Generalized Bochner-Riesz means on spaces generated by smooth blocks

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Abstract. We investigate generalized Bochner-Riesz means at the critical index on spaces generated by smooth blocks and give some approximation theorems.

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1. Introduction and statement of result

The Bochner-Riesz multiplier of order α is defined by

$$(S_R^{\alpha})^{\wedge}(x) = \left(1 - \frac{|x|^2}{R^2}\right)_+^{\alpha} \hat{f}(x),\ f \in \varphi(\mathbb{R}^n),$$

where $\varphi(\mathbb{R}^n)$ is the Schwartz class.

It is known that for the Bochner-Riesz means at critical index there exists a function $f \in L^1(\mathbb{R}^n)$ such that $\limsup_{R\to\infty}(S_R^{(n-1)/2} f)(x) = \infty$, a.e. (see [1]). Shanzhen Lu and Shiming Wang [2] introduced so-called *spaces generated by smooth blocks*, a subspace of $L^1(\mathbb{R}^n)$. On this space, the Bochner-Riesz means at the critical index $S_R^{(n-1)/2} f$ converge to f a.e. as $R \to \infty$.

Now let us turn to the definition of *smooth blocks*. A (q,λ)-block $(1 < q \le \infty)$ is a function b that is supported on a cube Q satisfying

$$\|b\|_{\mathcal{L}_\lambda^q} \le |Q|^{\frac{1}{q}-1},$$

where $\mathcal{L}_\lambda^q(\mathbb{R}^n)$ denotes the Bessel potential space ([3]). We define the *spaces generated by smooth blocks* as

$$B_q^{\lambda}(\mathbb{R}^n) = \Big\{f : f = \sum_k m_k b_k,\ b_k \text{ is } (q,\lambda)\text{-block},\ N(\{m_k\}) < \infty\Big\},$$

where $N(\{m_k\}) = \sum_k |m_k| \left(1 + \log \frac{\sum_l |m_l|}{|m_k|}\right)$, and $N_q(f) = \inf\{N(\{m_k\}) : f = \sum_k m_k b_k\}$ is a quasinorm on B_q^{λ}.

Shanzhen Lu and Shiming Wang [2] get the following result.

Theorem A. *If* $f \in B_q^1(\mathbb{R}^n)(1 < q < \infty)$, *then*

$$(S_R^{(n-1)/2} f)(x) - f(x) = o(\frac{1}{R}) \quad a.e. \quad as \quad R \to \infty.$$

Let α be complex number, $\operatorname{Re}\alpha > -1$, $b > 0$. The generalized Bochner-Riesz multiplier of order α is defined by

$$(S_R^{\alpha,b} f)^\wedge(x) = \left(1 - \frac{|x|^b}{R^b}\right)_+^\alpha \hat{f}(x), \quad f \in \varphi(\mathbb{R}^n).$$

The main result of this paper is

Theorem 1. *If* $f \in B_q^1(\mathbb{R}^n)(1 < q < \infty)$, $b > 1$, *then*

$$(S_R^{\frac{n-1}{2},b} f)(x) - f(x) = o(1/R) \quad a.e. \quad as \quad R \to \infty.$$

2. Proof of Theorem 1

Let

$$(M_\lambda^{\alpha,b})(x) = \sup_{R>0} |R^\lambda\{(S_R^{\alpha,b} f)(x) - f(x)\}|.$$

To prove Theorem 1, we need the following theorem.

Theorem 2. *Let* $0 \le \lambda \le 2$, $1 < p < \infty$, $b > 2$, $\alpha = \sigma + i\tau$ *and* $\sigma > \frac{n-1}{2}|\frac{2}{p} - 1|$. *If* $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, *then*

$$\|M_\lambda^{\sigma,b} f\|_p \le C\|f\|_{\mathcal{L}_\lambda^p},$$

where C *is independent of* f.

First we give some lemmas.

Lemma 1 (Xuean Zheng [6, p. 1342]). *Let* $\phi(t) = (1 - t^{\alpha_1})^{\delta_1} \cdots (1 - t^{\alpha_s})^{\delta_s}$, $0 \le t < 1$, α *be the minimum of* $\alpha_1, \cdots, \alpha_s$ *except* 2, $\delta = \delta_1 + \delta_2 + \cdots + \delta_s$, *and*

$$H_l^{\alpha,\delta}(t) = \int_0^1 \phi(u)(ut)^{l+1} J_\tau(ut)\, du, \quad \tau = \frac{1}{2}(n-2).$$

Then

$$H_l^{\alpha,\delta}(t) = (\frac{\alpha_1}{2})^{\delta_1} \cdots (\frac{\alpha_s}{2})^{\delta_s} H_l^\delta(t) + \sum_{k=1}^m C_k H_l^{\delta+k}(t) + R(t),$$

where $\alpha \neq 2$, $m > l + \alpha_1 + \cdots \alpha_s + 2 + \delta$ *and* m *is a positive integer,* $H_l^\delta(t) = H_l^{2,\delta}(t) = 2^\delta \Gamma(\delta+1) J_{l+\delta+1}(t) t^{l-\delta}$, $|R(t)| \le Ct^{2l+1} (t \to 0)$, $|R(t)| \le t^{-\alpha-1} (t \to \infty)$, $J_k(t)$ *is the Bessel function* (see [7]).

Setting

$$\phi_{\alpha,b}(x) = \begin{cases} (1-|x|^b)^\alpha, & |x| < 1, \\ 0, & |x| \ge 1, \end{cases}$$

where $\operatorname{Re}\alpha > -1$, we have

$$\mathcal{F}(\phi_{\alpha,b})(y) = \frac{1}{(2\pi)^{n/2}|y|^{n/2-1}} \int_0^1 (1-r^b)^\alpha r^{n/2} J_{n/2-1}(|y|r)\,dr$$

(see [9]), where $\mathcal{F}$ denotes the Fourier transform.

It is known ([8]) that for $\alpha > \frac{n-1}{2}$ we have

$$\int_{\mathbb{R}^n} \mathcal{F}(\phi_{\alpha,b})(y)\,dy = 1.$$

Lemma 2. *Let* $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 2$. *If* $f \in L_2^p(\mathbb{R}^n)$, *then*

$$\|M_2^{\alpha,b} f\|_p \le Ce^{|\tau|^2} \|f\|_{L_2^p},$$

where C *is independent of* τ *and* f.

Proof: Setting $\phi(t) = (1-t^b)^\alpha, l = \frac{n}{2} - 1$ in Lemma 1, we have

$$H^{\alpha,b}(t) = \left(\frac{b}{2}\right)^\alpha H^\alpha(t) + \sum_{k=1}^m C_k H^{\alpha+k}(t) + R^{\alpha,b}(t), \tag{1}$$

where

$$\begin{gathered} H^{\alpha,b}(t) = H^{\alpha,b}_{\frac{n}{2}-1}(t) = \int_0^1 (1-u^b)^\alpha (ut)^{\frac{n}{2}} J_{\frac{n}{2}-1}(ut)\,du, \\ H^\alpha(t) = H^{\alpha,2}(t), \quad m > \frac{n}{2} - 1 + b + 2 + \operatorname{Re}\alpha \ \text{ is a positive integer}, \\ |R^{\alpha,b}(t)| \le Ct^{n-1} (t \to 0), \quad |R^{\alpha,b}(t)| \le Ct^{-b-1} (t \to \infty). \end{gathered} \tag{2}$$

By (1), we have

$$\begin{aligned} \mathcal{F}(\phi_{\alpha,b})(y) &= \frac{1}{(2\pi)^{\frac{n}{2}}|y|^{\frac{n}{2}-1}} \int_0^1 (1-r^b)^\alpha r^{\frac{n}{2}} J_{\frac{n}{2}-1}(|y|r)\,dr \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}|y|^{n-1}} H^{\alpha,b}(|y|) \\ &= \frac{1}{(2\pi)^{\frac{n}{2}}} \left\{ \left(\frac{b}{2}\right)^\alpha \frac{H^\alpha(|y|)}{|y|^{n-1}} + \sum_{k=1}^m C_k \frac{H^{\alpha+k}(|y|)}{|y|^{n-1}} + \frac{R^{\alpha,b}(|y|)}{|y|^{n-1}} \right\} \\ &= \left(\frac{b}{2}\right)^\alpha \mathcal{F}(\phi_\alpha)(y) + \sum_{k=1}^m C_k \mathcal{F}(\phi_{\alpha+k})(y) + \frac{R^{\alpha,b}(|y|)}{(2\pi)^{\frac{n}{2}}|y|^{n-1}}, \end{aligned} \tag{3}$$

where $\phi_\alpha(x) = \phi_{\alpha,2}$ and $\operatorname{Re}\alpha > -1$. Denote $(S_R^\alpha f)(x) = (S_R^{\alpha,2} f)(x)$. We have

$$(S_R^{\alpha,b} f)(x) = \int_{\mathbb{R}^n} f(y+x) R^n \mathcal{F}(\phi_{\alpha,b})(Ry)\, dy$$

$$= \left(\frac{b}{2}\right)^\alpha (S_R^\alpha f)(x) + \sum_{k=1}^m C_k (S_R^{\alpha+k} f)(x) \tag{4}$$

$$+ \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n\, dy.$$

Let $v_0 = \frac{1}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} \frac{R^{\alpha,b}(|y|)}{|y|^{n-1}}\, dy$ and note that

$$\int_{\mathbb{R}^n} \mathcal{F}(\phi_{\alpha,b})(y)\, dy = \int_{\mathbb{R}^n} \mathcal{F}(\phi_\alpha)(y)\, dy = 1.$$

Integrating both sides of (3), we get $v_0 = 1 - \sum_{k=1}^m C_k - (\frac{b}{2})^\alpha$. By (4), we have

$$|R^2\{(S_R^{\alpha,b} f)(x) - f(x)\}|$$
$$\le |R^2 \left(\frac{b}{2}\right)^\alpha \{(S_R^\alpha f)(x) - f(x)\}| + \sum_{k=1}^m |R^2 C_k \{(S_R^{\alpha+k} f)(x) - f(x)\}|$$
$$+ \left| \left\{ \frac{R^2}{(2\pi)^{\frac{n}{2}}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n\, dy - R^2 v_0 f(x) \right\} \right|.$$

By Lemma 1 of [2], to prove Lemma 2, we must set up the following inequality:

$$\left\| \sup_{R>0} \left\{ R^2 \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n\, dy - v_0 f(x) \right] \right\} \right\|_p \le C e^{|\tau|^2} \|f\|_{L_2^p}.$$

In fact,

$$R^2 \left[\frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n\, dy - v_0 f(x) \right]$$
$$= R^2 \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} [f(x+y) - f(x)] \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n\, dy$$
$$= R^2 \frac{1}{(2\pi)^{n/2}} \int_0^\infty \int_{\Sigma_{n-1}} [f(x+ty') + f(x-ty') - 2f(x)]\, dy' \cdot R^{\alpha,b}(Rt) R\, dt$$
$$= R^2 \frac{1}{(2\pi)^{n/2}} \int_0^\infty g(x, \frac{t}{R}) R^{\alpha,b}(t)\, dt,$$

where $g(x,t) = \int_{\Sigma_{n-1}} [f(x+ty') + f(x-ty') - 2f(x)]\,dy'$. Denote $A(t) = \int_t^\infty R^{\alpha,b}(\tau)\,d\tau$, $B(t) = \int_t^\infty A(\tau)\,d\tau$. It is easy to see that $g(x,\frac{t}{R})|_{t=0} = 0$, $\frac{d}{dt}g(x,\frac{t}{R})|_{t=0} = 0$. Using integration by parts, we get

$$\begin{aligned} &R^2\left[\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x+y)\frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}}R^n\,dy - v_0 f(x)\right] \\ &= \int_0^\infty R^2\frac{d^2}{dt^2}g(x,\frac{t}{R})B(t)\,dt. \end{aligned}$$

Let

$$g_{ij}(x,t) = \int_{\Sigma_{n-1}} |D_{ij}f(x+ty')|\,dy', \text{ where } D_{ij}f(x) = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}f(x),$$

$$(\mathcal{M}_2^{\alpha,b}f)(x) = \sup_{R>0} R^2\left|\frac{1}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} f(x-y)\frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}}R^n\,dy - v_0 f(x)\right|.$$

Then

$$(\mathcal{M}_2^{\alpha,b}f)(x) \le \sum_{i,j=1}^n \sup_{R>0}\int_0^\infty |g_{ij}(x,\frac{t}{R})||B(t)|\,dt.$$

For $t \ge 1$,

$$\begin{aligned} |A(t)| &= \Big|\int_t^\infty R^{\alpha,b}(\tau)\,d\tau\Big| \le C\int_t^\infty \tau^{-b-1}\,d\tau = Ct^{-b}, \\ |B(t)| &= \Big|\int_t^\infty A(\tau)\,d\tau\Big| \le C\int_t^\infty \tau^{-b} = Ct^{-b+1} \quad (b>1). \end{aligned}$$

For $t < 1$,

$$|A(t)| \le C\Big|\int_1^\infty R^{\alpha,b}(\tau)\,d\tau\Big| + \Big|\int_t^1 R^{\alpha,b}(\tau)\,d\tau\Big| \le C$$

and by (2),

$$|B(t)| \le \Big|\int_1^\infty A(\tau)\,d\tau\Big| + \Big|\int_t^1 A(\tau)\,d\tau\Big| \le \int_1^\infty C\tau^{-b}\,d\tau + \int_t^1 C\,d\tau \le C.$$

By this, we have

$$\begin{aligned}
&\int_0^1 g_{ij}(x,\frac{t}{R})|B(t)|\,dt \le C\int_0^1 g_{ij}(x,\frac{t}{R})\,dt\\
&\le CR\int_0^{\frac{1}{R}}\int_{\Sigma_{n-1}}|D_{ij}f(x+ty')|\,dy'\,dt\\
&= CR\int_0^{\frac{1}{R}} t^{1-n} d\left[\int_0^t\left(\int_{\Sigma_{n-1}}|D_{ij}f(x+\tau y')|\,dy'\tau^{n-1}\right)\,d\tau\right]\\
&\le CR\left(\frac{1}{R}\right)^{1-n}\int_0^{\frac{1}{R}}\int_{\Sigma_{n-1}}|D_{ij}f(x+ty')|\,dy't^{n-1}\,dt\\
&\quad + CR\int_0^{\frac{1}{R}} t^{-n}\left[\int_0^t\left(\int_{\Sigma_{n-1}}|D_{ij}f(x+\tau y')|\,dy'\right)\tau^{n-1}\,d\tau\right]\,dt\\
&\le C\left(\frac{1}{R}\right)^{-n}\int_{|y|<\frac{1}{R}}|D_{ij}f(x+y)|\,dy\\
&\quad + CR\int_0^{\frac{1}{R}} t^{-n}\left(\int_{|y|<t}|D_{ij}f(x+y)|\,dy\right)\,dt\\
&\le C\,\mathrm{HL}(D_{ij}f)(x),
\end{aligned}$$

where $\mathrm{HL}(g)$ is the Hardy-Littewood maximal function of g. Thus, we get

$$\left\|\sup_{R>0}\int_0^1 g_{ij}(x,\frac{t}{R})|B(t)|\,dt\right\|_p \le C\|f\|_{L_2^p} \le Ce^{|\tau|^2}\|f\|_{L_2^p},$$

where C is independent of τ. Meanwhile, we have

$$\int_1^\infty g_{ij}(x,\frac{t}{R})|B(t)|\,dt \le \int_1^\infty g_{ij}(x,\frac{t}{R})t^{-b+1}\,dt = R^{-b+1}\int_{\frac{1}{R}}^\infty g_{ij}(x,t)t^{-b+1}R\,dt.$$

Using the inequality $\int_0^t \tau^{n-1}g_{ij}(x,\tau)\,d\tau \le Ct^n\,\mathrm{HL}(D_{ij}f)(x)$, we obtain

$$\begin{aligned}
\int_1^\infty g_{ij}(x,\frac{t}{R})|B(t)|\,dt &\le R^{-b+2}\int_{\frac{1}{R}}^\infty t^{-b+1-(n-1)}d\left(\int_0^t \tau^{n-1}g_{ij}(x,\tau)\,d\tau\right)\\
&\le R^{-b+2}\left[R^{b+n-2}\int_0^{\frac{1}{R}}\tau^{n-1}g_{ij}(x,\tau)\,d\tau\right.\\
&\qquad\left. +C\int_{\frac{1}{R}}^\infty t^{-b-n+1}\left(\int_0^t\tau^{n-1}g_{ij}(x,\tau)\,d\tau\right)\,dt\right]\\
&\le C\,\mathrm{HL}(D_{ij}f)(x).
\end{aligned}$$

Thus

$$\Big\| \sup_{R>0} \int_1^\infty g_{ij}(x, \frac{t}{R})|B(t)|\,dt \Big\|_p \le C\|f\|_{L_2^p} \le Ce^{|\tau|^2}\|f\|_{L_2^p}.$$

□

Using the same method, we have

Lemma 2′. *Let $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 1$. If $f \in L_1^p(\mathbb{R}^n)$, then $\|M_1^{\alpha,b} f\|_p \le Ce^{|\tau|^2}\|f\|_{L_1^p}$, where C is independent of τ and f.*

Lemma 3. *Let $0 \le \lambda \le 2$, $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 2$. If $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, then*

$$\|M_\lambda^{\alpha,b}\|_p \le Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p},$$

where C is independent of τ and f.

Proof: Suppose $\{r_k\}$ is a sequence consisting of all positive rational numbers. Let $A_k = \{r_1, r_2, \cdots, r_k\}$. Define

$$(F_\lambda^{\alpha,b,k} f)(x) = \sup_{r_j \in A_k} r_j^\lambda |(S_{r_j}^{\alpha,b} f)(x) - f(x)|.$$

Then

$$(F_\lambda^{\alpha,b,k} f)(x) \le (F_\lambda^{\alpha,b,k+1} f)(x)$$

and

$$(M_\lambda^{\alpha,b} f)(x) = \lim_{k\to\infty} (F_\lambda^{\alpha,b,k} f)(x).$$

For any fixed $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, there exists $g \in L^p(\mathbb{R}^n)$ such that $f = \jmath_\lambda g$. Fix k and let $S_j (1 \le j \le k)$ be a set such that for $x \in S_j$,

$$(F_\lambda^{\alpha,b,k} f)(x) = r_j^\lambda |(S_{r_j}^{\alpha,b} f)(x) - f(x)|$$

and

$$(F_\lambda^{\alpha,b,k} f)(x) > r_i^\lambda |(S_{r_i}^{\alpha,b} f)(x) - f(x)|, \quad i < j.$$

It is easy to see that the sets $\{S_j\}$ do not intersect each other. Let $\Omega = \{z \in \mathbb{C}, 0 \le \operatorname{Re} z \le 1\}$ and define

$$\phi_j(x) = \operatorname{sign}\{(S_{r_j}^{\alpha,b} f)(x) - f(x)\},$$

$$(T_z g)(x) = \sum_{r_j \in A_k} r_j^{2z} \chi_{S_j}(x) \{S_{r_j}^{\alpha,b}(\jmath_{2z} g)(x) - (\jmath_{2z} g)(x)\}\phi_j.$$

Now we assert that $\{T_z\}$ is an admissible family of operators in the sense of E.M. Stein (see [7]). Indeed, $\{\jmath_{2z}g\}$ is an admissible family of operators in the sense of E.M. Stein, so we only set up the following inequality:

$$\|S_R^{\alpha,b}f\|_p \le C\|f\|_p, \quad f \in L^p(\mathbb{R}^n), \quad p > 1. \tag{5}$$

By (4), we must prove that

$$\left\| \int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n \, dy \right\|_p \le \|f\|_p. \tag{6}$$

In fact,

$$\begin{aligned}
&\int_{\mathbb{R}^n} f(x+y) \frac{R^{\alpha,b}(R|y|)}{|Ry|^{n-1}} R^n \, dy \\
&= \int_0^{1/R} \left(\int_{\Sigma_{n-1}} f(x-ty')\, dy' \right) R^{\alpha,b}(Rt) R \, dt \\
&\quad + \int_{1/R}^{\infty} \left(\int_{\Sigma_{n-1}} f(x-ty')\, dy' \right) R^{\alpha,b}(Rt) R \, dt \\
&= I_1 + I_2.
\end{aligned}$$

We have

$$\begin{aligned}
|I_1| &\le \int_0^{1/R} \left(\int_{\Sigma_{n-1}} f(x-ty')\, dy' \right) (Rt)^{n-1} R \, dt \\
&= (\frac{1}{R})^{-n} \int_{|y|<1/R} f(x-y)\, dy \le C\,\mathrm{HL}(f)(x)
\end{aligned}$$

and

$$\begin{aligned}
|I_2| &\le \left| \int_{1/R}^{\infty} (Rt)^{-b-1} R \left(\int_{\Sigma_{n-1}} f(x-ty')\, dy' \right) dt \right| \\
&= \left| \int_{1/R}^{\infty} (Rt)^{-b-1} R t^{-n+1} d \left(\int_0^t \tau^{n-1} (\int_{\Sigma_{n-1}} f(x-\tau y')\, dy')\, d\tau \right) \right| \\
&\le \left| R^n \int_0^{1/R} \tau^{n-1} \left(\int_{\Sigma_{n-1}} f(x-\tau y')\, dy' \right) d\tau \right| \\
&\quad + \left| C \int_{1/R}^{\infty} R^{-b} t^{-n-b-1} \int_0^t \tau^{n-1} \left(\int_{\Sigma_{n-1}} f(x-\tau y')\, dy' \right) d\tau \right| \\
&\le C\,\mathrm{HL}(f)(x) + \mathrm{HL}(f)(x) C \int_{1/R}^{\infty} R^{-b} t^{-b-1}\, dt \le C\,\mathrm{HL}(f)(x).
\end{aligned}$$

Thus (6) is true. So (5) is also true, therefore $\{T_z\}$ is an admissible family of operators in the sense of E.M. Stein.

Now we write

$$\jmath_\lambda g = G_\lambda * g, \quad \hat{G}_\lambda(x) = (1+4\pi^2|x|^2)^{-\frac{\lambda}{2}}.$$

A multiplier theorem (see [3, p. 96]) implies that

$$\|\jmath_{i\eta} g\|_p \le P(\eta)\|g\|_p,$$

where $P(x)$ is a polynomial of degree $k > n/2$. Note that $\operatorname{Re}\alpha > (n-1)/2$ and we have

$$\begin{aligned}\|T_{i\eta} g\|_p &\le \|F_0^{\alpha,b,k}(\jmath_{2i\eta} g)\|_p \le \|M_0^{\alpha,b}(\jmath_{2i\eta} g)\|_p \\ &\le Ce^{|\tau|^2}\|\jmath_{2i\eta} g\|_p \le Ce^{|\tau|^2}P(2\eta)\|g\|_p.\end{aligned}$$

By Lemma 2, we get

$$\begin{aligned}\|T_{1+i\eta} g\|_p &< \|F_2^{\alpha,b,k}(\jmath_{2+2i\eta} g)\|_p \le \|M_2^{\alpha,b}(\jmath_{2+2i\eta} g)\|_p \\ &\le Ce^{|\tau|^2}\|\jmath_{2+2i\eta} g\|_{L_2^p} \le Ce^{|\tau|^2}\|\jmath_{2i\eta} g\|_p \le Ce^{|\tau|^2}P(2\eta)\|g\|_p.\end{aligned}$$

Using Stein's interpolation theorem of analytic operators, we obtain

$$\|F_\lambda^{\alpha,b,k} f\|_p \le \|T_{\frac{\lambda}{2}} g\|_p \le Ce^{|\tau|^2}\|g\|_p \le Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p}.$$

Finally, by Lebesgue's monotone convergence theorem, we get

$$\|M_\lambda^{\alpha,b} f\|_p \le Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p}.$$

□

If we define T_z in the proof of Lemma 3 as

$$(T_z g)(x) = \sum_{j\in A_k} r_j^z \chi_{S_j}(x)\{S_{r_j}^{\alpha,b}(\jmath_z g)(x) - (\jmath_z g)(x)\}\phi_j$$

and using the same method of proof of $2'$, we have

Lemma 3′. *Let* $0 \le \lambda \le 1$, $1 < p < \infty$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}$, $b > 1$. *If* $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, *then*

$$\|M_\lambda^{\alpha,b} f\|_p \le Ce^{|\tau|^2}\|f\|_{\mathcal{L}_\lambda^p},$$

where C *is independent of* τ *and* f.

Define

$$(N_\lambda^{\alpha,b} f)(x) = \sup_{R>0} R^\lambda |(S_R^{\alpha+1,b} f)(x) - (S_R^{\alpha,b} f)(x)|.$$

Lemma 4. *Let $\alpha = \sigma + i\tau$, $\sigma > 0$, $0 \le \lambda \le \tau$, $b > \lambda$. If $f \in \mathcal{L}^2_\lambda(\mathbb{R}^n)$, then*

$$\|N^{\alpha,b}_\lambda f\|_2 \le \|f\|_{\mathcal{L}^2_\lambda}.$$

Proof: Let $\beta \in \mathbb{C}$, $\operatorname{Re}\beta > \frac{1}{2}$, $\delta > -\frac{1}{2}$, $0 < \lambda < 2$. Then

$$\begin{aligned}
&(S^{\beta+\delta+1,b}_R f)(x) - (S^{\beta+\delta,b}_R f)(x) \\
&= \int_0^R t^{n-1} \left(\int_{\Sigma_{n-1}} \hat{f}(x - ty')\, dy'\right) \left[\left(1 - \frac{t^b}{R^b}\right)^{\beta+\delta+1} - \left(1 - \frac{t^b}{R^b}\right)^{\beta+\delta}\right] dt \\
&= \int_0^R t^{n-1} \left(1 - \frac{t^b}{R^b}\right)^{\beta} \left(\int_{\Sigma_{n-1}} \hat{f}(x - ty')\, dy'\right) \\
&\quad \times \left[\left(1 - \frac{t^b}{R^b}\right)^{1+\delta} - \left(1 - \frac{t^b}{R^b}\right)^{\delta}\right] dt \\
&= \int_0^R (1 - \frac{t^b}{R^b})^{\beta} d \left(\int_0^t \tau^{n-1} \int_{\Sigma_{n-1}} \hat{f}(x - \tau y')\, dy' \right.\\
&\quad \left. \times \left[\left(1 - \frac{\tau^b}{R^b}\right)^{1+\delta} - \left(1 - \frac{\tau^b}{R^b}\right)^{\delta}\right] d\tau\right) \\
&= \beta \int_0^R \left(1 - \frac{t^b}{R^b}\right)^{\beta-1} b\frac{t^{b-1}}{R^b} \left\{(S^{\delta+1,b}_t f)(x) - (S^{\delta,b}_t f)(x)\right\} dt \\
&= b\beta \frac{1}{R^{b(\beta-1)+b}} \int_0^R (R^b - t^b)^{\beta-1} t^{b-1} \left\{(S^{\delta+1,b}_t f)(x) - (S^{\delta,b}_t f)(x)\right\} dt.
\end{aligned}$$

Writing $(G^{\delta,b}_\lambda f)(x) = \left\{\int_0^\infty t^{2\lambda-1} |(S^{\delta+1,b}_t f)(x) - (S^{\delta,b}_t f)(x)|^2\, dt\right\}^{\frac{1}{2}}$, we have

$$\begin{aligned}
&R^\lambda |(S^{\beta+\delta+1,b}_R f)(x) - (S^{\beta+\delta,b}_R f)(x)| \\
&\le b|\beta| \frac{1}{R^{b\operatorname{Re}\beta - \lambda}} \left\{\int_0^R (R^b - t^b)^{2\operatorname{Re}\beta-2} t^{2b-2\lambda-1}\, dt\right\}^{\frac{1}{2}} \\
&\quad \times \left\{\int_0^R t^{2\lambda-1} \left|(S^{\delta+1,b}_t f)(x) - (S^{\delta,b}_t f)(x)\right|^2 dt\right\}^{\frac{1}{2}} \\
&\le b|\beta| \left\{\int_0^1 (1 - t^b)^{2\operatorname{Re}\beta-2} t^{2b-2\lambda-1}\, dt\right\}^{\frac{1}{2}} (G^{\delta,b}_\lambda f)(x) \le C(G^{\delta,b}_\lambda f)(x).
\end{aligned}$$

Choose $\delta = \frac{\sigma-1}{2} > -\frac{1}{2}$, $\beta = \alpha - \delta = \frac{\sigma+1}{2} + i\tau$. Then

$$\|N_\lambda^{\alpha,b} f\|_2^2 \le C \|G_\lambda^{\frac{\sigma-1}{2},b} f\|_2^2.$$

Further,

$$\begin{aligned}
&\|G_\lambda^{\frac{\sigma-1}{2},b} f\|_2^2 \\
&\le \int_{\mathbb{R}^n} \left(\int_0^\infty t^{2\lambda-1} |(S_t^{\frac{\sigma+1}{2},b} f)(x) - (S_t^{\frac{\sigma-1}{2},b} f)(x)|^2 \, dt \right) dx \\
&= \int_0^\infty t^{2\lambda-1} \int_{|y|<t} \left| \left(1 - \frac{|y|^b}{t^b}\right)^{\frac{\sigma+1}{2}} - \left(1 - \frac{|y|^b}{t^b}\right)^{\frac{\sigma-1}{2}} \right|^2 \\
&\quad \times |\hat{f}(y)|^2 \, dy \, dt \\
&= \int_0^\infty t^{2\lambda-1} \int_{|y|<t} \left(1 - \frac{|y|^b}{t^b}\right)^{\sigma-1} \frac{|y|^{2b}}{t^{2b}} |\hat{f}(y)|^2 \, dy \, dt \\
&= \int_{\mathbb{R}^n} |y|^{2b} |\hat{f}(y)|^2 \left[\int_{|y|}^\infty t^{2\lambda-2b-1} \left(1 - \frac{|y|^b}{t^b}\right)^{\sigma-1} dt \right] dy.
\end{aligned}$$

Setting $\frac{|y|}{t} = t$, we get

$$\int_{|y|}^\infty t^{2\lambda-2b-1} (1 - \frac{|y|^b}{t^b})^{\sigma-1} \, dt = \int_0^1 |y|^{2\lambda-2b-1} t^{-2\lambda+2b+1} (1-t^b)^{\sigma-1} \frac{|y|}{t^2} \, dt.$$

Therefore

$$\|G_\lambda^{\frac{\sigma-1}{2},b}\|_2^2 = \int_{\mathbb{R}^n} |y|^{2\lambda} |\hat{f}(y)|^2 \, dy \int_0^1 t^{2b-2\lambda-1} (1-t^b)^{\sigma-1} \, dt \le C \|f\|_{\mathcal{L}_\lambda^2}^2.$$

□

Proof of Theorem 2: Let $f = \jmath_\lambda g$, $g \in L^p(\mathbb{R}^n)$. For $\sigma > 0$, we choose $k \in \mathbb{N}$ such that $\sigma + k > \frac{n-1}{2}$. By Lemma 3, Lemma 4 and inequality

$$(M_\lambda^{\alpha,b} f)(x) \le \sum_{j=0}^{k-1} (N_\lambda^{\alpha+j,b} f)(x) + (M_\lambda^{\alpha+k,b} f)(x),$$

we have

$$\|M_\lambda^{\alpha,b} f\|_2 \le C e^{|\tau|^2} \|f\|_{\mathcal{L}_\lambda^2} = C e^{|\tau|^2} \|g\|_2. \tag{7}$$

Let $p_1 > 1$, $\sigma > \frac{n-1}{2}$. Then by Lemma 3

$$\|M_\lambda^{\alpha,b} f\|_{p_1} \le Ce^{|\tau|^2} \|f\|_{\mathcal{L}_\lambda^{p_1}} = Ce^{|\tau|^2} \|g\|_{p_1}.$$

For $p_1 < p < 2$, there exists $0 < t < 1$ such that $\frac{1}{p} = \frac{1-t}{2} + \frac{t}{p_1}$. Let $u_0 > 0$, $u_1 > \frac{n-1}{2}$, $\delta(z) = u_0(1-z) + u_1 z$, $0 \le \operatorname{Re} z \le 1$. Then

$$\delta(t) = u_0(1-t) + u_1 t = u_0 + (u_1 - u_0)t = u_0 + (u_1 - u_0)\frac{\frac{2}{p} - 1}{\frac{2}{p_1} - 1}.$$

Thus

$$\delta(t) \longrightarrow \frac{n-1}{2} \cdot \frac{\frac{2}{p} - 1}{\frac{2}{p_1} - 1} > \frac{n-1}{2}(\frac{2}{p} - 1) \text{ as } u_0 \searrow 0,\ u_1 \searrow \frac{n-1}{2},$$

and

$$\delta(t) \longrightarrow \frac{n-1}{2}(\frac{2}{p} - 1) \text{ as } p_1 \searrow 1.$$

So for $\sigma > \frac{n-1}{2}(|\frac{2}{p} - 1|)$, there exist u_0, u_1, p_1 such that $\delta(t) = \sigma$, where $t = \frac{1/p - 1/2}{1/p_1 - 1/2}$. Thus for given $1 < p < 2$ and $\sigma > \frac{n-1}{2}(\frac{2}{p} - 1)$, we can find u_0, u_1 and p_1 such that $1 < p_1 < p < 2$, $\delta(t) = \sigma$.

Fix such u_0, u_1, p_1. Let $\{R_j\}$ be a sequence consisting of all positive rational numbers. Denote $A_k = \{R_1, \cdots, R_k\}$, and

$$(F_\lambda^{\alpha,b,k} f)(x) = \sup_{R \in A_k} \{R^\lambda (S_R^{\alpha,b} f)(x) - f(x)\}.$$

Then we have $(F_\lambda^{\alpha,b,k} f)(x) \le (F_\lambda^{\alpha,b,k+1} f)(x)$, and,

$$(M_\lambda^{\alpha,b} f)(x) = \lim_{k \to \infty} (F_\lambda^{\alpha,b,k} f)(x).$$

For $1 \le j \le k$, let

$$E_j = \left\{ x \in \mathbb{R}^n : \sup_{R \in A_k} \left[R^\lambda |(S_R^{\alpha,b} f)(x) - f(x)| \right] = R_j^\lambda |(S_{R_j}^{\alpha,b} f)(x) - f(x)| \right\},$$

and $F_1 = E_1$, $F_j = E_j - \bigcup_{i=1}^{j-1} E_i$, $j = 2, 3, \cdots, k$. Define

$$(T_z g)(x) = \sum_{j=1}^{k} R_j^\lambda \chi_{F_j}(x) \{S_{R_j}^{\delta(z),b} (\jmath_\lambda g)(x) - (\jmath_\lambda g)(x)\} \phi_j(x),$$

where

$$\phi_j(x) = \operatorname{sign}\{S_{R_j}^{\sigma,b}(\jmath_\lambda g)(x) - (\jmath_\lambda g)(x)\}.$$

It is easy to verify that $\{T_z\}$ is an admissible family of linear operators. Since

$$\|T_{i\tau}g\|_2 \le \|F_\lambda^{\delta(i\tau),b,k} f\|_2 \le \|M_\lambda^{\delta(i\tau),b}\|_2 \le Ce^{(u_1-u_0)^2\tau^2}\|g\|_2,$$

$$\|T_{1+i\tau}g\|_{p_1} \le \|F_\lambda^{\delta(1+i\tau),b,k} f\|_{p_1} \le Ce^{(u_1-u_0)^2\tau^2}\|g\|_{p_1},$$

by Stein's interpolation theorem of analytic operators, we have

$$\|F_\lambda^{\sigma,b,k} f\|_p = \|F_\lambda^{\delta(t),b,k} f\|_p = \|T_t g\|_p \le C\|g\|_p = C\|f\|_{\mathcal{L}_\lambda^p}.$$

Hence, by the monotone convergence theorem we obtain

$$\|M_\lambda^{\sigma,b} f\|_p \le C\|f\|_{\mathcal{L}_\lambda^p}, 1 < p \le 2.$$

Finally, it should be pointed out that the proof in the case $2 < p < \infty$ is similar to the above. □

Applying the same method and using Lemma 2′, we have

Theorem 2′. *Let* $0 \le \lambda \le 1$, $1 < p < \infty$, $b > 1$, $\alpha = \sigma + i\tau$, $\sigma > \frac{n-1}{2}|\frac{2}{p} - 1|$. *If* $f \in \mathcal{L}_\lambda^p(\mathbb{R}^n)$, *then*

$$\|M_\lambda^{\sigma,b} f\|_p \le C\|f\|_{\mathcal{L}_\lambda^p},$$

where C *is independent of* f.

To prove Theorem 1, we first need to establish a weak type estimation of the maximal operator $M_1^{\frac{n-1}{2},b}$ on any block.

Lemma 5. *Let* $a(x)$ *be a* $(q,1)$*-block,* $b > 1$. *Then*

$$|\{x : (M_1^{\frac{n-1}{2},b} a)(x) > \lambda\}| \le C\lambda^{-1}, \tag{8}$$

where C *is independent of* λ *and* $a(x)$.

Proof: We have

$$\begin{aligned} &R\left\{(S_R^{\frac{n-1}{2},b} a)(x) - a(x)\right\} \\ &= R\left\{\left(\frac{b}{2}\right)^{\frac{n-1}{2}} (S_R^{\frac{n-1}{2}} a)(x) + \sum_{k=1}^m C_k (S_R^{\frac{n-1}{2}+k} a)(x) \right. \\ &\quad \left. + \int_{\mathbb{R}^n} a(x+y) \frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n\, dy - a(x)\right\}. \end{aligned}$$

Denote

$$v_0 = \int_{\mathbb{R}^n} \frac{R^{\frac{n-1}{2},b}(|y|)}{|y|^{n-1}}\,dy = 1 - \sum_{k=1}^{m} C_k - \left(\frac{b}{2}\right)^{\frac{n-1}{2}},$$

$$(\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) = \sup_{R>0} R\left|\int_{\mathbb{R}^n} [a(x+y)-a(x)]\frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n\,dy\right|.$$

By Lemma 4 in [2], we must prove that

$$|\{x : (\mathcal{M}_1^{\frac{n-1}{2},b} a)(x) > \lambda\}| \le C\lambda^{-1}.$$

In fact, setting $g(x,t) = \int_{\Sigma_{n-1}} [a(x-ty') - a(x)]\,dy'$, we have

$$\begin{aligned}
&R\left|\int_{\mathbb{R}^n} [a(x+y)-a(x)]\frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}} R^n\,dy\right| \\
&= R\left|\int_0^\infty \left\{\int_{\Sigma_{n-1}} [a(x-\frac{t}{R}y') - a(x)]\,dy'\right\} R^{\frac{n-1}{2},b}(t)\,dt\right| \\
&= R\left|\int_0^\infty g(x,\frac{t}{R}) R^{\frac{n-1}{2},b}(t)\,dt\right| \\
&= R\left|\int_0^\infty g(x,\frac{t}{R}) d\left(\int_\tau^\infty R^{\frac{n-1}{2},b}(\tau)\,d\tau\right)\right| \\
&= R\int_0^\infty \frac{d}{dt} g(x,\frac{t}{R}) A(t)\,dt,
\end{aligned}$$

where $A(t) = \int_t^\infty R^{\frac{n-1}{2},b}(\tau)\,d\tau$.

For $t > 1$,

$$|A(t)| \le \int_t^\infty |R^{\frac{n-1}{2},b}(\tau)|\,d\tau \le C\int_t^\infty t^{-b-1}\,d\tau = Ct^{-b} < C\frac{1}{t}$$

and for $0 < t < 1$,

$$|A(t)| \le |\int_1^\infty R^{\frac{n-1}{2},b}(\tau)| + \int_t^1 |R^{\frac{n-1}{2},b}(\tau)|\,d\tau \le C \le \frac{C}{t}.$$

Hence

$$
\begin{aligned}
&R\left|\int_{\mathbb{R}^n}[a(x+y)-a(x)]\frac{R^{\frac{n-1}{2},b}(|y|R)}{|Ry|^{n-1}}R^n\,dy\right| \\
&= C\int_0^\infty\left\{\int_{\Sigma_{n-1}}|D_x a(x+\frac{t}{R}y')|\,dy'\right\}|A(t)|\,dt \\
&= C\int_0^\infty\left\{\int_{\Sigma_{n-1}}|D_x a(x+ty')|\,dy'\right\}|A(Rt)|R\,dt \\
&\le C\int_0^\infty\int_{\Sigma_{n-1}}|D_x a(x+ty')|\,dy'\frac{dt}{t} \\
&= C\int_{\mathbb{R}^n}|D_x a(x+y)|\frac{dy}{|y|^n} = C\int_{\mathbb{R}^n}\frac{|Da(u)|}{|u-x|^n}\,dy.
\end{aligned}
$$

Therefore

$$(\mathcal{M}_1^{\frac{n-1}{2},b}a)(x) \le C\int_Q\frac{|Da(u)|}{|x-u|^n}\,du,$$

where supp $a(x)\subset Q$.

Let $\widetilde{Q}=2Q$. Then for $x\notin\widetilde{Q}$,

$$
\begin{aligned}
(\mathcal{M}_1^{\frac{n-1}{2},b}a)(x) &\le \frac{C}{|x|^n}\int_Q|Da(u)|\,du \le \frac{C}{|x|^n}\|Da(u)\|_q|Q|^{1-\frac{1}{q}} \\
&\le \frac{C}{|x|^n}\|a\|_{\mathcal{L}_1^q}|Q|^{1-\frac{1}{q}} \le \frac{C}{|x|^n}.
\end{aligned}
$$

So

$$\left|\left\{x\notin\widetilde{Q}:(\mathcal{M}_1^{\frac{n-1}{2},b}a)(x)>\lambda,\lambda\le\frac{1}{|Q|}\right\}\right|\le C\lambda^{-1}.$$

Clearly,

$$|\{x\in\widetilde{Q}:(\mathcal{M}_1^{\frac{n-1}{2},b}a)(x)>\lambda,\ \lambda\le\frac{1}{|Q|}\}|\le|\widetilde{Q}|\le C\lambda^{-1}.$$

By Theorem 2′, we have

$$\|\mathcal{M}_1^{\frac{n-1}{2},b}a\|_q\le C\|a\|_{\mathcal{L}_1^q},\ 1<q<\infty.$$

Thus

$$
\begin{aligned}
&\left|\left\{x\in\mathbb{R}^n:(\mathcal{M}_1^{\frac{n-1}{2},b}a)(x)>\lambda,\ \lambda>\frac{1}{|Q|}\right\}\right| \\
&\le C(\lambda^{-1}\|a\|_{\mathcal{L}_\lambda^q})^q\le C(\frac{|Q|^{\frac{1}{q}-1}}{\lambda})^q\le C\lambda^{-1}.
\end{aligned}
$$

Therefore (8) holds. □

Proof of Theorem 1: Suppose $f(x) \in B^1_q(\mathbb{R}^n)$. Then

$$f(x) = \sum_k m_k b_k(x) = \sum_{k=1}^{N} m_k b_k(x) + \sum_{k=N+1}^{\infty} m_k b_k(x) = g(x) + h(x),$$

where b_k is a $(q,1)$-block and $N\{m_k\} < \infty$.

To complete the proof of Theorem 1, we must prove for all $\lambda > 0$,

$$\left|\left\{x : \limsup_{R\to\infty} |R\{(S_R^{\frac{n-1}{2},b} f)(x) - f(x)\}| > \lambda\right\}\right| = 0.$$

Since $g \in L^q_1(\mathbb{R}^n)$, by theorem in [9] we have

$$(S_R^{\frac{n-1}{2},b} g)(x) - g(x) = o(\frac{1}{R}),\ a.e.$$

So

$$\left|\left\{x : \limsup_{R\to\infty} |R\{(S_R^{\frac{n-1}{2},b} g)(x) - g(x)\}| > \lambda/2\right\}\right| = 0.$$

Thus by Lemma 5 and Lemma 1.3 in [10], we get

$$\begin{aligned}
&|\{x : \limsup_{R\to\infty} |R\{(S_R^{\frac{n-1}{2},b} f)(x) - f(x)\}| > \lambda\}| \\
&\leq |\{x : (M_1^{\frac{n-1}{2},b} h)(x) > \lambda/2\}| \\
&\leq C\lambda^{-1} \sum_{k=N+1}^{\infty} |m_k| \left(1 + \log \frac{\sum_l |m_l|}{|m_k|}\right) \longrightarrow 0, \quad N \to \infty.
\end{aligned}$$

□

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