a subset of $\mathbb{R}^n$, we denote by $\overline{A}$ the closure of A in the Alexandroff compactification of $\mathbb{R}$.

2. Biharmonic measures

For the definition of finely biharmonic functions we need to use the notion of biharmonic measures on finely open subsets of $\mathbb{R}^n$. The definition of these measures is based on a result from the general theory of biharmonic spaces of Smyrnelis ([11] and [12]).

Let $(X, \mathcal{H})$ be a biharmonic space in the sense of Smyrnelis [11] and denote by $\mathcal{U}^+(X)$ the convex cone of $\mathcal{H}$-hyperharmonic pairs ≥ 0 on X. For every pair $\Phi = (f,g)$ of functions on X and every subset E of Ω, we denote by $\Phi^E = (\Phi_1^E, \Phi_2^E)$ the reduced pair of Φ relative to E. We recall that this pair is defined by

$$\Phi^E = \inf\{(u,v) \in \mathcal{U}^+(X);\ (u,v) \ge \Phi \ \text{ on } \ E\},$$

where the infimum is taken in the sense of order product. This pair is sometimes denoted by R_Φ^E. The balayage of Φ on E is denoted by $\widehat{\Phi}^E$ or $\widehat{R}_\Phi^E$ and defined by $\widehat{\Phi}^E = (\widehat{\Phi}_1^E, \widehat{\Phi}_2^E)$, where, for a function h on X, $\widehat{h}$ denotes the l.s.c. (lower semicontinuous) regularization of h, that is, the greatest l.s.c. minorant of h in X. We remark that we have $\Phi^E = (\Phi^+)^E$, where $\Phi^+ = \max(\Phi, 0)$.

As in the theory of harmonic spaces, it is the notion of balayage of a pair of measures which allows us to define finely hyperharmonic, superharmonic or biharmonic pairs of functions. That is why we recall the following result ([12, Theorem 7.11 and Theorem 7.12]):

Theorem 2.1. *For every pair (μ, λ) of positive Radon measures on X and every subset E of X, there exist three positive Radon measures μ^E, ν^E and λ^E on X*

such that, for every $\mathcal{H}$-potential $P = (p,q)$, one has

$$\int \widehat{P}_1^E \, d\mu = \int p \, d\mu^E + \int q \, d\nu^E,$$

$$\int \widehat{P}_2^E \, d\lambda = \int q \, d\lambda^E,$$

where $\widehat{P}^E = (\widehat{P}_1^E, \widehat{P}_2^E)$.

Remarks. 1. The above relations are true for any pair $P = (p,q) \in \mathcal{U}^+(X)$. This can be seen by realising that every pair $P \in \mathcal{U}^+(X)$ is the supremum of an increasing sequence (P_n) of $\mathcal{H}$-potentials in X.

2. The measures μ^E and λ^E are just the balayages of the measures μ and λ with respect to the harmonic spaces associated with the biharmonic space $(X,\mathcal{H})$ (see [11], [12] and the proof of Proposition 2.2 below). If these spaces are identical as in the case that will be considered in the sequel, one has $\mu^E = \lambda^E$ when $\mu = \lambda$.

When $\mu = \lambda = \varepsilon_x$, where ε_x denotes the Dirac measure in $x \in X$, we denote the corresponding measures μ^E, ν^E and λ^E in the above theorem by μ_x^E, ν_x^E and λ_x^E respectively. These are the measures which allow us to define finely biharmonic and finely hyperharmonic or superharmonic pairs of functions. Let us recall that these notions have been introduced and studied in [4] where we refer for more details.

Now let Ω be a domain in $\mathbb{R}^n$, $n \geq 1$. We shall deal with the biharmonic sheaf $\mathcal{H}_\Delta$ on Ω defined by the Laplacian:

$$\mathcal{H}_\Delta(\omega) = \{(u,v) \in [\mathcal{C}^2(\omega)]^2 : \Delta u = -v, \Delta v = 0\},$$

for any open subset ω of Ω. The pair $(\Omega, \mathcal{H}_\Delta)$ is a biharmonic space whose harmonic spaces associated are identical to the classical one defined by the Laplacian on Ω. An $\mathcal{H}_\Delta$-biharmonic (superharmonic) pair will be simply called a biharmonic (superharmonic) pair.

We say that Ω is strong if there exists a pair (p,q) of Green potentials on Ω such that $q > 0$ and $\Delta p \leq -q$ on Ω in the sense of distributions. We recall, following [3], that the biharmonic space $(\mathbb{R}^n, \mathcal{H}_\Delta)$ is strong if and only if $n \geq 5$. However, if Ω is bounded, the biharmonic space $(\Omega, \mathcal{H}_\Delta)$ is strong for every $n \geq 1$. In the following we assume that Ω is strong. Then Ω possesses a Green kernel that will be denoted by $G(x,y) = G_\Omega(x,y)$.

Proposition 2.2. *For any relatively compact finely open set ω, $\bar{\omega} \subset \Omega$, and any $x \in \omega$, one has $\mu_x^{C\omega} = \lambda_x^{C\omega} = \varepsilon_x^{C\omega}$, where $\varepsilon_x^{C\omega}$ is the balayage of the measure ε_x on $C\omega$ in the classical harmonic space associated with the Laplace operator.*

Proof: By applying Theorem 2.1 to the pairs $P = (p,0)$, where p is an arbitrary potential on Ω, we see that $\mu_x^{C\omega} = \varepsilon_x^{C\omega}$. To establish the relation $\lambda_x^{C\omega} = \varepsilon_x^{C\omega}$, it suffices to use the above lemma and observe that for any $\mathcal{H}$-potential $P = (p,q)$, the function P_2^E is just the reduced function of q relative to E. □

Lemma 2.3. *For any finely open set ω of Ω and any $x \in \omega$, we have $\int d\nu_x^{C\omega} > 0$.*

Proof: It follows easily from [4, Theorem 9.1], that the pair $(\int d\nu_\cdot^{C\omega}, 1)$ is non-negative finely superharmonic, not identically 0 in each finely connected component of ω, hence $\int d\nu_x^{C\omega} > 0$ for any $x \in \omega$.

□

For every finely open V we denote by $\partial_f V$ the fine boundary of V and by $\widetilde{V}$ its fine closure. It is well known that if a finely open set ω is regular, then the measure $\varepsilon_x^{C\omega}$ is supported by $\partial_f \omega$ (see [5]). According to [4, Theorem 9.4], the measure $\nu_x^{C\omega}$ is also supported by $\partial_f \omega$.

3. Finely biharmonic functions

Let U be a finely open subset of $\mathbb{R}^n$. We recall that a function $f : U \longrightarrow \mathbb{R}$ is said to be finely harmonic in U if

1. f is finely continuous in U,
2. for every $x \in U$, there exists a relatively compact finely open fine neighborhood ω of x such that $\overline{\omega} \subset U$, f is bounded on $\overline{\omega}$ and

$$f(x) = \int f \, d\varepsilon_x^{C\omega}.$$

Now let us consider the family $D(U)$ of finely continuous functions f on U such that the limit

$$Lf(x) = \lim_{\omega \downarrow x} \frac{f(x) - \int f \, d\varepsilon_x^{C\omega}}{\int d\nu_x^{C\omega}}$$

exists and is finite for every $x \in U$ (the fraction $\frac{f(x) - \int f \, d\varepsilon_x^{C\omega}}{\int d\nu_x^{C\omega}}$ is well defined by Lemma 2.3).

Definition 3.1. A finely continuous function on U is said to be finely biharmonic on U if $f \in D(U)$ and Lf is finely harmonic on U.

We also recall the definition of finely biharmonic pairs in a finely open subset of $\mathbb{R}^n$. This notion has been introduced and studied in [4].

Definition 3.2. A pair (u, v) of functions on U is said to be finely biharmonic in U if u and v are finely continuous with values in $\mathbb{R}$ and if for every relatively compact, finely open neighborhood $\omega \subset \bar{\omega} \subset U$, such that u and v are bounded on $\overline{\omega}$,

$$u(y) = \int u \, d\varepsilon_y^{C\omega} + \int v \, d\nu_y^{C\omega} \quad \text{and} \quad v(y) = \int v \, d\varepsilon_y^{C\omega}$$

for every $y \in \omega$.

The following propositions underline the link between the notion of finely biharmonic functions in the sense of Definition 3.1 and the notion of finely biharmonic pairs in the sense of [4]:

Proposition 3.3. *If a pair (u, v) is finely biharmonic in a finely open set U, then $u \in D(U)$ and $Lu = v$.*

Proof: Let $x \in U$ and $\varepsilon > 0$. Since v is finely continuous, there exists a finely open $\omega_0 \subset U$ such that $x \in \omega_0$ and $|v(x) - v(y)| < \varepsilon$ for any $y \in \omega_0$. Then, for any finely open $\omega \subset \widetilde{\omega} \subset \omega_0$, $x \in \omega$, we have

$$|u(x) - \int u \, d\varepsilon_x^{C\omega} - v(x) \int d\nu_x^{\omega}| < \varepsilon \int d\nu_x^{\omega}$$

and therefore $u \in D(U)$ and $Lu = v$. □

Conversely, we have:

Proposition 3.4. *A function $f \in D(U)$ is finely biharmonic in U if and only if the pair (f, Lf) is finely biharmonic on U.*

Proof: The "if" part is immediate from the definitions of biharmonic pairs and biharmonic functions and the above proposition. Conversely, let us suppose that the function f is finely biharmonic in U. Then the function Lf is finely harmonic in U. Let $x \in U$ and let V be a relatively compact fine domain in Ω such that $x \in V \subset \overline{V} \subset U$. Denote by $\mathcal{V}$ the potential kernel defined on V by

$$\mathcal{V}g = \int G_V(., y) g(y) \, dy,$$

where G_V is the Green kernel of V. By Proposition 7.11 and Theorem 7.13 of [4], the pair $(\mathcal{V}(Lf), Lf)$ is finely biharmonic in V. Hence, if ω is a relatively compact finely open subset of Ω such that $x \in \omega \subset \overline{\omega} \subset V$, we have

$$\int (f - \mathcal{V}(Lf)) \, d\varepsilon_x^{C\omega} = \int f \, d\varepsilon_x^{C\omega} - \mathcal{V}(Lf)(x) + \int Lf \, d\nu_x^{C\omega}.$$

Hence $L(f - \mathcal{V}(Lf))(x) = 0$. As x is arbitrary, we deduce from the next lemma that $f - \mathcal{V}f$ is finely harmonic in U; this shows that the pair (f, Lf) is finely biharmonic and the functions f and Lf are finely continuous in U. □

Lemma 3.5. *Let $f \in D(U)$ be finite and such that $Lf = 0$. Then f is finely harmonic in U.*

Proof: We have $L(f + \varepsilon \mathcal{V}1) = \varepsilon > 0$. Then for every $x \in U$ and every finely open neighborhood ω of x, there exists an open fine neighborhood ω' of x such that

$$f(x) + \varepsilon \mathcal{V}1(x) \geq \int (f + \varepsilon \mathcal{V}1) \, d\varepsilon_x^{C\omega'}.$$

Since x is arbitrary, we deduce from the definition of finely hyperharmonic functions (see [5]) that the function $f + \varepsilon \mathcal{V}1$ is finely superharmonic in U. Letting $\varepsilon \to 0$, we get that f is finely superharmonic in U. Replacing f by $-f$ we obtain the desired conclusion. □

4. Approximation of continuous functions by biharmonic functions

For any bounded open subset V of $\mathbb{R}^n$, we denote by G_V the Green kernel of V normalized in such a way that for every $y \in V$, we have $\Delta G_V(.,y) = -\varepsilon_y$. Let $\mathcal{V}_V$ be the potential kernel on V defined for a bounded Borel function f on V by

$$\mathcal{V}_V(f)(x) = \int G_W(x,y) f(y)\, dy$$

for any connected component W of V and any $x \in W$. Then, for any bounded harmonic function k on V, the function $\mathcal{V}_V(k)$ is biharmonic in V.

Lemma 4.1. *For every function g continuous on K and finely harmonic in K', the function $x \mapsto \int g\, d\nu_x^{CK'}$ can be extended to a continuous function on K.*

Proof: Assume first that $g \geq 0$. By Debiard-Gaveau's theorem, there exists a sequence (g_n) of harmonic functions on neighborhoods V_n such that $\overline{V}_{n+1} \subset V_n$ for every n and $\bigcap_n V_n = K$, which converges uniformly on K to g. Fix an integer n and let $m \geq n$. Then the pair $(V_m g_n, g_n)$ is biharmonic in V_m. Hence we have

$${}^nR^{CK}_{(\mathcal{V}_m g_n, g_n)} = \sup_{p \geq m} {}^nR^{CU_p}_{(\mathcal{V}_m g_n, g_n)},$$

and therefore

$$\int g_n\, d\nu_x^{CK} = \sup_{p \geq n} g_n d\nu_x^{CU_p}$$

for every $x \in K'$, where $\mathcal{V}_n = \mathcal{V}_{V_n}$. Here we have denoted by ${}^nR_f$ the reduced function of f relative to V_n. But the left hand side of the last equality is l.s.c. at x in K. This shows that the function $\int g_n\, d\nu_.^{CK}$ is l.s.c. in K. On the other hand, for $p \geq n$ we have:

$$\mathcal{V}_n 1 = \int \mathcal{V}_n 1\, d\varepsilon_.^{CU_p} + \int d\nu_.^{CU_p},$$

because the pair $(\mathcal{V}_n 1, 1)$ is biharmonic in a neighborhood of $\overline{V}_p$. Letting $p \to +\infty$, we obtain

$$\mathcal{V}_n 1 = \int \mathcal{V}_n 1\, d\varepsilon_.^{CK} + \int d\nu_.^{CK}.$$

The functions $\int \mathcal{V}_n 1\, d\mu_.^{CK}$ and $\int d\nu_.^{CK}$ are l.s.c. in K, and $\mathcal{V}_n 1$ is continuous in K, hence $\int d\nu_.^{CK}$ is continuous in K. The function g_n is continuous in $\overline{V}_{n+1}$. Thus it is bounded in V_{n+1}, and by multiplying it by a positive constant we can assume that $g_n \leq 1$ in V_{n+1}. By applying the above result concerning g_n to $1 - g_n$, we deduce that $\int 1\, d\nu_.^{CK} - \int g_n\, d\nu_.^{CK}$ is l.s.c. and therefore $\int g_n\, d\nu_.^{CK}$ is continuous in K. Since (g_n) converges uniformly to g in K and the measures

$\nu_{\cdot}^{CK}$ are of total mass bounded by $\sup_{x\in K}\mathcal{V}_1 1(x) < +\infty$ and supported by K, we conclude that the sequence $(\int g_n\, d\nu_{\cdot}^{CK})$ converges uniformly to a continuous function on K which equals $\int g\, d\nu_{\cdot}^{CK'}$ in K'. The general case can be obtained by adding to g a constant $c > 0$ such that $g + c \geq 0$ in K, and applying the above case to $g + c$. $\square$

Now we can prove the main theorem of this work:

Theorem 4.2. *Let f be a real function on a compact set K. Then the following statements are equivalent:*

1. *There exists a sequence (h_n) of biharmonic functions, each defined on an open neighborhood of K, such that (h_n) converges uniformly on K to f and (Δh_n) converges uniformly on K to a continuous function g.*
2. *f is continuous on K and finely biharmonic on K', and Lf can be extended continuously to K.*

Proof: 1. $\Longrightarrow$ 2: Since the pairs $(h_n, -\Delta h_n)$ are finely biharmonic in K' and converge uniformly in K, it results from the definition of biharmonic pairs ([4]) that the pair (f, g) is finely biharmonic in K', and clearly continuous on K.

2. $\Longrightarrow$ 1: Let g be a continuous extension of $-Lf$ to K and let (V_n) and (g_n) be as in the proof of the above lemma. The function $f + \int g\, d\nu_x^{K'}$ is finely harmonic in K'. On the other hand it follows from the above lemma that the function $f - \int g\, d\nu_{\cdot}^{K'}$ is the restriction to K' of a continuous function h in K. Then, by Debiard-Gaveau's theorem there exists a sequence (k_n) of functions such that, for every n, k_n is harmonic on an open neighborhood $U_n \subset \overline{U}_n \subset V_n$ of K, and (k_n) converges uniformly in K to h. The functions $k_n - \int g_n\, d\nu_{\cdot}^{CU_n}$ are biharmonic on U_n and converge uniformly in K to f, and we have seen that the harmonic functions $\Delta(k_n - \int g_n\, d\nu_{\cdot}^{CU_n}) = g_n$ converge uniformly on K to g. $\square$

The space $\overline{\mathcal{H}(K)}$ is identical with the space of finely harmonic functions on K' with a continuous extension to K. The above theorem can be stated as follows:

Theorem 4.2$'$. *Let f be a real function on a compact set K. Then the following statements are equivalent:*

1. *There exists a sequence (h_n) of biharmonic functions, each defined on an open neighborhood of K, such that (h_n) converges uniformly on K to f and (Δh_n) converges uniformly on K to a continuous function g.*
2. *f is continuous on K, finely biharmonic on K', and $Lf \in \overline{\mathcal{H}(K)}$.*

Corollary 1. *A function f on U is finely biharmonic if and only if, for every point $x \in U$, there exist a compact finely open neighborhood $K \subset U$ and a sequence (h_n) of biharmonic functions in neighborhoods of K such that (h_n) converges uniformly on K to f and the sequence (Δh_n) converges uniformly on K to a continuous function g.*

Corollary 2. *A pair of functions (f, g) on U is finely biharmonic if and only if, for every point $x \in U$, there exist a compact finely open neighborhood $K \subset U$ and a sequence (h_n, k_n) of biharmonic pairs of functions in neighborhoods of K such that (h_n) and (k_n) converge uniformly on K to f and g, respectively.*

5. Concluding remarks

Let Ω be a regular relatively compact open subset of $\mathbb{R}^n$. If (h_n) is a sequence of biharmonic functions in neighborhoods of $\overline{\Omega}$ which converges uniformly on $\overline{\Omega}$ to a function h, then h is obviously continuous in $\overline{\Omega}$. On the other hand, it follows from the mean value property of biharmonic functions that

$$h_n(x) = \frac{1}{|B|} \int_B h(y)\, dy - \frac{r^2}{2(n+1)} \Delta h_n(x)$$

for all balls $B \subset \overline{B} \subset \Omega$ of center x and radius $r > 0$, where $|B|$ denotes the volume of B, that the sequence of harmonic functions (Δh_n) converges locally uniformly in Ω to a harmonic function k and we have $\Delta h = k$ in Ω so that h is biharmonic in Ω.

This result leads to the following question: Let (h_n) be a sequence of biharmonic functions in neighborhoods of a compact set K of $\mathbb{R}^n$ which converges uniformly to a function h on K. Is h finely biharmonic in K'?

The answer to this question is not always positive. Indeed, let Ω be the Lebesgue spine at 0 in $\mathbb{R}^3$ (see [8, p. 175]) and $U = (\Omega \cup \{0\}) \cap B$, where B is the unit ball of $\mathbb{R}^3$, and let (h_n) be the sequence of finely biharmonic functions defined in U by

$$h_n(x) = 1 - \|x - (\frac{1}{n}, 0, 0)\|$$

for all $n \in \mathbb{N}^*$. Then the sequence (h_n) converges locally uniformly to the function h defined in U by $h(x) = 1 - \|x\|$. However the function h is not finely biharmonic in U because we have $\Delta h(x) = \frac{2}{\|x\|}$ for all $x \in U \setminus \{0\}$ and the function h is not bounded in fine neighborhoods of 0. This example shows that if we do not assume that the sequence (Δh_n) converges locally finely uniformly then the sequence (h_n) need not converge to a finely biharmonic function, hence the assumption of Theorem 4.2 that Δh_n converges to a continuous function g in K is necessary.

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