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Topological modules over strictly minimal topological rings

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Abstract. We study the validity of two basic results of the classical theory of topological vector spaces in the context of topological modules.

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The starting-point for this paper are the following facts:

(a) for every topological vector space E over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) and for every non-zero linear form φ on E, we have that φ is continuous if and only if its kernel is closed;

(b) for every positive integer n and for every n-dimensional separated topological vector space E over $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we have that every vector space isomorphism from $\mathbb{K}^n$ into E is a homeomorphism.

Various authors have established results in the same spirit of (b); see the Historical Notes of [3]. Finally, Nachbin [2] has found the widest classes of topological division rings for which (a) and (b) hold.

The purpose of this paper, which is strongly influenced by that of Nachbin, is to discuss the validity of (a) and (b) in the context of topological modules. In particular, it is shown that (a) and (b) hold for topological modules over the topological ring $\mathbb{Z}_p$ of p-adic integers.

Throughout this paper (R, τ_R) denotes a separated topological ring, where R is a ring with a non-zero identity element 1, and all modules under consideration are unitary left R-modules.

Consider R with its canonical left R-module structure and let θ be a topology on R such that (R, θ) is a topological (R, τ_R)-module. Then θ is coarser than τ_R. In fact, the continuity of the mapping

$$(\lambda, \mu) \in (R \times R, \tau_R \times \theta) \mapsto \lambda\mu \in (R, \theta)$$

implies the continuity of the identity mapping $1_R\colon (R, \tau_R) \to (R, \theta)$ (put $\mu = 1$). This observation motivates the following

Definition 1. (R, τ_R) is said to be *strictly minimal* if there is no separated (R, τ_R)-module topology on R which is strictly coarser than τ_R (R endowed with its canonical left R-module structure).

To say that (R, τ_R) is strictly minimal is equivalent to saying that, for every separated topology θ on R such that (R, θ) is a topological (R, τ_R)-module, we have $\theta = \tau_R$.

Remark 2. In the special case where (R, τ_R) is a topological division ring the notion just defined is due to Nachbin [2], who proved that every non-discrete valuable topological division ring is strictly minimal. In particular, every separated non-discrete locally compact topological division ring is strictly minimal.

Example 3. If (R, τ_R) is compact, then (R, τ_R) is obviously strictly minimal. In particular, the topological ring $\mathbb{Z}_p$ of p-adic integers is strictly minimal and every finite discrete ring with a non-zero identity element is strictly minimal.

Theorem 4. *The following conditions are equivalent:*

(a) (R, τ_R) *is strictly minimal;*

(b) *for every separated topological* (R, τ_R)*-module* (F, τ') *which is a free* R*-module with a basis of* 1 *element, every* R*-module isomorphism from* R *into* F *is a homeomorphism from* (R, τ_R) *into* (F, τ')*;*

(c) *for every free* R*-module* F *with a basis of* 1 *element, there is only one separated* (R, τ_R)*-module topology on* F*;*

(d) *for every topological* (R, τ_R)*-module* (E, τ) *and for every separated topological* (R, τ_R)*-module* (F, τ') *which is a free* R*-module with a basis of* 1 *element, every surjective* R*-linear mapping from* E *into* F *with a closed kernel is continuous;*

(e) *for every topological* (R, τ_R)*-module* (E, τ) *and for every separated topological* (R, τ_R)*-module* (F, τ') *which is a free* R*-module with a basis of* 1 *element, every* R*-linear mapping from* E *into* F *with a closed graph is continuous.*

In order to prove the theorem we shall need two lemmas which are valid for arbitrary (R, τ_R). But, before we proceed, let us recall that if (X, τ) is a topological space, Y is a set and $f: X \to Y$ is a surjective mapping, then the *direct image of* τ *under* f is the topology τ_f on Y defined as follows: for $Z \subset Y$, $Z \in \tau_f$ if $f^{-1}(Z) \in \tau$.

Lemma 5. *Let* (E, τ) *be a topological* (R, τ_R)*-module,* F *an* R*-module and* $u: E \to F$ *a surjective* R*-linear mapping. Then* τ_u*, the direct image of* τ *under* u*, is an* (R, τ_R)*-module topology on* F*. Moreover,* τ_u *is separated if and only if the kernel* $\mathrm{Ker}(u)$ *of* u *is* τ*-closed. In particular, if* M *is a submodule of* E *and* τ_1 *is the quotient topology on* E/M*, then* $(E/M, \tau_1)$ *is a topological* (R, τ_R)*-module which is separated if and only if* M *is* τ*-closed.*

Proof: First, let us observe that a subset Y of F is τ_u-open if and only if there is a τ-open subset X of E such that $u(X) = Y$. In fact, if Y is τ_u-open, then $X = u^{-1}(Y)$ is τ-open and $u(X) = Y$. Conversely, if X is τ-open and $u(X) = Y$, then the equality

$$u^{-1}(Y) = \operatorname{Ker}(u) + X = \bigcup_{t \in \operatorname{Ker}(u)} (t + X)$$

implies that Y is τ_u-open. By using this fact it is easily verified that (F, τ_u) is a topological (R, τ_R)-module.

Finally, $\operatorname{Ker}(u)$ is obviously τ-closed if τ_u is separated. Conversely, if $\operatorname{Ker}(u)$ is τ-closed, then $u(C_E \operatorname{Ker}(u))$ is τ_u-open. But $\{0\} = C_F\, u(C_E \operatorname{Ker}(u))$, from which we conclude that τ_u is separated. $\square$

Lemma 6. *Let (F, τ') be a topological (R, τ_R)-module. If for every topological (R, τ_R)-module (E, τ) we have that every surjective R-linear mapping from E into F with a closed kernel is continuous, then for every topological (R, τ_R)-module (E, τ) we have that every R-linear mapping from E into F with a closed graph is continuous.*

Proof: Let (E, τ) be a topological (R, τ_R)-module and let $u: E \to F$ be an R-linear mapping whose graph $\operatorname{Gr}(u)$ is $(\tau \times \tau')$-closed. Define $v: E \times F \to F$ by $v(x, y) = u(x) - y$. Then v is a surjective R-linear mapping such that $\operatorname{Ker}(v) = \operatorname{Gr}(u)$. By hypothesis, $v: (E \times F, \tau \times \tau') \to (F, \tau')$ is continuous. Therefore $u: (E, \tau) \to (F, \tau')$ is continuous because $u(x) = v(x, 0)$ for all $x \in E$. $\square$

Let us now turn to the

Proof of Theorem 4: Let us prove that (a) implies (b). Indeed, let (F, τ') be as in (b) and let $u: R \to F$ be an R-module isomorphism. Then $u(\lambda) = \lambda u(1)$ for all $\lambda \in R$, and thus u is obviously continuous from (R, τ_R) into (F, τ'). Let θ be the inverse image of τ' under u; by Theorem 12.5 of [3], (R, θ) is a topological (R, τ_R)-module, θ being clearly separated. By hypothesis, $\theta = \tau_R$, and hence $u^{-1}: (F, \tau') \to (R, \tau_R)$ is continuous. Therefore $u: (R, \tau_R) \to (F, \tau')$ is a homeomorphism, proving (b).

Let us prove that (b) implies (c). Indeed, let F be as in (c) and let τ_1 and τ_2 be two separated (R, τ_R)-module topologies on F. If $u: R \to F$ is an R-module isomorphism, then $u: (R, \tau_R) \to (F, \tau_1)$ and $u: (R, \tau_R) \to (F, \tau_2)$ are homeomorphisms by (b), from which we conclude that $1_F: (F, \tau_1) \to (F, \tau_2)$ is a homeomorphism. Therefore $\tau_1 = \tau_2$, proving (c).

Let us prove that (c) implies (d). Indeed, let (E, τ) and (F, τ') be as in (d) and let $u: E \to F$ be a surjective R-linear mapping whose kernel $\operatorname{Ker}(u)$ is τ-closed. By Lemma 5, (F, τ_u) is a separated topological (R, τ_R)-module. By (c), $\tau_u = \tau'$, which implies that $u: (E, \tau) \to (F, \tau')$ is continuous. This proves (d).

Finally, since Lemma 6 guarantees that (d) implies (e), it remains to prove that (e) implies (a). Indeed, assume that (R, τ_R) is not strictly minimal. Then there exists a separated (R, τ_R)-module topology θ on R which is strictly coarser than τ_R; thus $1_R\colon (R, \theta) \to (R, \tau_R)$ is discontinuous. On the other hand, if $\lambda, \mu \in R$ and $\lambda \neq \mu$, there are a θ-neighborhood U of λ in R and a θ-neighborhood V of μ in R so that $U \cap V = \emptyset$. Since θ is coarser than τ_R, $U \times V$ is a $(\theta \times \tau_R)$-neighborhood of (λ, μ) in $R \times R$ with $(U \times V) \cap \mathrm{Gr}(1_R) = \emptyset$. Therefore $\mathrm{Gr}(1_R)$ is $(\theta \times \tau_R)$-closed. This completes the proof. □

Corollary 7. *Assume that (R, τ_R) is strictly minimal and let (E, τ) be a separated topological (R, τ_R)-module. Let M and N be two submodules of E such that N is a free R-module with a basis of 1 element, M is τ-closed and $E = M \oplus N$. Then (E, τ) is the topological direct sum of M and N.*

Proof: Let $p_N\colon E \to N$ be the projection of E onto N along M. Then $\mathrm{Ker}(p_N) = M$ is τ-closed by hypothesis. By Theorem 4, $p_N\colon (E, \tau) \to (N, \tau_N)$ is continuous, τ_N being the separated (R, τ_R)-module topology induced by τ on N. This completes the proof.

□

Corollary 8. *Assume that (R, τ_R) is strictly minimal. Let (E, τ) be a topological (R, τ_R)-module and let (F, τ') be a separated topological (R, τ_R)-module which is a free R-module with a basis of 1 element. If $u\colon (E, \tau) \to (F, \tau')$ is a surjective continuous R-linear mapping, then u is open.*

Proof: Let $\pi\colon E \to E/\mathrm{Ker}(u)$ be the canonical surjection and let $\tilde{u}\colon E/\mathrm{Ker}(u) \to F$ be the unique R-module isomorphism such that $u = \tilde{u} \circ \pi$. If τ_1 denotes the quotient topology on $E/\mathrm{Ker}(u)$, then $(E/\mathrm{Ker}(u), \tau_1)$ is a separated topological (R, τ_R)-module by Lemma 5, $E/\mathrm{Ker}(u)$ being a free R-module with a basis of 1 element. By Theorem 4, $\tilde{u}\colon (E/\mathrm{Ker}(u), \tau_1) \to (F, \tau')$ is a homeomorphism. Therefore $u\colon (E, \tau) \to (F, \tau')$ is open because $\pi\colon (E, \tau) \to (E/\mathrm{Ker}(u), \tau_1)$ is open. □

Corollary 9 ([2])**.** *If (R, τ_R) is a topological division ring, then the following conditions are equivalent:*

- (a) *(R, τ_R) is strictly minimal;*
- (b) *for every one-dimensional separated topological vector space (F, τ') over (R, τ_R), every R-vector space isomorphism from R into F is a homeomorphism from (R, τ_R) into (F, τ');*
- (c) *for every one-dimensional vector space F over R, there is only one separated (R, τ_R)-vector topology on F;*
- (d) *for every topological vector space (E, τ) over (R, τ_R) and for every one-dimensional separated topological vector space (F, τ') over (R, τ_R), every non-zero R-linear mapping from E into F with a closed kernel is continuous;*

(e) *for every topological vector space* (E, τ) *over* (R, τ_R) *and for every one-dimensional separated topological vector space* (F, τ') *over* (R, τ_R), *every* R*-linear mapping from* E *into* F *with a closed graph is continuous.*

Proof: If E and F are two vector spaces over R, F being one-dimensional, and if $u: E \to F$ is an R-linear mapping such that $u \neq 0$, then u is necessarily surjective. In view of this obvious remark, the result follows immediately from Theorem 4. □

Corollary 10. *Let* (E, τ) *be a topological vector space over a strictly minimal topological division ring* (R, τ_R). *In order that a hyperplane* $H = \{x \in E; \varphi(x) = \lambda\}$ *in* E *be* τ*-closed, it is necessary and sufficient that the non-zero linear form* φ *on* E *be continuous.*

Proof: The necessity of the condition follows immediately from Corollary 9 because we may assume $\lambda = 0$. The sufficiency of the condition is obvious. □

Theorem 11. *Assume that* (R, τ_R) *is complete and strictly minimal. Then, for every* $n \in \mathbb{N}^*$ *and for every separated topological* (R, τ_R)*-module* (F, τ') *which is a free* R*-module with a basis of* n *elements, every* R*-module isomorphism from* R^n *into* F *is a homeomorphism,* R^n *being endowed with the product topology.*

Proof: We shall argue by induction on n, the case $n = 1$ being a consequence of Theorem 4. Let $n \geq 2$ and assume the result true for $n - 1$. Let (F, τ') be a separated topological (R, τ_R)-module which is a free R-module with a basis of n elements, and let $u: R^n \to F$ be an R-module isomorphism. Let $e_1 = (1, 0, \ldots, 0)$, $e_2 = (0, 1, 0, \ldots, 0), \ldots, e_n = (0, 0, \ldots, 0, 1)$ and, for each $i = 1, \ldots, n$, put $f_i = u(e_i)$; then $\{f_1, \ldots, f_n\}$ is a basis of F. Let M be the submodule of F generated by $f_1, \ldots, f_{n-1}$ and let N be the submodule of F generated by f_n ; then $F = M \oplus N$. Since the mapping

$$(\lambda_1, \ldots, \lambda_{n-1}) \in R^{n-1} \mapsto \sum_{i=1}^{n-1} \lambda_i f_i \in M$$

is an R-module isomorphism, the induction hypothesis implies that it is a homeomorphism, R^{n-1} being endowed with the product topology and M with the topology induced by τ'. Therefore M is τ'-complete because R^{n-1} is complete ((R, τ_R) is complete), and hence M is τ'-closed. By Corollary 7, (F, τ') is the topological direct sum of M and N. Consequently, the mapping

$$(\lambda_1, \ldots, \lambda_{n-1}, \lambda_n) \in R^{n-1} \times R \mapsto \sum_{i=1}^{n} \lambda_i f_i \in F$$

is a homeomorphism, that is, u is a homeomorphism. This completes the proof. □

Corollary 12. *Let (R, τ_R) be as in Theorem 11. Then, for every $n \in \mathbb{N}^*$ and for every free R-module F with a basis of n elements, there is only one separated (R, τ_R)-module topology on F.*

Proof: Let $n \in \mathbb{N}^*$ and F a free R-module with a basis of n elements, and let τ_1 and τ_2 be two separated (R, τ_R)-module topologies on F. If $u: R^n \to F$ is an R-module isomorphism, then $u: R^n \to (F, \tau_1)$ and $u: R^n \to (F, \tau_2)$ are homeomorphisms by Theorem 11. Therefore the identity mapping $1_F: (F, \tau_1) \to (F, \tau_2)$ is a homeomorphism, that is, $\tau_1 = \tau_2$. This completes the proof. $\square$

Corollary 13. *Let (R, τ_R) be as in Theorem 11 and let (E, τ) be a separated topological (R, τ_R)-module. If M is a submodule of E which is a free R-module with a basis of n elements ($n \in \mathbb{N}$), then M is τ-closed.*

Proof: We may assume that $n \in \mathbb{N}^*$. Let τ_M be the topology induced by τ on M. If $u: R^n \to M$ is an R-module isomorphism, then $u: R^n \to (M, \tau_M)$ is a homeomorphism by Theorem 11. Therefore M is τ-complete, hence τ-closed, as asserted. $\square$

Corollary 14. *Let (R, τ_R) be as in Theorem 11. Then, for every topological (R, τ_R)-module (E, τ), for every $n \in \mathbb{N}^*$ and for every separated topological (R, τ_R)-module (F, τ') which is a free R-module with a basis of n elements, we have that every surjective R-linear mapping from E into F with a closed kernel is continuous.*

Proof: Let (E, τ), n and (F, τ') be as in the statement of the corollary, and let $u: E \to F$ be a surjective R-linear mapping with a closed kernel. By Lemma 5, (F, τ_u) is a separated topological (R, τ_R)-module, where τ_u is the direct image of τ under u. By Corollary 12, $\tau' = \tau_u$, and so $u: (E, \tau) \to (F, \tau')$ is continuous. This completes the proof. $\square$

Corollary 15. *Let (R, τ_R) be as in Theorem 11. Then, for every (E, τ), n and (F, τ') as in Corollary 14, we have that every R-linear mapping from E into F with a closed graph is continuous.*

Proof: Follows immediately from Corollary 14 and Lemma 6. $\square$

Corollary 16. *Let (R, τ_R) be as in Theorem 11 and let (E, τ) be a separated topological (R, τ_R)-module. Let M and N be two submodules of E such that N is a free R-module with a basis of n elements ($n \in \mathbb{N}^*$), M is τ-closed and $E = M \oplus N$. Then (E, τ) is the topological direct sum of M and N.*

Proof: Analogous to that of Corollary 7, by applying Corollary 14 in place of Theorem 4. $\square$

Corollary 17. *Let* (R, τ_R) *be as in Theorem* 11. *Then, for every* (E, τ), n *and* (F, τ') *as in Corollary* 14, *we have that every surjective continuous* R*-linear mapping from* (E, τ) *into* (F, τ') *is open.*

PROOF: Analogous to that of Corollary 8, by applying Theorem 11 in place of Theorem 4. □

Corollary 18. *Assume that* R *is a principal ring and that* (R, τ_R) *is complete and strictly minimal. Let* $n \in \mathbb{N}^*$, (E, τ) *a separated topological* (R, τ_R)*-module which is a free* R*-module of dimension* n *and* (F, τ') *a separated topological* (R, τ_R)*-module which is a free* R*-module. Then every* R*-linear mapping from* E *into* F *is continuous.*

PROOF: Let $u: E \to F$ be an R-linear mapping. If $u = 0$, the result is clear. So, let us assume that $u \neq 0$. Since $\mathrm{Im}(u) = \{u(x); x \in E\}$ is a finitely generated submodule of F, it follows from Proposition 1, p. 80 of [1] that $\mathrm{Im}(u)$ is a free submodule of F of finite dimension. Moreover, $(\mathrm{Im}(u), \tau_2)$ is a separated topological (R, τ_R)-module, where τ_2 is the topology induced by τ' on $\mathrm{Im}(u)$.

Let $\pi: E \to E/\mathrm{Ker}(u)$ be the canonical surjection and let $\tilde{u}: E/\mathrm{Ker}(u) \to F$ be the unique injective R-linear mapping such that $u = \tilde{u} \circ \pi$. Since $\mathrm{Im}(\tilde{u}) = \mathrm{Im}(u)$, $E/\mathrm{Ker}(u)$ and $\mathrm{Im}(u)$ are isomorphic as R-modules. Let τ_1 be the quotient topology on $E/\mathrm{Ker}(u)$. By Corollary 13 and Lemma 5, $(E/\mathrm{Ker}(u), \tau_1)$ is a separated topological (R, τ_R)-module. Thus, by considering $\tilde{u}$ as an R-module isomorphism from $E/\mathrm{Ker}(u)$ into $\mathrm{Im}(u)$, Theorem 11 implies that $\tilde{u}: (E/\mathrm{Ker}(u), \tau_1) \to (\mathrm{Im}(u), \tau_2)$ is a homeomorphism. Consequently, $u: (E, \tau) \to (F, \tau')$ is continuous, as was to be shown. □

In Corollary 18 (R, τ_R) may be taken, for example, as being any complete strictly minimal topological field (Corollary 18 remains valid if (R, τ_R) is any complete strictly minimal topological division ring; see Theorem 24.13(2) of [3]) or the topological ring $\mathbb{Z}_p$ of p-adic integers.

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