Convolution operators on the dual of hypergroup algebras

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Abstract. Let X be a hypergroup. In this paper, we define a locally convex topology β on $L(X)$ such that $(L(X), \beta)^*$ with the strong topology can be identified with a Banach subspace of $L(X)^*$. We prove that if X has a Haar measure, then the dual to this subspace is $L_C(X)^{**} = \mathrm{cl}\{F \in L(X)^{**}; F \text{ has compact carrier}\}$. Moreover, we study the operators on $L(X)^*$ and $L_0^\infty(X)$ which commute with translations and convolutions. We prove, among other things, that if $\mathrm{wap}(L(X))$ is left stationary, then there is a weakly compact operator T on $L(X)^*$ which commutes with convolutions if and only if $L(X)^{**}$ has a topologically left invariant functional. For the most part, X is a hypergroup not necessarily with an involution and Haar measure except when explicitly stated.

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1. Introduction and notations

The theory of hypergroups was initiated by Dunkl [4], Jewett [8] and Spector [19] and has received a good deal of attention from harmonic analysts. It is still unknown if an arbitrary hypergroup admits a left Haar measure, but commutative hypergroups with an involution [1] and compact hypergroups with an involution have a Haar measure. The lack of Haar measure and involution presents many difficulties, however, we succeed to get some results.

Let X be a locally compact Hausdorff space with convolution measure algebra $M(X)$ and probability measures $M_p(X)$ ([4], [5], [6]). Also, let $L(X) = \{\mu \in M(X);\ x \mapsto |\mu| * \delta_x \text{ is norm continuous}\}$ ([5], [15]). We assume that X is a foundation, i.e.

$$X = \mathrm{cl}(\bigcup\{\mathrm{supp}\,\mu;\ \mu \in L(X)\}).$$

It is known that $L(X)$ is an L-ideal of $M(X)$ and $L(X)$ has a positive bounded approximate identity with norm one ([5, Lemma 1]).

If $L(X)^*$, $L(X)^{**}$ are the first and second duals of $L(X)$ respectively, the first Arens product in $L(X)^{**}$ is defined by

$$\langle FG, f\rangle = \langle F, Gf\rangle, \langle Gf, \mu\rangle = \langle G, f\mu\rangle,$$

where $\mu, \nu \in L(X)$, $f \in L(X)^*$ and $F, G \in L(X)^{**}$. In addition, we define

$$\langle \mu f, \nu \rangle = \langle f, \nu * \mu \rangle, \langle f\mu, \nu \rangle = \langle f, \mu * \nu \rangle$$

where $\mu \in M(X)$, $\nu \in L(X)$ and $f \in L(X)^*$. Most of our notation in this paper is taken from [4], [14].

The paper is organized as follows. In Section 2, we introduce a locally convex topology β on $L(X)$, and prove that the strong topology on $(L(X), \beta)^*$ can be identified with a Banach subspace of $L(X)^*$, and the dual to this subspace is $L_C(X)^{**}$ (when X has Haar measure) where

$$L_C(X)^{**} = \mathrm{cl}\{F \in L(X)^{**};\ F \text{ has compact carrier}\}$$

is defined in [14].

In Section 3, we deal with the operators on $L(X)^*$ and $L_0(X)^*$ which commute with translations and convolutions, and we show that if wap$(L(X))$ is left stationary, then there is a weakly compact operator T on $L(X)^*$ which commutes with convolutions if and only if $L(X)^{**}$ has a topologically left invariant functional.

2. Locally convex topology on $L(X)$

Let X be a hypergroup. If (K_n) is an increasing sequence of compact subsets of X and (a_n) is a sequence in $(0, \infty)$ with $a_n \longrightarrow \infty$, then we define

$$U((K_n), (a_n)) = \{\mu \in L(X);\ \|\mu\chi_{K_n}\| \le a_n, n \in \mathbb{N}\}.$$

It is clear that the set of all $U((K_n), (a_n))$ is a base of neighbourhoods of zero for a locally convex topology β on $L(X)$. We write $L_0(X)^*$ for the dual $(L(X), \beta)$.

If $f \in L(X)^*$, we define

$$\|f\|_A = \sup\{|\langle f, \mu \rangle|, \mu \in L(X) \text{ and } \operatorname{supp} \mu \subseteq A,\ \|\mu\| \le 1\}$$

where A is a Borel subset of X. Also, we take

$$L_0^\infty(X) = \{f \in L(X)^*; \|f\|_{X \setminus K} \longrightarrow 0 \text{ where } K \text{ is compact and } K \uparrow X\}$$

([12, Definition 2.4]). In this paper for $f \in L(X)^*$ and $\mu \in L(X)$, we define $\langle f\chi_A, \mu \rangle = \langle f, \mu\chi_A \rangle$ (A is a Borel subset of X).

Lemma 2.1. *Let X be a hypergroup. Then $L_0^\infty(X) = L_0(X)^*$.*

Proof: Let $f \in L_0(X)^*$, and $\epsilon > 0$ be given. There exists $U((K_n), (a_n))$ such that for $\mu \in L(X)$ with $\|\mu\chi_{K_n}\| \le a_n$ ($n \in \mathbb{N}$), we have $|\langle f, \mu \rangle| < \epsilon$. Now, if $\mu \in L(X)$ and $\|\mu\| \le 1$,

$$|\langle f, \mu \rangle| < \epsilon/a$$

where $a = \inf\{a_n, n \in \mathbb{N}\}$. Consequently $f \in L(X)^*$. On the other hand, there exists $n_\circ \in \mathbb{N}$ such that for all $n \geq n_\circ$ ($n \in \mathbb{N}$), $a_n \geq 1$. Therefore if $\mu \in L(X)$ with $\|\mu\| \leq 1$, then for every $n > n_\circ$ we have $\|\mu\chi_{K_n}\| \leq a_n$. Hence $\|f\|_{X\setminus K_n} < \epsilon$ ($n \geq n_\circ$) and it follows that $f \in L_0^\infty(X)$.

To prove the converse, let $f \in L_0^\infty(X)$. There exists an increasing sequence (K_n) of compact subsets X such that $b_n = \|f\|_{X\setminus K_n} \longrightarrow 0$. Now for $i \in \mathbb{N}$, there exists $n_i \in \mathbb{N}$ such that $b_{n_i} \leq 1/(1+i)2^i$. For all $\mu \in U((K_{n_i}),(i))$, we can write

$$|\langle f, \mu\rangle| \leq \sum_{i=1}^{\infty} |\langle f\chi_{K_{n_i}\setminus K_{n_{i-1}}}, \mu\chi_{K_{n_i}\setminus K_{n_{i-1}}}\rangle|$$

where $K_{n_\circ} = \emptyset$. Hence

$$\begin{aligned} |\langle f, \mu\rangle| &\leq \sum_{i=1}^{\infty} \|f\chi_{K_{n_i}\setminus K_{n_{i-1}}}\| \|\mu\chi_{K_{n_i}\setminus K_{n_{i-1}}}\| \\ &\leq \sum_{i=1}^{\infty} \|f\|_{X\setminus K_{n_{i-1}}} \|\mu\chi_{K_{n_i}}\| \leq \|f\| + 1. \end{aligned}$$

Consequently $f \in L_0(X)^*$. □

Lemma 2.2. *Let β be as above. Then the following statements hold:*

(1) *$H \subseteq L(X)$ is β bounded if and only if H is norm bounded;*
(2) *the strong topology on $(L(X),\beta)^*$ can be identified with the norm topology on $L_0^\infty(X)$.*

Proof: (1) Let H be β bounded. If (μ_n) is a sequence in H and (α_n) is a sequence of scalars such that $\alpha_n \longrightarrow 0$ as $n \longrightarrow \infty$, then $\alpha_n\mu_n \longrightarrow 0$ as $n \longrightarrow \infty$. Indeed, we can find an increasing sequence (K_n) of compact subsets X such that $\|\mu\chi_{X\setminus K_n}\| \leq 1/\sqrt{|\alpha_n|}$ (without loss of generality we can assume that $\alpha_n \neq 0$ for all $n \in \mathbb{N}$). But H is β bounded, so there exists $m \in \mathbb{N}$ with $H \subseteq mU((K_n),(1/\sqrt{|\alpha_n|}))$. It follows that for every $n \in \mathbb{N}$, we have

$$\|\alpha_n\mu_n\| \leq \|\alpha_n\mu_n\chi_{K_n}\| + \|\alpha_n\mu_n\chi_{X\setminus K_n}\| \leq (m+1)\sqrt{|\alpha_n|}.$$

Consequently H is norm bounded ([16]). The converse is obvious.

(2) Let $B = \{\mu \in L(X); \|\mu\| < 1\}$, $f \in L_0^\infty(X)$ and $\|f\| < 1$. We consider $\delta = 1 - \|f\|$. Since B is norm bounded, B is β bounded. Hence B is weak bounded $(\sigma(L(X),(L(X),\beta)^*))$. But

$$\{g \in L_\circ^\infty(X); \rho_B(g-f) < \delta\} \subseteq \{h \in L_\circ^\infty(X); \|h\| < 1\}$$

where for $h \in L(X)^*$, $\rho_B(h) = \sup\{|\langle h, \mu\rangle|; \mu \in B\}$. So $\{f \in L_0^\infty(X); \|f\| < 1\}$ is open in the strong topology on $L_0^\infty(X)$.

Now, let A be a weak bounded subset of $(L(X), \beta)$. So A is β bounded, by (1) A is norm bounded. Therefore there exists $m \in \mathbb{N}$ such that $\|\mu\| < m$ for all $\mu \in A$. If $\epsilon > 0$, $f \in L_0^\infty(X)$ and $\rho_A(f) < \epsilon$, then

$$\{h \in L_0^\infty(X); \|h - f\| < (\epsilon - \rho_A(f))/m\} \subseteq \{g \in L_\circ^\infty(X); \rho_A(g) < \epsilon\}.$$

Consequently the strong topology is identified with the norm topology. □

Let H be a subspace of $L_0^\infty(X)$. H is called left topologically introverted if for each $F \in H^*$, $f \in H$ and $\mu \in L(X)$, both Ff and $f\mu$ are also in H.

For $\psi \in C(X)$ and $\mu \in M(X)$ we define $\langle \psi, \mu\rangle = \int \psi(x)\, d\mu(x)$. So $C(X) \subseteq M(X)^*$. Now, let $f \in C_\circ(X)$ and $\mu \in M(X)$. Then the map $\psi(x) = \langle f, \mu * \delta_x\rangle$ is in $C_\circ(X)$ [4], and

$$\int \psi(x) d\nu(x) = \int \langle f, \mu * \delta_x\rangle\, d\nu(x) = \langle f\mu, \nu\rangle$$

where $\nu \in L(X)$. So we can regard $f\mu$ as a continuous function vanishing at infinity. Consequently $C_\circ(X)$ is a left topologically introverted subspace of $L_0^\infty(X)$.

Definition 2.3. A compact subset K of X is said to be a carrier for $F \in L_0^\infty(X)^*$ (respectively $F \in L(X)^{**}$) if for all $f \in L_0^\infty(X)$ (respectively $f \in L(X)^*$) $\langle F, f\chi_K\rangle = \langle F, f\rangle$.

We know that if X is a hypergroup with an involution and Haar measure ([1], [2]), then $L^1(X)$ is an FC-algebra [11]. If X has an involution and Haar measure, then by an argument similar to the proof of ([12, Proposition 2.6]), the set of all functionals in $L_0^\infty(X)^*$ with compact carrier is dense in $L_0^\infty(X)^*$ (in the norm topology). In addition, if K_1 and K_2 are compact subsets of X, then for $\mu \in L(X) \cap M_p(X)$, $x \notin \bar{K}_2 * K_1$, we have $(\mu\chi_{K_2} * \delta_x)\chi_{K_1} = 0$ ([1, Lemma 1.2.11]). Hence for $f \in L^*(X)$, $\langle f\chi_{K_1}, \mu\chi_{K_2} * \delta_x\rangle = 0$. So for all $\nu \in L(X)$ with $\operatorname{supp}\nu \cap \bar{K}_2 * K_1 = \emptyset$,

$$\langle f\chi_{K_1}, \mu\chi_{K_2} * \nu\rangle = \int \langle f\chi_{K_1}, \mu\chi_{K_2} * \delta_x\rangle\, d\nu(x) = 0$$

(since for all $\mu \in L(X)$, $\nu \in M(X)$, we have $\mu * \nu = \int \mu * \delta_x\, d\nu(x)$ ([16, Theorem 3.20 and Theorem 3.27])). It follows that $\|f\chi_{K_1}\mu\chi_{K_2}\|_{X\setminus\bar{K}_2 * K_1} = 0$. Consequently $f\chi_{K_1}\mu\chi_{K_2} \in L_0^\infty(X)$. It is easy to see that for all $f \in L_0^\infty(X)$ and $\mu \in L(X)$ we have $f\mu \in L_0^\infty(X)$. Similarly $Ff \in L_0^\infty(X)$ whenever $F \in L_0^\infty(X)^*$ and $f \in L_0^\infty(X)$.

Lemma 2.4. *Let X be a hypergroup as above. If K_1 is a carrier for $F \in L_0^\infty(X)^*$ and K_2 is a carrier for $H \in L_0^\infty(X)^*$, then $K_1 * K_2$ is a carrier for FH.*

Proof: Let K_1 be a carrier for $F \in L_0^\infty(X)^*$ and K_2 be a carrier for $H \in L_0^\infty(X)^*$. For $\mu, \nu \in L(X)$ and $f \in L_0^\infty(X)$, since $\mu\chi_{K_1} * \nu\chi_{K_2} = (\mu\chi_{K_1} * \nu\chi_{K_2})\chi_{K_1 * K_2}$ ([1]), we have

$$\begin{aligned}\langle (f\mu\chi_{K_1})\chi_{K_2}, \nu\rangle &= \langle f\mu\chi_{K_1}, \nu\chi_{K_2}\rangle = \langle f, \mu\chi_{K_1} * \nu\chi_{K_2}\rangle \\ &= \langle f, (\mu\chi_{K_1} * \nu\chi_{K_2})\chi_{K_1*K_2}\rangle = \langle (f\chi_{K_1*K_2}\mu\chi_{K_1})\chi_{K_2}, \nu\rangle.\end{aligned}$$

So $(f\mu\chi_{K_1})\chi_{K_2} = (f\chi_{K_1*K_2}\mu\chi_{K_1})\chi_{K_2}$. But

$$\begin{aligned}\langle (Hf)\chi_{K_1}, \mu\rangle &= \langle H, (f\mu\chi_{K_1})\chi_{K_2}\rangle \\ &= \langle H, f\chi_{K_1*K_2}\mu\chi_{K_1}\rangle = \langle (Hf\chi_{K_1*K_2})\chi_{K_1}, \mu\rangle.\end{aligned}$$

Consequently

$$\langle FH, f\rangle = \langle F, (Hf)\chi_{K_1}\rangle = \langle F, (Hf\chi_{K_1*K_2})\chi_{K_1}\rangle = \langle (FH)\chi_{K_1*K_2}, f\rangle.$$

Therefore $K_1 * K_2$ is a compact carrier for FH. □

If X has an involution and Haar measure, then $L_0^\infty(X)$ is left topologically introverted and the first Arens product is well defined. Also there is an algebra isomorphism from $L_C(X)^{**}$ onto $L_\circ^\infty(X)^*$. Indeed, the restriction map is an isometric isomorphism.

We recall that a Banach algebra A is Arens regular if two Arens products on A^{**} coincide [3]. In the following theorem, we prove that if $L_\circ^\infty(X)^*$ is Arens regular, then $L(X)^{**}$ is unital.

Theorem 2.5. *Let X be a hypergroup such that the first and the second Arens multiplications are both well defined on $L_0^\infty(X)^*$. If $L_0^\infty(X)^*$ is Arens regular, then $L(X)^{**}$ is unital.*

Proof: If $L_0^\infty(X)^*$ is Arens regular, then $L(X)$ is Arens regular. Therefore by [3] $\mathrm{wap}(L(X)) = L(X)^*$ where $\mathrm{wap}(L(X)) = \{f \in L(X)^*; \{f\mu; \mu \in L(X), \|\mu\| \le 1\}$ is relatively weakly compact$\}$. Now, let $f \in L(X)^*$ and (e_α) be a bounded approximate identity of norm one ([5, Lemma 1]). Since $fe_\alpha \longrightarrow f$ in the weak*-topology and $f \in \mathrm{wap}(L(X))$, we have $f \in L(X)^*L(X)$. Consequently $L(X)^*$ factors on the left ([13]). It follows that $L(X)^{**}$ is unital. □

Medghalchi ([14], [15]) has defined $B = L(X)^*L(X)$ which is a Banach subspace of $L(X)^*$ and has shown B^* is a Banach algebra by Arens type product. For $\mu \in M(X)$ and $f\nu \in B$ we define $\langle \mu, f\nu\rangle = \langle f, \nu * \mu\rangle$, hence $\mu \in B^*$. We can show that if $L(X)$ is an ideal in B^*, then X is compact. Indeed if X is not compact and Σ is the family of all compact subsets of X, then Σ is a directed set under the set inclusion. Now we take $x_K \notin K$ (K is a compact subset of X). Let $m \in B^*$ be a weak*-limit of a subnet of $\{\delta_{x_K}\}$. Then for $\psi \in C_\circ(X)$, we have $\langle m, \psi\rangle = 0$. Hence $m \in C_\circ(X)^\perp$ ([14, Theorem 4]). On the other hand for all $\mu \in L(X)$, we have $\mu m \in L(X)$ and $\mu m \in C_\circ(X)^\perp$. So $\mu m = 0$ ([14, Theorem 4]). Consequently $m = 0$, which is a contradiction.

3. Convolution of operators

We know that for a locally compact abelian group G, $L(X) = L^1(G)$, $L(X)^* = L^\infty(G)$ and $f\delta_x = L_x f$ where $L_x(f)(y) = f(xy)$ ($f \in L^\infty(G)$, $x, y \in G$). Also, if $\psi \in L^1(G)$, $f\psi = \psi^\vee * f$ where $\psi^\vee(x) = \psi(x^{-1})$. The operators on $L^\infty(G)$ which commute with translations and convolutions have been studied by Lau and Pym in [12]. In [9], Larsen has studied some operators on $L^\infty(G)$, and has proved that if $\mathbb{Z}$ is the additive group of integers, then there exists $T \in M(L^\infty(\mathbb{Z}))$ ($M(L^\infty(\mathbb{Z}))$ is the set of all operators on $L^\infty(\mathbb{Z})$ which commute with translations [9]) which cannot be written as convolution with an element of $M(\mathbb{Z})$ ([9, p. 78]). Indeed, T is not weak*-weak* continuous. We show that if X is a hypergroup which has an involution and Haar measure and $T : L_0^\infty(X) \longrightarrow L_0^\infty(X)$ commutes with convolutions, i.e. $T(f\mu) = T(f)\mu$ for $f \in L_\circ^\infty(X)$ and $\mu \in L(X)$, then for some $\mu \in M(X)$ we have $T = \lambda_\mu{}^*$ where λ_μ is a left multiplier on $L(X)$ defined by $\lambda_\mu(\nu) = \mu * \nu$ for $\nu \in L(X)$. In this section, we may assume that all operators are bounded.

Theorem 3.1. *Let X be a hypergroup. Then the following statements hold:*

(1) *If $T : L(X)^* \longrightarrow L(X)^*$ is weak*-weak* continuous and $T(\delta_x f) = \delta_x T(f)$ for every $f \in L(X)^*$ and $x \in X$, then there exists a unique measure $\mu \in M(X)$ such that $T = \lambda_\mu{}^*$ and $\|T\| = \|\mu\|$. Indeed, the correspondence between T and μ defines an isometric isomorphism from $\{T; T : L(X)^* \longrightarrow L(X)^*$ is weak*-weak* continuous and $T(\delta_x f) = \delta_x T(f)$, $x \in X$, $f \in L(X)^*\}$ onto $M(X)$.*

(2) *If $T : L(X) \longrightarrow L(X)^*$ commutes with translations, i.e. $T(\mu * \delta_x) = T(\mu)\delta_x$ ($x \in X$, $\mu \in L(X)$), then there exists a unique $f \in L(X)^*$ such that $T(\mu) = f\mu$ for all $\mu \in L(X)$. In addition, $\|T\| = \|f\|$.*

(3) *Let X be a hypergroup with involution and Haar measure. If T is an operator on $L_0^\infty(X)$ such that $T(f\mu) = T(f)\mu$ for $f \in L_0^\infty(X)$ and $\mu \in L(X)$, then there exists a unique measure $\mu \in M(X)$ such that $T = \lambda_\mu{}^*$ and $\|T\| = \|\mu\|$. In addition, if T is compact then $\mu \in L(X)$. Moreover, there exists an isometric isomorphism from $\{T; T : L_0^\infty(X) \longrightarrow L_0^\infty(X), T(f\mu) = T(f)\mu$ for $f \in L_0^\infty(X)$ and $\mu \in L(X)\}$ onto $M(X)$.*

Proof: We know that $T^* : L(X)^{**} \longrightarrow L(X)^{**}$ is a bounded linear map. On the other hand, since T is weak*-weak* continuous, for $\mu \in L(X)$, $T^*(\mu) \in L(X)^{**}$ is weak* continuous. Hence $T^*(\mu) \in L(X)$ ([16, Chapter 3]). But for $x \in X$ and

$\mu \in L(X)$, $T^*(\mu * \delta_x) = T^*(\mu) * \delta_x$. Consequently, for $f \in L(X)^*$ and $\nu \in L(X)$ we have

$$\langle f, T^*(\mu * \nu)\rangle = \langle T(f), \mu * \nu\rangle = \int \langle T(f), \mu * \delta_x\rangle \, d\nu(x) = \int \langle f, T^*(\mu * \delta_x)\rangle \, d\nu(x)$$
$$= \int \langle f, T^*(\mu) * \delta_x\rangle \, d\nu(x) = \langle f, T^*(\mu) * \nu\rangle.$$

Consequently for all $\mu, \nu \in L(X)$, we have $T^*(\mu*\nu) = T^*(\mu)*\nu$. Hence there exists a measure $\mu \in M(X)$ such that for $\nu \in L(X)$, $T^*(\nu) = \mu * \nu$ ([5, Proposition 1]). It is clear that μ is unique and $\|T^*\| = \|\mu\|$. Also, it is obvious that $T = {\lambda_\mu}^*$ and the correspondence between T and μ is an isometric isomorphism.

(2) Let $T^* : L(X)^{**} \longrightarrow L(X)^*$ be adjoint to T. For all $\mu, \nu, \eta \in L(X)$, since $T(\mu * \delta_x) = T(\mu)\delta_x$ $(x \in X)$, we have

$$\langle T(\mu * \nu), \eta\rangle = \langle T^*(\eta), \mu * \nu\rangle = \int \langle T^*(\eta), \mu * \delta_x\rangle \, d\nu(x)$$
$$= \int \langle T(\mu * \delta_x), \eta\rangle \, d\nu(x) = \int \langle T(\mu), \delta_x * \eta\rangle \, d\nu(x) = \langle T(\mu)\nu, \eta\rangle.$$

Consequently $T(\mu * \nu) = T(\mu)\nu$.

Now, let (e_α) be a bounded approximate identity with norm one. Then without loss of generality, we may assume that $T(e_\alpha) \longrightarrow f$ $(f \in L(X)^*)$ in the weak*-topology. It is clear that $T(\mu) = f\mu$ for all $\mu \in L(X)$. Since $L(X)$ has a bounded approximate identity, f is unique. Now, let $\epsilon > 0$ be given. We take $\nu \in L(X)$ $(\|\nu\| = 1)$ such that $\|f\| \le |\langle f, \nu\rangle| + \epsilon$. Since

$$|\langle f, \nu\rangle| \le \lim_\alpha |\langle f e_\alpha, \nu\rangle| = \lim_\alpha |\langle T(e_\alpha), \nu\rangle| \le \|T\|,$$

we have $\|f\| \le \|T\| + \epsilon$. But $\|T\| \le \|f\|$. Consequently $\|T\| = \|f\|$.

(3) We know that $L_0^\infty(X)^* = L_C(X)^{**}$. Now if $T^* : L_0^\infty(X)^* \longrightarrow L_0^\infty(X)^*$ is adjoint to T, then for $\mu, \nu \in L(X)$ we have $T^*(\mu * \nu) = \mu T^*(\nu)$. But $\mu T^*(\nu) = \mu\pi(T^*(\nu))$ ([14, Proposition 6]) and $\pi(T^*(\nu)) \in M(X)$ ([14, Proposition 13]). So $T^*(\mu*\nu) \in L(X)$. Since $L(X)$ has a bounded approximate identity, by the Cohen-Hewitt factorization theorem, we have $L(X) * L(X) = L(X)$. Consequently for every $\mu \in L(X)$, $T^*(\mu) \in L(X)$. A similar proof as above can be used to show that for some $\mu \in M(X)$, $T = {\lambda_\mu}^*$, and μ is unique with $\|T\| = \|\mu\|$.

Now, if T is compact then $\lambda_\mu : L(X) \longrightarrow L(X)$ is compact. So $\mu \in L(X)$ ([5, Theorem 1]). It is obvious that the correspondence between T and μ is an isometric isomorphism. This completes our proof. □

Skantharajah has proved there are some hypergroups X with $\mathrm{LIM}(L^\infty(X)) \setminus \mathrm{TLIM}(L^\infty(X)) \neq \emptyset$ ([17], [18]). In general, if G is a nondiscrete abelian group, then $\mathrm{LIM}(L^\infty(G)) \setminus \mathrm{TLIM}(L^\infty(G)) \neq \emptyset$ ([7]). If $m \in \mathrm{LIM}(L^\infty(G)) \setminus \mathrm{TLIM}(L^\infty(G))$, then the map $T : L^\infty(G) \longrightarrow L^\infty(G)$ given by $T(f) = m(f)$ commutes with translations, but T is not weak*-weak* continuous. Therefore there is no $\mu \in M(X)$ such that $T = {\lambda_\mu}^*$.

For a hypergroup X with an involution and Haar measure Wolfenstetter in [20] has defined $\mathrm{wap}(X) = \{f \in C(X); \{L_x f; x \in X\}$ is relatively weakly compact in $C(X)\}$. Also, Lasser has studied $\mathrm{ap}(X)$ [10]. In this paper, for an arbitrary hypergroup X, we define $\mathrm{wap}(L(X)) = \{f \in L(X)^*; \{f\mu; \mu \in L(X)$ and $\|\mu\| \leq 1\}$ is relatively weakly compact in $L(X)^*\}$ ([13]). It is easy to see that if $f \in \mathrm{wap}(L(X))$ and $\mu \in M(X)$, then $f\mu \in \mathrm{wap}(L(X))$. Also, the map $1 : L(X) \longrightarrow \mathbb{C}$ given by $\langle 1, \mu\rangle = \mu(X)$ is a weakly almost periodic functional on $L(X)$, i.e. $\{1\mu; \mu \in L(X)$ and $\|\mu\| \leq 1\}$ is relatively weakly compact.

Theorem 3.2. *Let X be a hypergroup and $f \in \mathrm{wap}(L(X))$. Then the following statements hold:*

(1) *the weak-closure of $\{f \sum_{i=1}^n \alpha_i \delta_{x_i}; x_i \in X, \alpha_i \in \mathbb{C}, n \in \mathbb{N}, \sum_{i=1}^n |\alpha_i| \leq 1\}$ is equal to the weak-closure of $\{f\mu; \mu \in L(X), \|\mu\| \leq 1\}$.*

(2) *Let T be an operator on $L(X)^*$ and $T(f\delta_x) = T(f)\delta_x$ for $f \in L(X)^*$, $x \in X$. Then $T(f\mu) = T(f)\mu$ for all $f \in \mathrm{wap}(L(X))$ and $\mu \in L(X)$.*

Proof: If $f \in \mathrm{wap}(L(X))$, then $\{f\mu; \mu \in L(X), \|\mu\| \leq 1\}$ is relatively weakly compact. Now for a bounded approximate identity (e_α) of norm one, (fe_α) has a convergence subsequence to f in the weak topology (since $fe_\alpha \longrightarrow f$ in the weak*-topology). But $B = L(X)^* L(X)$ is a Banach space, hence $f \in B$. For $x \in X$, let $m \in L(X)^{**}$ be an extension of δ_x with norm one. So there exists a net (μ_α) in $L(X)$ with $\|\mu_\alpha\| \leq 1$ such that $\mu_\alpha \longrightarrow m$ in the weak*-topology. Hence for every $\nu \in L(X)$

$$\langle \nu f, \mu_\alpha\rangle \longrightarrow \langle m, \nu f\rangle.$$

But $f \in \mathrm{wap}(L(X))$ and we may assume without loss of generality that $f\mu_\alpha \longrightarrow g$ ($g \in L(X)^*$) in the weak topology. On the other hand, $\langle m, \nu f\rangle = \langle \delta_x, \nu f\rangle = \langle f\delta_x, \nu\rangle$, and so $g = f\delta_x$. It follows that

$$\Big\{\sum_{i=1}^n f\alpha_i\delta_{x_i}; \alpha_i \in \mathbb{C}, n \in \mathbb{N}, x_i \in X, \sum_{i=1}^n |\alpha_i| \leq 1\Big\} \subseteq \text{weak-closure } \{f\mu; \mu \in L(X), \|\mu\| \leq 1\}.$$

To prove the converse, let $\mu \in L(X)$ and $\|\mu\| \leq 1$. By the Hahn Banach theorem, there exists a net (μ_α) in $\{\sum_{i=1}^n \alpha_i\delta_{x_i}, x_i \in X, \alpha_i \in \mathbb{C}, \sum_{i=1}^n |\alpha_i| \leq$

$1, n \in \mathbb{N}\}$ such that $\mu_\alpha \longrightarrow \mu$ in the $\sigma(B^*, B)$ topology. It is obvious to realize that

$$f\mu \in \text{weak-closure } \Big\{\sum_{i=1}^{n} f\alpha_i\delta_{x_i}; \alpha_i \in \mathbb{C}, x_i \in X, n \in \mathbb{N}, \sum_{i=1}^{n} |\alpha_i| \leq 1\Big\}.$$

This completes the proof.

(2) Let $f \in \text{wap}(L(X))$. Since $T(f\delta_x) = T(f)\delta_x$ and $f \in B$, so $T(f) \in B$. Indeed, for $\epsilon > 0$ there exists a neighbourhood U of e such that $\|T(f)\delta_x - T(f)\| \leq \epsilon/\|T(f)\|$ $(x \in U)$. Now for $\nu \in L(X)$ $(\|\nu\| \leq 1)$ and $\mu \in L(X) \cap M_p(X)$ with $\operatorname{supp}\mu \subseteq U$, we have

$$\Big|\int \langle T(f), \delta_x * \nu\rangle - \langle T(f), \nu\rangle \, d\mu(x)\Big| < \epsilon.$$

So $|\langle T(f)\mu, \nu\rangle - \langle T(f), \nu\rangle| < \epsilon$, i.e. $\|T(f)\mu - T(f)\| < \epsilon$. But $T(f)\mu \in B$ and B is a Banach space, hence $T(f) \in B$. Now if $\mu \in L(X)$, it is easy to see that $T(f\mu) = T(f)\mu$. □

Definition 3.3. Let X be a hypergroup; $\text{wap}(L(X))$ is said to be left stationary if for every $f \in \text{wap}(L(X))$

$$\text{weak}^*\text{-closure } \{\mu f; \mu \in M_p(X) \cap L(X)\} \cap \{c1; c \in \mathbb{C}\} \neq \emptyset.$$

$m \in L(X)^{**}$ is said to be topologically left invariant, if $\langle m, f\mu\rangle = \langle m, f\rangle$ for all $f \in L(X)^*$ and $\mu \in L(X) \cap M_p(X)$. In the following theorem we can find a relation between the set of all weakly compact operators which commute with convolutions and the set of all topologically left invariant functionals on $L(X)^*$. It is interesting for $L^1(X)$ when X has a Haar measure.

Theorem 3.4. *Let X be a hypergroup such that* $\text{wap}(L(X))$ *is left stationary. Then $L(X)^{**}$ has a topologically left invariant if and only if there exists a weakly compact operator $T : L(X)^* \longrightarrow L(X)^*$ such that $T(f\mu) = T(f)\mu$ for $f \in L(X)^*$ and $\mu \in L(X) \cap M_p(X)$.*

Proof: Let $m \in L(X)^{**}$ be topologically left invariant. Then the linear operator $T : L(X)^* \longrightarrow L(X)^*$ given by $T(f) = \langle m, f\rangle 1$ is a weakly compact operator and $T(f\mu) = T(f)\mu$ for all $\mu \in L(X) \cap M_p(X)$ and $f \in L(X)^*$.

Conversely, let $T : L(X)^* \longrightarrow L(X)^*$ be a weakly compact operator and $T(f\mu) = T(f)\mu$ for $f \in L(X)^*$, $\mu \in L(X) \cap M_p(X)$. So $T(L(X)^*) \subseteq \text{wap}(L(X))$. Now, let $f \in \text{wap}(L(X))$. There is a net (μ_α) in $L(X) \cap M_p(X)$ and $c \in \mathbb{C}$ such that $\mu_\alpha f \longrightarrow c1$ in the weak*-topology. Passing to a subnet if necessary, we can assume that (μ_α) converges to some m in $L(X)^{**}$ in the weak* topology. So, $mf = c1$. We take

$$\Sigma(f) = \{m \in L(X)^{**}; \|m\| \leq 1, \ mf = c1 \ \text{ for some } \ c \in \mathbb{C} \ \text{ and } \ \langle m, 1\rangle = 1\}.$$

For $f \in \text{wap}(L(X))$, $\Sigma(f) \neq \emptyset$. It is easy to see that $\Sigma(f)$ is weak* compact.

Now, if $f_1, f_2, \ldots, f_n \in \text{wap}(L(X))$ and $m_1 \in \bigcap_{i=1}^{n-1} \Sigma(f_i)$, then we can take $m_2 \in L(X)^{**}$ such that $m_2 m_1 f_n = c_n 1$ and $\langle m_2, 1\rangle = 1$ for some $c_n \in \mathbb{C}$ (since $m_1 f_n \in \text{wap}(L(X))$). Let $c_1, c_2, \ldots, c_{n-1} \in \mathbb{C}$ such that $m_1 f_i = c_i 1$ $(1 \leq i \leq n-1)$. We have $m_2 m_1 f_i = c_i 1$ $(1 \leq i \leq n-1)$, so that $m_2 m_1 \in \bigcap_{i=1}^{n} \Sigma(f_i)$. Consequently

$$\bigcap\{\Sigma(f); f \in \text{wap}(L(X))\} \neq \emptyset.$$

If $m_\circ \in \bigcap\{\Sigma(f); f \in \text{wap}(L(X))\}$, it is clear that $m = m_\circ m_\circ$ is a topologically left invariant on $\text{wap}(L(X))$. It follows that $m \circ T$ is a topologically left invariant on $L(X)^*$. □

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References

[1] Bloom W.R., Heyer H., *Harmonic Analysis of Probability Measures on Hypergroups*, de Gruyter, Berlin, 1995.

[2] Bloom W.R., Walter M.E., *Isomorphism of hypergroups*, J. Austral. Math. Soc. **52** (1992), 383–400.

[3] Duncan J., Hosseiniun S.A.R., *The second dual of Banach algebra*, Proc. Royal Soc. Edinburgh **84** (1979), 309–325.

[4] Dunkl C.F., *The measure algebra of a locally compact hypergroup*, Trans. Amer. Math. Soc. **179** (1973), 331–348.

[5] Ghahramani F., Medghalchi A.R., *Compact multipliers on hypergroup algebras*, Math. Proc. Cambridge Philos. Soc. **98** (1985), 493–500.

[6] Ghahramani F., Medghalchi A.R., *Compact multiplier on hypergroup algebras II*, Math. Proc. Cambridge Philos. Soc. **100** (1986), 145–149.

[7] Granirer E.E., *Criteria for compactness and for discreteness of locally compact amenable groups*, Proc. Amer. Math. Soc. **40** (1973), 615–624.

[8] Jewett R.I., *Spaces with an abstract convolution of measures*, Adv. Math. **18** (1975), 1–101.

[9] Larsen R., *An Introduction to the Theory of Multipliers*, Springer Verlag, Berlin, Heidelberg, New York, 1960.

[10] Lasser R., *Almost periodic functions on hypergroups*, Math. Ann. **252** (1980), 183–196.

[11] Lau A.T., *Analysis on a class of Banach algebras with applications to harmonic analysis on locally compact groups*, Fund. Math. **118** (1983), 161–175.

[12] Lau A.T., Pym J.S., *Concerning the second dual of the group algebra of a locally compact group*, J. London Math. Soc. **41** (1990), 445–460.

[13] Lau A.T., Ulger A., *Topological centers of certain dual algebras*, Trans. Amer. Math. Soc. **348** (1996), 1191–1212.

[14] Medghalchi A.R., *The second dual of a hypergroup*, Math. Z. **210** (1992), 615–624.

[15] Medghalchi A.R., Modarres S.M.S., *Amenability of the second dual of hypergroup algebras*, Acta. Math. Hungar. **86** (2000), 335–342.

[16] Rudin W., *Functional Analysis*, McGraw Hill, New York, 1991.

[17] Skantharajah M., *Amenable hypergroups*, Ph.D. Thesis, The University of Alberta, 1989.

[18] Skantharajah M., *Amenable hypergroups*, Illinois J. Math. **36** (1992), 15–46.
[19] Spector R., *Apercu de la theorie des hypergroups in analyse harmonique sur les groups de Lie*, Lecture Notes in Math. **497**, Springer Verlag, New York, 1975.
[20] Wolfenstetter S., *Weakly almost periodic functions on hypergroups*, Monatsh. Math. **96** (1983), 67–79.

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