Topological characterization of the small cardinal i

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Abstract. We show that the small cardinal number $i = \min\{|\mathcal{A}| : \mathcal{A}$ is a maximal independent family$\}$ has the following topological characterization: $i = \min\{\kappa \leq c : \{0,1\}^\kappa$ has a dense irresolvable countable subspace$\}$, where $\{0,1\}^\kappa$ denotes the Cantor cube of weight κ. As a consequence of this result, we have that the Cantor cube of weight c has a dense countable submaximal subspace, if we assume (ZFC plus $i = c$), or if we work in the Bell-Kunen model, where $i = \aleph_1$ and $c = \aleph_{\omega_1}$.

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1. Introduction

In this paper we will explore the relationship between the independent families of the power set of ω and the canonical subbasis of the Cantor cubes of weight $\leq c$. Let $\{0,1\}^I$ be the Cantor cube of weight $\aleph_0 \leq |I| \leq c$. The elements of the canonical basis of this topological product space will be denoted by $W(p)$, which by definition, is $W(p) = \{s \in {}^I2 : s \restriction \mathrm{dom}(p) = p\}$, for each $p \in \mathrm{Fn}(I, 2)$, where $\mathrm{Fn}(I, 2)$ is the set of all finite partial functions from the set I into 2.

Let us recall the definitions of independent families and irresolvable spaces.

Definition 1.1. $\mathcal{A} \subset \mathcal{P}(\omega)$ is an independent family if and only if for all $n, m \in \omega$ and all pairwise distinct elements $a_0, \dots, a_n, b_0, \dots, b_m$ of $\mathcal{A}$ we have $|a_0 \cap \dots \cap a_n \cap (\omega \setminus b_0) \cap \dots \cap (\omega \setminus b_m)| = \omega$.

We will always assume $|\mathcal{A}| > \omega$ and we say that $\mathcal{A}$ is a maximal independent family if for all $x \in \mathcal{P}(\omega) \setminus \mathcal{A}$, $\mathcal{A} \cup \{x\}$ is not an independent family.

Definition 1.2. Let (X, τ) be a Hausdorff dense-in-itself space. X is an irresolvable space if and only if for all dense subset $D \subset X$ we have $\mathrm{int}(D) \neq \emptyset$.

Definition 1.3. Let us define the following small cardinal numbers:
$i = \min\{|\mathcal{A}| : \mathcal{A} \subset \mathcal{P}(\omega)$ is a maximal independent family$\}$ and
$\lambda = \min\{\kappa \leq c : \exists A \subset \{0,1\}^\kappa$ dense irresolvable countable subspace$\}$.

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2. Main theorem

Definition 2.1. (1) If $\mathcal{A} = \{A_i : i \in I\}$ is any family of subsets of ω, we define the mapping $\psi_{\mathcal{A}} : \omega \longrightarrow \{0,1\}^I$ by: $(\forall x \in \omega)$ $(\forall i \in I)$

$$\psi_{\mathcal{A}}(n)(i) = \begin{cases} 0 & \text{if } n \in A_i \\ 1 & \text{if } n \in \omega \setminus A_i. \end{cases}$$

(2) If $A = \{a_n : n \in \omega\} \subset \{0,1\}^I$, let $\mathcal{A}^*(A) = \{A_i : i \in I\}$, where $A_i = \{n : a_n(i) = 0\}$.

To show that $i = \lambda$ we need to prove the following lemma.

Lemma 2.2. (a) *If $\mathcal{A} \subset \mathcal{P}(\omega)$, then $\mathcal{A}^*(\psi_{\mathcal{A}}(\omega)) = \mathcal{A}$.*

(b) *If $A = \{a_n : n \in \omega\} \subset \{0,1\}^I$, then $\psi_{\mathcal{A}^*(A)}(\omega) = A$.*

(c) *A is dense in $\{0,1\}^I$ if and only if $\mathcal{A}^*(A)$ is an independent family. Further, if A is dense, then $|I| = |\mathcal{A}^*(A)|$.*

(d) *A is dense and irresolvable if and only if $\mathcal{A}^*(A)$ is maximal independent.*

Proof: (a) We define $A = \{\psi_{\mathcal{A}}(n) : n \in \omega\}$. Given any $A_i \in \mathcal{A}$, then we have $A_i = \{n : \psi_{\mathcal{A}}(n)(i) = 0\}$.

(b) As $\psi_{\mathcal{A}^*(A)}(n)(i) = a_n(i)$, $\forall\, i \in I$ we have that $\psi_{\mathcal{A}^*(A)}(n) = a_n$.

(c) Assume that A is dense in $\{0,1\}^I$.
Given $n, m \in \omega$ and $\mathcal{B} = \{U_1, \dots, U_n, V_1, \dots, V_m\}$ a set of pairwise distinct elements of $\mathcal{A}^*(A)$, pick $p \in \mathrm{Fn}(I,2)$ such that: $\mathrm{dom}(p) = \{i \in I : A_i \in \mathcal{B}\}$, and

$$p(i) = \begin{cases} 0 & \text{if } A_i \in \{U_1, \dots, U_n\} \\ 1 & \text{if } A_i \in \{V_1, \dots, V_m\}. \end{cases}$$

Since A is dense in $\{0,1\}^I$ we have $|W(p) \cap A| = \omega$ and

$$\{n \in \omega : a_n \in W(p)\} = U_1 \cap \dots \cap U_n \cap (\omega \setminus V_1) \cap \dots \cap (\omega \setminus V_m)$$

which shows that $\mathcal{A}^*(A)$ is independent.

If A is not dense, there is $p \in \mathrm{Fn}(I,2)$ such that $W(p) \cap A = \emptyset$. Then $\bigcap\{A_i : p(i) = 0\} \cap \bigcap\{A_i : p(i) = 1\} = \emptyset$, which shows that if $\mathcal{A}^*(A)$ is independent, then A is dense in $\{0,1\}^I$. If A is dense, then $A_i \neq A_j$ whenever $i \neq j$. Indeed, if $i \neq j$ and $A_i = A_j$, then $a_n(i) = a_n(j)$ for each $n \in \omega$, hence A is not dense.

(d) Assume that there exist two dense subsets D_0, D_1 in the space A such that $D_0 \cup D_1 = A$ and $D_0 \cap D_1 = \emptyset$. Then, for instance, $\{n : a_n \in D_0\} \notin \mathcal{A}^*(A)$ and $\mathcal{A}^*(A) \cup \{\{n : a_n \in D_0\}\} = \mathcal{B}$ would be an independent family.

Now suppose that $B \subset \omega$, $B \notin \mathcal{A}^*(A)$ and $\mathcal{A}^*(A) \cup \{B\}$ is an independent family. Then for all $p \in \mathrm{Fn}(I,2) \setminus \{\emptyset\}$ we have $W(p) \cap \{a_n : n \in B\} \neq \emptyset \neq W(p) \cap \{a_n : n \in A \setminus B\}$.

This shows that the space A is resolvable if and only if the independent family $\mathcal{A}$ is not maximal. □

Theorem 2.3. $i = \lambda$.

PROOF: We first prove $i \leq \lambda$. Pick $A \subset \{0,1\}^\lambda$ a countable dense irresolvable subspace of the Cantor cube of weight λ. By Lemma 2.2(d), we have that $\mathcal{A}^*(A)$ is a maximal independent family of cardinality $|\mathcal{A}^*(A)| = \lambda$, hence $i \leq \lambda$.

Now we prove $\lambda \leq i$. Let $\mathcal{A} \subset \mathcal{P}(\omega)$ be a maximal independent family of cardinality $|\mathcal{A}| = i$. By Lemma 2.2(d), $A = \psi_\mathcal{A}(\omega)$ is a dense, countable, irresolvable subspace of the Cantor cube $\{0,1\}^i$. Hence $\lambda \leq i$. □

3. Submaximal spaces

We say that a topological space (X, τ) is a submaximal space if and only if every dense subset of X is open in X.

In [ASTTW] the authors show that the Tychonoff cube $[0,1]^c$ in a model of ZFC plus BL (Booth's Lemma or equivalently $\mathbf{p} = c$), has a dense countable submaximal subspace. It is well known that $\mathbf{p} \leq i$ and that it is consistent that $\mathbf{p} < i$. In [Ma], Malykhin shows that in the Bell-Kunen's model, the Cantor cube of weight ω_1 has a dense countable irresolvable subspace. Thus, in this model, by Theorem 2.3, it is true that $i = \omega_1$. Also we will show that the Cantor cube of weight $c = \aleph_{\omega_1}$ has a dense countable submaximal subspace. Hence the existence of these dense countable subspaces of these cubes is independent of $i = c$ or $\mathbf{p} = c$.

Let $\mathcal{A}$ be an independent family of $\mathcal{P}(\omega)$. Follows from Lemma 2.2(c) that $\psi_\mathcal{A}(\omega)$ is the dense subspace of the Cantor cube $\{0,1\}^\mathcal{A}$.

Definition 3.1. We say that $M \subset \omega$ is dense (open) in $\mathcal{A}$ if $\psi_\mathcal{A}(M)$ is dense (open) in $\psi_\mathcal{A}(\omega)$.

Lemma 3.2. (a) *D is dense in $\mathcal{A}$ if and only if $\forall p \in \mathrm{Fn}(\mathcal{A}, 2) \setminus \{\emptyset\}$ holds $D \cap V(p) \neq \emptyset$.*

(b) *G is open in $\mathcal{A}$ if and only if for each $x \in G$ there exists $p \in \mathrm{Fn}(\mathcal{A}, 2)$ such that $x \in V(p) \subset G$.*

PROOF: Let $X = (\omega, \tau)$ be the topological space whose basis of open set is: $\{V(p) : p \in \mathrm{Fn}(\mathcal{A}, 2) \setminus \{\emptyset\}\}$, and each $V(p)$ is defined by:

$$V(p) = \bigcap\{V : p(V) = 0\} \cap \bigcap\{\omega \setminus V : p(V) = 1\}.$$

By Lemma 2.2(c), $\psi_\mathcal{A}$ is a continuous open mapping from X onto the dense subspace $\psi_\mathcal{A}(\omega)$ of $\{0,1\}^\mathcal{A}$. □

The following example is important for construction subsets of ω which are dense and open in a given independent family, like in Lemma 3.5 and Lemma 3.7(ii).

Example 3.3. Let $Z = \{z \in 2^\omega : \{n : z(n) = 1\}$ is finite$\}$; Z is a countable dense subspace of the Cantor set.

Choose $s \in 2^\omega$ such that $|\{n : s(n) = 0\}| = |\{n : s(n) = 1\}| = \omega$, and define the family $\mathcal{A} = \{A_\alpha : \alpha \in \omega\}$, where $A_\alpha = \{z \in Z : z(\alpha) = s(\alpha)\}$, $\forall\, \alpha \in \omega$. Then follows easily from the definition, that $\mathcal{A}$ satisfies: $\bigcap \mathcal{A} = \emptyset$ and $\bigcup \mathcal{A} = Z$. Now we define the independent family $\mathcal{T} = \{T_\alpha : \alpha \in \omega\}$, where for each $\alpha \in \omega$, $T_\alpha = \{n \in \omega : z_n \in A_\alpha\}$ and $Z = \{z_n : n \in \omega\}$ is an enumeration of the set Z.

Lemma 3.4. *Let $\mathcal{A} \subset \mathcal{B}$ be two independent families. Then*

(i) *$D \subset \omega$, dense in $\mathcal{B}$, implies D is also dense in $\mathcal{A}$,*
(ii) *$G \subset \omega$, open in $\mathcal{A}$, implies G is also open in $\mathcal{B}$.*

Proof: If we pick $p \in \mathrm{Fn}(\mathcal{B}, 2)$ such that $\mathrm{dom}(p) \subset \mathcal{A}$, then both (i) and (ii) follow easily from the Lemma 3.2. □

Lemma 3.5. *Let D be a subset of ω such that:*

(i) *$D = \bigcup_{n\in\omega} D_n$ and for all $n \in \omega$, $|D_n| \geq 1$,*
(ii) *$n \neq m \Longrightarrow D_n \cap D_m = \emptyset$.*

Let $\mathcal{T}$ be the independent family of Example 3.3. Define $\mathcal{D} = \{V_\alpha : \alpha < \omega\}$, where $V_\alpha = \bigcup\{D_n : n \in T_\alpha\}$. Then the family $\mathcal{D}$ is independent as the family $\mathcal{T}$ is independent; D is open in $\mathcal{D}$ as $\bigcup \mathcal{T} = \omega$.

Lemma 3.6. *Let $\mathcal{A}$ be an independent family and suppose that $|\mathcal{A}| < i$. If $D \subset \omega$ is dense in $\mathcal{A}$, then there exists a sequence $\{D_n : n \in \omega\}$ of subsets of D such that:*

(i) *for each $n \in \omega$, D_n is dense in $\mathcal{A}$,*
(ii) *$n \neq m \Longrightarrow D_n \cap D_m = \emptyset$,*
(iii) *$\bigcup_{n\in\omega} D_n = D$.*

Proof: Follows from the Main Theorem, $|\mathcal{A}| < i \Longrightarrow |\mathcal{A}| < \lambda$ and hence any dense countable subset of $\{0,1\}^{\mathcal{A}}$ is resolvable. □

Lemma 3.7. *Let $D \subset \omega$ be dense in $\mathcal{A}$ and suppose that $|\mathcal{A}| < i$. Then there exists an independent family $\mathcal{B} \subset \mathcal{P}(\omega)$ such that:*

(i) *$\mathcal{A} \subset \mathcal{B}$ and $|\mathcal{B} \setminus \mathcal{A}| = \omega$,*
(ii) *D is dense and open in $\mathcal{B}$.*

Proof: Let $\{D_n : n \in \omega\}$ be a sequence of subsets of D, as in Lemma 3.6. By Lemma 3.5, we may define an independent family $\mathcal{D}$, such that D is open in $\mathcal{D}$. The family $\mathcal{B} = \mathcal{A} \cup \mathcal{D}$ is independent by definition of $\mathcal{D}$ and density of each D_n, which also imply that D is dense in $\mathcal{B}$. □

Theorem 3.8 ($i = c$)**.** *There exists an independent family $\mathcal{A} \subset \mathcal{P}(\omega)$ such that:*

(i) *$|\mathcal{A}| = c$ and*
(ii) *if $D \subset \omega$ is dense in $\mathcal{A}$, then D is open in A.*

PROOF: Let $\{E_\alpha : \alpha < c\}$ be an enumeration of all infinite subsets of ω with $E_0 = \omega$, and choose an independent family $\mathcal{A}_0$ such that $|\mathcal{A}_0| < c$ and $\bigcup \mathcal{A}_0 = \omega$. By transfinite induction on $c \setminus \{0\}$, we can choose a sequence of independent families $\{\mathcal{A}_\alpha : \alpha \in c \setminus \{0\}\}$, such that, by Lemma 3.7:

(i) $\mathcal{A}_\alpha \subset \mathcal{A}_{\alpha+1}$,
(ii) $|\mathcal{A}_{\alpha+1} \setminus A_\alpha| = \omega$,
(iii) if E_α is dense in $\mathcal{A}_\alpha$, then E_α is dense and open in $\mathcal{A}_{\alpha+1}$, and
(iv) if β is a limit ordinal, then $\mathcal{A}_\beta = \bigcup_{\alpha\in\beta} \mathcal{A}_\alpha$.

The independent family $\mathcal{A} = \bigcup_{\alpha\in c} \mathcal{A}_\alpha$ has $|\mathcal{A}| = c$. Also, if $D \subset \omega$ is dense in $\mathcal{A}$ then $\exists \alpha \in c$ such that $D = E_\alpha$ and this set is dense in $\mathcal{A}_\alpha$ as it is dense in $\mathcal{A}$, by Lemma 3.4(i). Therefore by construction, it is open in $\mathcal{A}_{\alpha+1}$, so it is also open in $\mathcal{A}$, by Lemma 3.4(ii). □

Corollary 3.9 ($i = c$). *The Cantor cube $\{0,1\}^c$ has a dense countable submaximal subspace.*

PROOF: Let $\mathcal{A}$ be an independent family which satisfies Theorem 3.8 and let $\psi_{\mathcal{A}}(\omega)$ be a dense countable subspace of the Cantor cube of weight c. If D is dense in A, then ${\psi_{\mathcal{A}}}^{-1}(D)$ is dense in $\mathcal{A}$, so it is also open in $\mathcal{A}$, which implies that D is open in $\psi_{\mathcal{A}}(\omega)$. □

4. Bell-Kunen's model

Let M be a countable transitive model of ZFC plus GCH. In [BK], Bell and Kunen construct in M an increasing family of partial ordered sets $\{P_\alpha : \alpha \leq \omega_1\}$ such that:

(i) each P_α has c.c.c.,
(ii) if β is limit, $P_\beta = \bigcup\{P_\alpha : \alpha < \beta\}$,
(iii) if α is not a limit ordinal, then P_α is such that both MA (Martin's axiom) and $2^\omega = \aleph_\alpha$ hold in M^{P_α}.

Let $G = G_{\omega_1}$ be a $P_{\omega_1} - generic$ over M and $G_\alpha = G \cap P_\alpha$ for each $\alpha \leq \omega_1$. In M_{ω_1} there is a transfinite increasing sequence of models $\{M_\alpha = M[G_\alpha] : \alpha \leq \omega_1\}$, and if $\alpha > 0$ is a non-limit ordinal, then the assertion "MA plus $c = \aleph_\alpha$ " is true in M_α. Let us also note that, in M_{ω_1}, the power set of all subset of ω is the union of the increasing sequence $\{\mathcal{P}(\omega) \cap M_{\alpha+1} : \alpha < \omega_1\}$.

Theorem 4.1. *In the Bell-Kunen's model there is an independent family $\mathcal{A} \subset \mathcal{P}(\omega)$ such that every $D \subset \omega$ dense in $\mathcal{A}$ is also open in $\mathcal{A}$. Further $|\mathcal{A}| = c$.*

PROOF: By transfinite induction in M_{ω_1}, we construct an increasing sequence of independent families $\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ such that, in $M_{\alpha+1}$, the independent family $\mathcal{A}_\alpha$ satisfies Theorem 3.8. This is possible because MA holds in $M_{\alpha+1}$, thus $i = c = \aleph_{\alpha+1}$. Now, we look at this family in $M_{\alpha+2}$, and in this model we have that $|\mathcal{A}_{\alpha+1}| \leq \aleph_{\alpha+1} < \aleph_{\alpha+2} = c$. Then, by the prove of the Theorem 3.8, in $M_{\alpha+2}$, we can choose an independent family $\mathcal{A}_{\alpha+2} \supset \mathcal{A}_{\alpha+1}$, such that (in $M_{\alpha+2}$) if D is dense in $\mathcal{A}_{\alpha+2}$, then it is open too. (For a limit ordinal we take the union.) The family $\mathcal{A} = \bigcup\{\mathcal{A}_\alpha : \alpha < \omega_1\}$ is an independent family in M_{ω_1}. Also, if $D \subset \omega$ is dense in $\mathcal{A}$, then $\exists \alpha < \omega_1$ such that $D \in \mathcal{P}(\omega) \cap M_{\alpha+1}$ and D is open in $\mathcal{A}_{\alpha+2}$ (in $M_{\alpha+2}$), so it is open in $\mathcal{A}$ in M_{ω_1}. Further, for each $\alpha \in \omega_1$ we have that $|\mathcal{A}_{\alpha+2}| = c$ (in $M_{\alpha+2}$) hence, follows $|\mathcal{A}| = c$ in Bell-Kunen's model. □

Corollary 4.2. *In the Bell-Kunen's model there is a countable dense submaximal subspace X of the Cantor cube $\{0,1\}^c$.*

Proof: Let $\mathcal{A}$ be an independent family as in Theorem 4.1 and take $X = \psi_{\mathcal{A}}(\omega)$ be the countable dense submaximal subspace of $\{0,1\}^c$. □

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