An inequality in Orlicz function spaces with Orlicz norm

Jincai Wang

Abstract. We use Simonenko quantitative indices of an $\mathcal{N}$-function Φ to estimate two parameters q_Φ and Q_Φ in Orlicz function spaces $L^\Phi[0,\infty)$ with Orlicz norm, and get the following inequality: $\frac{B_\Phi}{B_\Phi-1} \le q_\Phi \le Q_\Phi \le \frac{A_\Phi}{A_\phi-1}$, where A_Φ and B_Φ are Simonenko indices. A similar inequality is obtained in $L^\Phi[0,1]$ with Orlicz norm.

Keywords: Orlicz spaces, Simonenko indices, $\triangle_2$-condition

Classification: 46B20, 46E30

1. Introduction

Definition 1.1. A function $M:\mathbb{R}\longrightarrow\mathbb{R}$ is called an $\mathcal{N}$-function, if

(i) M is continuous, convex and even;
(ii) $M(u)>0$ for $u\neq 0$, $M(0)=0$;
(iii) $\lim\limits_{u\to 0} M(u)/u=0$, $\lim\limits_{u\to\infty} M(u)/u=\infty$.

Let

$$\Phi(u)=\int_0^{|u|}\phi(t)\,dt \quad\text{and}\quad \Psi(v)=\int_0^{|v|}\psi(s)\,ds$$

be a pair of complementary $\mathcal{N}$-functions. The Orlicz function space is defined as follows: $L^\Phi[0,1]=\{x(t): x(t)$ is measurable on $[0,1]$ and $\rho_\Phi(\lambda x(t))\,dt<\infty$ for some $\lambda>0\}$, where $\rho_\Phi(x(t))=\int_{[0,1]}\Phi(x(t))\,dt$; $L^\Phi[0,\infty)=\{x(t): x(t)$ is measurable on $[0,\infty)$, $\rho_\Phi(\lambda x(t))\,dt<\infty$ for some $\lambda>0\}$, and $\rho_\Phi(x(t))=\int_{[0,\infty)}\Phi(x(t))\,dt$. We define the Orlicz norm on the Orlicz space as

$$\|x\|_\Phi=\inf_{k>0}\frac{1}{k}[1+\rho_\Phi(kx)].$$

An $\mathcal{N}$-function $\Phi(u)$ is said to satisfy the $\triangle_2$-*condition for small* u (in symbol $\Phi\in\triangle_2(0)$), if there exists $u_0>0$ and $C>0$, such that $\Phi(2u)\le C\Phi(u)$ for $0\le u\le u_0$. $\Phi(u)$ is said to satisfy the $\triangle_2$-*condition for large* u (in symbol $\Phi\in\triangle_2(\infty)$), if there exists $u_0>0$ and $C>0$ such that $\Phi(2u)\le C\Phi(u)$ for $u\ge u_0$. $\Phi(u)$ is said to satisfy the $\triangle_2$-*condition for all* $u\ge 0$ (in symbol $u\in\triangle_2$), if there exist $C>0$ such that $\Phi(2u)\le C\Phi(u)$ for $u\ge 0$. An $\mathcal{N}$-function

$\Phi(u)$ is said to satisfy the ∇_2-*condition for small* u (*for large* u, *for all* $u \geq 0$), in symbol $\Phi \in \nabla_2(0)$ ($\Phi \in \nabla_2(\infty)$, $\Phi \in \nabla_2$), if its complementary $\mathcal{N}$-function $\Psi \in \triangle_2(0)$ ($\Psi \in \triangle_2(\infty)$, $\Psi \in \triangle_2$).

The basic results on Orlicz spaces can be found in Krasnosel'skii and Rutickii [2], Lindenstrauss and Tzafriri [3], Rao and Ren [6], Chen [1].

The Simonenko indices of an $\mathcal{N}$-function Φ are defined as

$$A_\Phi = \inf_{t>0} \frac{t\phi(t)}{\Phi(t)}, \qquad B_\Phi = \sup_{t>0} \frac{t\phi(t)}{\Phi(t)}. \tag{1}$$

Simonenko introduced these indices in [9] and [8], and we can find a detailed description in Maligranda [4].

Clearly, $1 \leq A_\Phi \leq B_\Phi \leq \infty$.

Proposition 1.1. *Let* Φ *be an* $\mathcal{N}$*-function. Then*
$\Phi \in \nabla_2 \iff 1 < A_\Phi$; $\Phi \in \triangle_2 \iff B_\Phi < \infty$.

The proof of the proposition can be found in Krasnosel'skii and Rutickii [2, p. 24–26].

Lemma 1.2. *Let* Φ *and* Ψ *be a pair of complementary* $\mathcal{N}$*-functions. Then*

$$\frac{1}{A_\Phi} + \frac{1}{B_\Psi} = 1. \tag{2}$$

The proof of Lemma 1.2 can be found in Simonenko [9] or Rao & Ren [6].

The next lemma can be found in [1], [10] or [5].

Lemma 1.3. *Let* $\Phi(u) = \int_0^{|u|} \phi(t)\,dt$ *and* $\Psi(v) = \int_0^{|v|} \psi(s)\,ds$ *be a pair of complementary* $\mathcal{N}$*-functions. We denote*

$$k_x^* = \inf\{k > 0 : \rho_\Psi[\phi(k|x|)] \geq 1\}, \quad k_x^{**} = \sup\{k > 0 : \rho_\Psi[\phi(k|x|)] \leq 1\}.$$

Then $k \in [k_x^*, k_x^{**}]$ *if and only if*

$$\|x\|_\Phi = \frac{1}{k}[1 + \rho_\Phi(kx)].$$

2. Main results

Y. Yan estimated the two parameters Q_Φ and q_Φ in the Orlicz sequence space l^Φ, and got the following result (see [11], [7] or [13]).

Proposition 2.1. *Let Φ and Ψ be a pair of complementary $\mathcal{N}$-functions. Then*

$$\frac{b_\Phi^*}{b_\Phi^*-1} \le q_\Phi \le Q_\Phi \le \frac{a_\Phi^*}{a_\Phi^*-1}\,, \tag{3}$$

where

$$a_\Phi^* = \inf\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t \le \psi[\Psi^{-1}(1)]\right\},$$
$$b_\Phi^* = \sup\left\{\frac{t\phi(t)}{\Phi(t)} : 0 < t \le \psi[\Psi^{-1}(1)]\right\}.$$

The upper estimate in (3) can also be found in [12]. Now we establish a similar inequality in the Orlicz function space with Orlicz norm. Firstly, we have

Theorem 2.1. *Let Φ, Ψ be a pair of complementary $\mathcal{N}$-functions. For $L^\Phi[0,\infty)$, we denote*

$$Q_\Phi = \sup_{\|x\|_\Phi=1} k_x^{**} = \sup_{\|x\|_\Phi=1}\left\{k>0 : \|x\|_\Phi = \frac{1}{k}(1+\rho_\Phi(kx))\right\},$$
$$q_\Phi = \inf_{\|x\|_\Phi=1} k_x^{*} = \inf_{\|x\|_\Phi=1}\left\{k>0 : \|x\|_\Phi = \frac{1}{k}(1+\rho_\Phi(kx))\right\}.$$

Then

$$A_\Psi = \frac{B_\Phi}{B_\Phi-1} \le q_\Phi \le Q_\Phi \le \frac{A_\Phi}{A_\Phi-1} = B_\Psi, \tag{4}$$

where A_Φ, B_Φ, A_Ψ and B_Ψ are defined by (1).

Proof: The left and right equations in (4) follow from Lemma 1.2. Now we prove

$$q_\Phi \ge \frac{B_\Phi}{B_\Phi-1}\,. \tag{5}$$

For $\Phi \notin \triangle_2$, by Proposition 1.1, we have $B_\Phi = \infty$ or $A_\Psi = 1$. The result is obvious.

For $\Phi \in \triangle_2$, we only prove that for every $x \in L^\Phi[0,\infty)$ which satisfies $\|x\|_\Phi = 1$, we have $k_x^* \ge \frac{B_\Phi}{B_\Phi-1}$. Firstly, we have $\rho_\Psi(\phi(k_x^*|x(t)|)) \ge 1$. In fact, if $\Phi \in \triangle_2$, then $\rho_\Phi[(k_x^*+1)x] < \infty$. So

$$\begin{aligned}\rho_\Psi(\phi((k_x^*+1)|x(t)|)) &\le \rho_\Psi(\phi((k_x^*+1)|x(t)|)) + \rho_\Phi((k_x^*+1)|x(t)|)\\ &= \int_G (k_x^*+1)|x(t)|\cdot\phi((k_x^*+1)|x(t)|)\,dt\\ &\le B_\Phi\rho_\Phi((k_x^*+1)|x(t)|) < \infty.\end{aligned}$$

Choose $k_x^* < k_n < k_x^* + 1$ such that $k_n \searrow k_x^*$. By the right continuity of ϕ and Lebesgue dominated convergence theorem, we have

$$\rho_\Psi(\phi(k_x^*|x(t)|)) = \lim_{n\to\infty} \rho_\Psi(\phi(k_n|x(t)|)) \geq 1.$$

For every $x \in L^\Phi[0,\infty)$ which satisfies $\|x\|_\Phi = 1$, we have

$$\begin{aligned} 1 + \rho_\Phi(k_x^* x) &\leq \rho_\Psi(\phi(k_x^*|x(t)|)) + \rho_\Phi(k_x^*|x(t)|) \\ &= \int_{[0,\infty)} \Psi\{\phi[(k_x^*|x(t)|)]\}\, dt + \int_{[0,\infty)} \Phi(k_x^*|x(t)|)\, dt \\ &= \int_{[0,\infty)} k_x^*|x(t)|\phi(k_x^*|x(t)|)\, dt \\ &\leq B_\Phi \int_{[0,\infty)} \Phi(k_x^*|x(t)|)\, dt = B_\Phi \rho_\Phi(k_x^* x). \end{aligned}$$

This implies

$$\rho_\Phi(k_x^* x) \geq \frac{1}{B_\Phi - 1}\,. \tag{6}$$

By Lemma 1.3, we get

$$1 = \|x\|_\Phi = \frac{1}{k_x^*}\{1 + \rho_\Phi(k_x^* x)\}.$$

So $\rho_\Phi(k_x^* x) = k_x^* - 1$. By (6)

$$k_x^* \geq \frac{B_\Phi}{B_\Phi - 1}\,.$$

Next, we prove

$$Q_\Phi \leq \frac{A_\Phi}{A_\Phi - 1}\,. \tag{7}$$

If $\Phi \notin \nabla_2$, then $A_\Phi = 1$ or $B_\Psi = \infty$. The result is obvious.

If $\Phi \in \nabla_2$, then $A_\Phi > 1$. For every $x \in L^\Phi[0,\infty)$ which satisfies $\|x\|_\Phi = 1$, and for any $k \in [k_x^*, k_x^{**}]$, we have

$$1 = \|x\|_\Phi = \frac{1}{k}[1 + \rho_\Phi(kx)].$$

For any $0 < \varepsilon < 1 < k$, we have

$$1 = \|x\|_\Phi = \inf_{t>0} \frac{1}{t}[1 + \rho_\Phi(tx)] \leq \frac{1}{k-\varepsilon}[1 + \rho_\Phi((k-\varepsilon)x)]. \tag{8}$$

By the definition of k_x^{**} and $k-\varepsilon < k_x^{**}$, we have

$$
\begin{aligned}
1+\rho_\Phi[(k-\varepsilon)x] &\geq \rho_\Psi\{\phi[(k-\varepsilon)x]\}+\rho_\Phi[(k-\varepsilon)x] \\
&= \int_{[0,\infty)} (k-\varepsilon)x(t)\phi[(k-\varepsilon)x(t)]\,dt \\
&\geq A_\Phi \rho_\Phi((k-\varepsilon)x(t)).
\end{aligned} \tag{9}
$$

Therefore by (8) and (9), we have

$$1 \geq (A_\Phi - 1)\rho_\Phi((k-\varepsilon)x(t)) \geq (A_\Phi - 1)(k-\varepsilon-1))$$

or

$$k-\varepsilon \leq \frac{A_\Phi}{A_\Phi - 1}\,.$$

Since ε is arbitrary, we have

$$k \leq \frac{A_\Phi}{A_\Phi - 1}\,.$$

This implies (7) since x and k are arbitrary. □

Corollary 2.1. (i) *If* $\Phi \in \nabla_2$, *then* $Q_\Phi < \infty$; (ii) *If* $\Phi \in \triangle_2$, *then* $q_\Phi > 1$.

For $0 \neq x \in L^\Phi[0,1]$, we still denote

$$
\begin{aligned}
k_x^* &= \inf\{k>0 : \rho_\Psi[\phi(kx)] \geq 1\}, \\
k_x^{**} &= \sup\{k>0 : \rho_\Psi[\phi(kx)] \leq 1\}, \\
Q_\Phi &= \sup_{\|x\|_\Phi = 1} k_x^{**} = \sup_{\|x\|_\Phi = 1} \left\{k>0 : \|x\|_\Phi = \frac{1}{k}(1+\rho_\Phi(kx))\right\}, \\
q_\Phi &= \inf_{\|x\|_\Phi = 1} k_x^{*} = \inf_{\|x\|_\Phi = 1} \left\{k>0 : \|x\|_\Phi = \frac{1}{k}(1+\rho_\Phi(kx))\right\}.
\end{aligned}
$$

Let $\varepsilon_0 = \min\{\frac{1}{2\phi(1)}, 1\}$. Denote

$$
\begin{aligned}
A_\Phi^* &= \inf\left\{\frac{t\phi(t)}{\Phi(t)} : t \in [\varepsilon_0, \infty)\right\}, \\
B_\Phi^* &= \sup\left\{\frac{t\phi(t)}{\Phi(t)} : t \in [\varepsilon_0, \infty)\right\}.
\end{aligned}
$$

Obviously, $\varepsilon_0\phi(\varepsilon_0) \leq \frac{\phi(\varepsilon_0)}{2\phi(1)} \leq \frac{1}{2}$.

Theorem 2.2. *If Φ, Ψ is a pair of complementary $\mathcal{N}$-functions, then*

$$\frac{B^*_\Phi - \varepsilon_0\phi(\varepsilon_0)}{B^*_\Phi - 1} \le q_\Phi \le Q_\Phi \le \frac{A^*_\Phi + A^*_\Phi\Phi(\varepsilon_0)}{A^*_\Phi - 1}\,.$$

Proof: Firstly, we prove $q_\Phi \ge \frac{B^*_\Phi - \varepsilon_0\phi(\varepsilon_0)}{B^*_\Phi - 1}$. If $\Phi \notin \triangle_2(\infty)$, then $B^*_\Phi = \infty$, and the result is clear. If $\Phi \in \triangle_2(\infty)$, then $B^*_\Phi < \infty$. By the proof of Theorem 2.1, for $x \in L^\Phi[0,1]$ with $\|x\|_\Phi = 1$, we have $\rho_\Psi(\phi(k^*_x x)) \ge 1$. So

$$\begin{aligned}
1 + \rho_\Phi(k^*_x x) &\le \rho_\Psi(\phi(k^*_x x)) + \rho_\Phi(k^*_x x)\\
&= \int_{[0,1]} k^*_x|x(t)|\phi(k^*_x|x(t)|)\,dt\\
&\le \int_{G_1=\{t:k^*_x|x(t)|<\varepsilon_0\}} \varepsilon_0\phi(\varepsilon_0)\,dt + \int_{G\setminus G_1} k^*_x|x(t)|\phi(k^*_x|x(t)|)\,dt\\
&< \varepsilon_0\phi(\varepsilon_0) + B^*_\Phi\rho_\Phi(k^*_x x).
\end{aligned}$$

Therefore

$$1 - \varepsilon_0\phi(\varepsilon_0) \le (B^*_\Phi - 1)\rho_\Phi(k^*_x x).$$

Noting that $\rho_\Phi(k^*_x x) = k^*_x - 1$, we have

$$\frac{1 - \varepsilon_0\phi(\varepsilon_0)}{B^*_\Phi - 1} \le k^*_x - 1,$$

i.e.

$$k^*_x \ge \frac{B^*_\Phi - \varepsilon_0\phi(\varepsilon_0)}{B^*_\Phi - 1}\,.$$

Since x is arbitrary,

$$q_\Phi \ge \frac{B^*_\Phi - \varepsilon_0\phi(\varepsilon_0)}{B^*_\Phi - 1}\,.$$

Next we prove $Q_\Phi \le \frac{A^*_\Phi(1+\Phi(\varepsilon_0))}{A^*_\Phi - 1}$. If $\Phi \notin \nabla_2(\infty)$, the result is obvious. If $\Phi \in \nabla_2(\infty)$, then $\forall x \in S(L^\Phi[0,1])$, $\forall k \in [k^*_x, k^{**}_x]$ and $0 < \varepsilon < 1$, we get

$$\begin{aligned}
1 + \rho_\Phi[(k-\varepsilon)x] &\ge \rho_\Psi\{\phi[(k-\varepsilon)|x|]\} + \rho_\Phi[(k-\varepsilon)x]\\
&= \int_{[0,1]} (k-\varepsilon)|x(t)|\phi[(k-\varepsilon)|x(t)|]\,dt\\
&\ge \int_{\{t\in[0,1]:(k-\varepsilon)|x(t)|\ge\varepsilon_0\}} (k-\varepsilon)|x(t)|\phi[(k-\varepsilon)|x(t)|]\,dt\\
&\ge A^*_\Phi \int_{\{(k-\varepsilon)|x(t)|\ge\varepsilon_0\}} \Phi((k-\varepsilon)|x(t)|)\,dt\\
&= A^*_\Phi\{\rho_\Phi[(k-\varepsilon)x(t)] - \int_{\{t\in[0,1]:(k-\varepsilon)|x(t)|<\varepsilon_0\}} \Phi((k-\varepsilon)x(t))\,dt\}\\
&\ge A^*_\Phi\{\rho_\Phi[(k-\varepsilon)x(t)] - \Phi(\varepsilon_0)\}.
\end{aligned}$$

So

$$1 + A_\Phi^* \Phi(\varepsilon_0) \geq (A_\Phi^* - 1)\rho((k-\varepsilon)x(t)) \geq (A_\Phi^* - 1)(k - \varepsilon - 1),$$

i.e.

$$k \leq \frac{A_\Phi^*[1 + \Phi(\varepsilon_0)]}{A_\Phi^* - 1} + \varepsilon.$$

Therefore,

$$k \leq \frac{A_\Phi^*[1 + \Phi(\varepsilon_0)]}{A_\Phi^* - 1}.$$

Since $x \in S(L^\Phi[0,1])$ is arbitrary,

$$Q_\Phi \leq \frac{A_\Phi^*(1 + \Phi(\varepsilon_0))}{A_\Phi^* - 1}.$$

□

Corollary 2.2 (S.T. Chen [1, p. 21])**.**

(i) *If* $\Phi \in \triangle_2(\infty)$*, then* $q_\Phi > 1$.

(ii) *If* $\Phi \in \nabla_2(\infty)$*, then* $Q_\Phi < \infty$.

From the proof of Theorem 2.2, we know Theorem 2.2 is true for any $0 < \varepsilon < \varepsilon_0$. Letting ε to tend to 0, we get

Corollary 2.3. *Let* Φ, Ψ *be a pair of complementary* $\mathcal{N}$*-functions. Then*

$$A_\Psi = \frac{B_\Phi}{B_\Phi - 1} \leq q_\Phi \leq Q_\Phi \leq \frac{A_\Phi}{A_\Phi - 1} = B_\Psi, \tag{10}$$

where A_Φ, B_Φ, A_Ψ *and* B_Ψ *are defined by* (1).

Example 1. For the $\mathcal{N}$-function $\Phi(u) = |u|^p$, which generates $L^p[0,\infty)$, we have $A_\Phi = B_\Phi = p$. By Theorem 2.1 and Corollary 2.3, we have $q_\Phi = Q_\Phi = \frac{p}{p-1}$.

Example 2. For the $\mathcal{N}$-function $\Phi(u) = e^{|u|} - |u| - 1$, we have

$$1 \leq q_\Phi \leq Q_\Phi \leq 2. \tag{11}$$

Indeed, $F_\Phi(t) = \frac{t(e^t-1)}{e^t-t-1}$ is increasing in $(0,+\infty)$. So $A_\Phi = \lim\limits_{t\to 0^+} F_\Phi(t) = 2$ and $B_\Phi = \lim\limits_{t\to+\infty} F_\Phi(t) = \infty$. Therefore (11) follows from Theorem 2.1 and Corollary 2.3.

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Department of Mathematics, Suzhou University, Suzhou 215006, P.R. China

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