

## The limit lemma in fragments of arithmetic

VÍTĚZSLAV ŠVEJDAR

*Abstract.* The recursion theoretic limit lemma, saying that each function with a  $\Sigma_{n+2}$  graph is a limit of certain function with a  $\Delta_{n+1}$  graph, is provable in  $B\Sigma_{n+1}$ .

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Let  $\mathbb{N}$  be the set of all natural numbers and let a function  $G : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  be such that for each  $x_1, \dots, x_k$  the function  $s \mapsto G(\underline{x}, s)$ , where  $\underline{x}$  is a shorthand for  $x_1, \dots, x_k$ , is eventually constant. Then we use  $\lim_s G(\underline{x}, s)$  to denote the value the function  $s \mapsto G(\underline{x}, s)$  assumes in each sufficiently large  $s$ . The *limit lemma* says that for each set  $A \subseteq \mathbb{N}^k$  such that  $A \in \Delta_2$  there exists a recursive function  $G : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  such that  $\lim_s G(\underline{x}, s) = 1$  whenever  $[x_1, \dots, x_k] \in A$ , and  $\lim_s G(\underline{x}, s) = 0$  whenever  $[x_1, \dots, x_k] \notin A$ . For the definition of  $\Sigma_n$ ,  $\Pi_n$ , and  $\Delta_n$ , where  $n \geq 1$ , see e.g. [5], and recall that a set is  $\Delta_1$  if and only if it is recursive, and that  $\Delta_n = \Sigma_n \cap \Pi_n$ . The version of the limit lemma for functions says that for each function  $F : \mathbb{N}^k \rightarrow \mathbb{N}$  whose graph is  $\Sigma_2$  there exists a recursive  $G : \mathbb{N}^{k+1} \rightarrow \mathbb{N}$  such that  $F(\underline{x}) = \lim_s G(\underline{x}, s)$  for each  $k$ -tuple  $[x_1, \dots, x_k]$ . As can be seen e.g. from [4] and [2], the limit lemma is a useful tool in recursion theory.

*Peano arithmetic* PA is an axiomatic theory formulated in the arithmetical language  $\{+, \cdot, 0, S, \leq, <\}$ ; its axioms can be described as a finite set of base axioms plus the induction scheme. For details see e.g. [3]. *Bounded quantifiers* are quantifiers of the form  $\forall v \leq x$ ,  $\exists v \leq x$ ,  $\forall v < x$ , and  $\exists v < x$ . A *bounded formula*, or a  $\Delta_0$ -*formula*, is a formula all quantifiers of which are bounded. A  $\Sigma_n$ -*formula* is a formula having the form  $\exists v_1 \forall v_2 \exists \dots v_n \varphi$ , with  $n$  alternating quantifiers, where the first quantifier is existential and the matrix  $\varphi$  is a  $\Delta_0$ -formula. A  $\Pi_n$ -*formula* is a formula of the form  $\forall v_1 \exists v_2 \forall \dots v_n \varphi$  where again  $\varphi \in \Delta_0$ . So  $\Sigma_0 = \Pi_0 = \Delta_0$ . The *theory*  $\Pi$ , where  $\Gamma$  is  $\Sigma_n$  or  $\Pi_n$ , is PA with the induction scheme restricted to  $\Gamma$ -formulas. The *collection scheme* is the scheme

$$\forall \underline{y} \forall x (\forall v \leq x \exists z \varphi(v, z, \underline{y}) \rightarrow \exists t \forall v \leq x \exists z \leq t \varphi(v, z, \underline{y})).$$

The *theory*  $B\Gamma$ , where again  $\Gamma$  is  $\Sigma_n$  or  $\Pi_n$ , is  $I\Delta_0$  extended by the collection scheme restricted to  $\Gamma$ -formulas. It is known that for each  $n$  the theories  $I\Sigma_n$

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and  $\text{III}_n$  are equivalent, and also  $\text{B}\Pi_n$  and  $\text{B}\Sigma_{n+1}$  are equivalent.  $\text{B}\Sigma_{n+1}$  is a theory stronger than  $\text{I}\Sigma_n$ , but weaker than  $\text{I}\Sigma_{n+1}$ . For details and proofs, see again e.g. [3]. A useful property of  $\text{I}\Sigma_n$  is that it proves induction for  $\Sigma_0(\Sigma_n)$ -formulas, i.e. for formulas built up from  $\Sigma_n$ -formulas using logical connectives and bounded quantification. Also the least number principle for  $\Sigma_0(\Sigma_n)$ -formulas is provable in  $\text{I}\Sigma_n$ . A useful property of  $\text{B}\Sigma_{n+1}$  is that any formula obtained from  $\Sigma_{n+1}$ -formulas by bounded quantification is  $\text{B}\Sigma_{n+1}$ -equivalent to a  $\Sigma_{n+1}$ -formula. This fact can be used to verify that each  $\Sigma_0(\Sigma_n)$ -formula is  $\text{B}\Sigma_{n+1}$ -equivalent to a  $\Sigma_{n+1}$ -formula. We will also use the fact that  $\Sigma_0(\Sigma_n)$ -induction is provable in  $\text{B}\Sigma_{n+1}$ .

P. Hájek and A. Kučera show in [2] that the limit lemma for sets is provable in  $\text{I}\Sigma_1$ . P. Clote in an earlier paper [1] uses a version of the limit lemma for  $\Sigma_{n+2}$  functions, saying that any function having a  $\Sigma_{n+2}$  graph is a limit of a function having a  $\Delta_{n+1}$  graph, and proves this version in  $\text{B}\Sigma_{n+2}$ . I show that the results from [2] and [1] can be considerably improved: the limit lemma for  $\Sigma_{n+2}$  functions is provable already in  $\text{B}\Sigma_{n+1}$ .

Note that speaking about sets definable in a model, in the formulation of Lemma 1 and Theorem 1 below, is a way to overcome the difficulty that one cannot directly speak about sets and functions in the arithmetical language. In proofs of Lemma 1 and Theorem 1 we are less careful and ignore this difficulty. Recall that if  $n \geq 1$  then a set is  $\Sigma_n$  if and only if it is  $\Sigma_n$ -definable in the standard model of arithmetic. So a set simultaneously  $\Sigma_n$ - and  $\Pi_n$ -definable in a model corresponds to a set which, on metamathematical level, is  $\Delta_n$ .

**Lemma 1.** *Let  $\mathbf{M}$  be a model of  $\text{B}\Sigma_{n+1}$  with domain  $M$  and let  $A \subseteq M^k$  be simultaneously  $\Sigma_{n+2}$ - and  $\Pi_{n+2}$ -definable in  $\mathbf{M}$ . Then there exists a function  $G : M^{k+1} \rightarrow M$  with a graph  $\Sigma_0(\Sigma_n)$ -definable in  $\mathbf{M}$  such that  $\lim_s G(\underline{x}, s) = 1$  whenever  $[x_1, \dots, x_k] \in A$  and  $\lim_s G(\underline{x}, s) = 0$  whenever  $[x_1, \dots, x_k] \notin A$ .*

PROOF: Let the set  $A$  be as specified and let  $\varphi$  and  $\psi$  be  $\Sigma_n$ -formulas such that  $A = \{ [x_1, \dots, x_k]; \exists u \forall v \varphi(\underline{x}, u, v) \}$  and  $\bar{A} = \{ [x_1, \dots, x_k]; \exists u \forall v \psi(\underline{x}, u, v) \}$ , where  $\bar{A}$  is the complement of  $A$ . Think of the  $k$ -tuple  $\underline{x}$  as fixed and think of  $\varphi$  and  $\psi$  as two zero-one tables unbounded in two directions, with  $u$  running down and  $v$  running to the right. One and only one of the two tables contains rows consisting entirely of ones. Let the function  $H$  be defined as follows:

$$H(\underline{x}, s) = \begin{cases} 1 & \text{if } \forall u \leq s (\forall v \leq s \psi(\underline{x}, u, v) \rightarrow \exists u' \leq u \forall v \leq s \varphi(\underline{x}, u', v)) \\ 0 & \text{otherwise.} \end{cases}$$

Assume that  $[x_1, \dots, x_k] \notin A$ . Then  $\exists u \forall v \psi(\underline{x}, u, v)$  and  $\forall u \exists v \neg \varphi(\underline{x}, u, v)$ . Let  $u_0$  be some number satisfying  $\forall v \psi(\underline{x}, u_0, v)$ ; note that the existence of least such number is not guaranteed in  $\text{B}\Sigma_{n+1}$ . By  $\text{B}\Sigma_{n+1}$  there exists a number  $s_0$  such that  $\forall u \leq u_0 \exists v \leq s_0 \neg \varphi(\underline{x}, u, v)$ . We can assume  $s_0 \geq u_0$ . If  $s \geq s_0$  then there exists a number  $u \leq s$ , namely  $u_0$ , such that  $\forall v \leq s \psi(\underline{x}, u, v)$  and simultaneously

$\forall u' \leq u \exists v \leq s \neg \varphi(\underline{x}, u', v)$ . So  $H(\underline{x}, s) = 0$  for all such  $s$ , i.e.  $\lim_s H(\underline{x}, s) = 0$ . The proof that  $\lim_s H(\underline{x}, s) = 1$  whenever  $[x_1, \dots, x_n] \in A$  is similar. The graph of  $H$  is  $\Sigma_0(\Sigma_n)$ . So the function  $H$  is as desired.  $\square$

**Theorem 1.** *Let  $\mathbf{M}$  be a model of  $\text{B}\Sigma_{n+1}$  with domain  $M$  and let  $F : M^k \rightarrow M$  have a graph  $\Sigma_{n+2}$ -definable in  $\mathbf{M}$ . Then there exists a function  $G : M^{k+1} \rightarrow M$  with a graph  $\Sigma_0(\Sigma_n)$ -definable in  $\mathbf{M}$  such that  $F(\underline{x}) = \lim_s G(\underline{x}, s)$  for each  $\underline{x}$ .*

PROOF: Let  $F \in \Sigma_{n+2}$  with  $k$  variables be given. It is clear that  $F \in \Delta_{n+2}$  since for the complement of its graph we have  $[\underline{x}, y] \notin F \Leftrightarrow \exists y' (y' \neq y \ \& \ [\underline{x}, y'] \in F)$ . By Lemma 1 applied to the graph of  $F$  there exists a function  $H \in \Sigma_0(\Sigma_n)$  such that  $\lim_t H(\underline{x}, y, t) = 1$  whenever  $F(\underline{x}) = y$  and  $\lim_t H(\underline{x}, y, t) = 0$  whenever  $F(\underline{x}) \neq y$ . As in the proof of Lemma 1, let  $\underline{x}$  be fixed and think of the function  $H$  as a table with  $t$  running down and  $y$  running to the right. Let the *score of a number  $y$  at stage  $s$*  be defined as the length of maximal contiguous segment of ones which lies in column  $y$ , the bottom end of which is in row  $s$  and the top end of which is in a row  $t \geq y$ .

	0	1	2	3	4	5	6	7	8
0	0	0	0	0	0	0	0	0	...
1	0	0	0	0	0	0	0	0	...
2	0	0	0	0	0	1	0	0	...
3	0	0	0	0	0	1	0	0	...
4	0	0	1	1	0	1	0	0	...
5	0	0	1	1	0	1	0	0	...
6	0	0	0	1	0	1	0	0	...
7	0	0	1	1	0	1	0	0	...
8	0	0	1	1	0	1	0	0	...

Figure 1: Computing scores

If  $H$  is, for example, as in Figure 1 then the scores of numbers 2, 3, and 5 at stage 5 are 2, 2, and 1 respectively, and the score of any other number at stage 5 is zero. The scores of numbers 2, 3, and 5 at stage 8 are 2, 5, and 4. Let  $G(\underline{x}, s)$  be defined as the least  $y$  having maximal possible score at stage  $s$ . So in our example from Figure 1 we have  $G(\underline{x}, 5) = 2$  and  $G(\underline{x}, 8) = 3$ . It is evident that a score of a number  $y \leq s$  at stage  $s$  is a number not exceeding  $s + 1 - y \leq s + 1$  and that all  $y$ 's greater than  $s$  have zero score at stage  $s$ . The formula

$$\exists u \leq s + 1 (z + u = s + 1 \ \& \ y \leq u \ \& \ \forall t \leq s (u \leq t \rightarrow H(\underline{x}, y, t) = 1)),$$

i.e. the formula the score of  $y$  at stage  $s$  is at least  $z$ , is a  $\Sigma_0(\Sigma_n)$ -formula. So by  $\Sigma_0(\Sigma_n)$ -induction available in  $\text{B}\Sigma_{n+1}$ , there exists a greatest  $z$  satisfying this formula, and the score of a number  $y$  at stage  $s$  is correctly defined. Also,

the formulas the number  $z$  is the maximal score at stage  $s$  and the number  $y$  is the least number having the maximal score at stage  $s$  are  $\Sigma_0(\Sigma_n)$ -formulas. So again by  $\Sigma_0(\Sigma_n)$ -induction, the maximal score exists, and the function  $G$  is correctly defined. We have to verify that  $\lim_s G(\underline{x}, s) = F(\underline{x})$ . Let  $y_0 = F(\underline{x})$ . We know that  $\lim_s H(\underline{x}, y_0, t) = 1$ . So let the number  $t_0$  be such that  $t_0 \geq y_0$  and  $\forall t(t \geq t_0 \rightarrow H(\underline{x}, y_0, t) = 1)$ . We also know that  $\lim_s H(\underline{x}, y, t) = 0$  for each  $y \leq t_0$  such that  $y \neq y_0$ . Thus

$$\forall y \leq t_0 (y \neq y_0 \rightarrow \exists t(t \geq t_0 \ \& \ H(\underline{x}, y, t) = 0)).$$

By  $\Sigma_{n+1}$ -collection (more precisely, by  $\Sigma_0(\Sigma_n)$ -collection available in  $B\Sigma_{n+1}$ ) there exists an  $s_0$  such that

$$\forall y \leq t_0 (y \neq y_0 \rightarrow \exists t \leq s_0 (t \geq t_0 \ \& \ H(\underline{x}, y, t) = 0)).$$

This means that if  $s \geq s_0$  then the score of all numbers  $y \leq t_0$  such that  $y \neq y_0$  at stage  $s$  is lower than the score of  $y_0$ . Since ones occurring in column  $y$  above the diagonal line do not count, the score of any  $y > t_0$  at stage  $s$  is automatically lower than the score of  $y_0$ . So  $G(\underline{x}, s) = y_0$  for each  $s \geq s_0$ , and thus  $\lim_s G(\underline{x}, s) = y_0$ .  $\square$

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PHILOSOPHICAL FACULTY OF CHARLES UNIVERSITY, PALACHOVO NÁM. 2, 116 38 PRAHA 1, CZECH REPUBLIC

*E-mail:* vitezslav.svejdar@cuni.cz

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