On m-sectorial Schrödinger-type operators with singular potentials on manifolds of bounded geometry

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Abstract. We consider a Schrödinger-type differential expression $H_V = \nabla^*\nabla + V$, where ∇ is a C^∞-bounded Hermitian connection on a Hermitian vector bundle E of bounded geometry over a manifold of bounded geometry (M, g) with metric g and positive C^∞-bounded measure $d\mu$, and V is a locally integrable section of the bundle of endomorphisms of E. We give a sufficient condition for m-sectoriality of a realization of H_V in $L^2(E)$. In the proof we use generalized Kato's inequality as well as a result on the positivity of $u \in L^2(M)$ satisfying the equation $(\Delta_M + b)u = \nu$, where Δ_M is the scalar Laplacian on M, $b > 0$ is a constant and $\nu \geq 0$ is a positive distribution on M.

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1. Introduction and the main result

1.1 The setting. Let (M, g) be a C^∞ Riemannian manifold without boundary, with metric g and $\dim M = n$. We will assume that M is connected. We will also assume that M has bounded geometry. Moreover, we will assume that we are given a positive C^∞-bounded measure $d\mu$, i.e. in any local coordinates $x^1, x^2, \dots, x^n$ there exists a strictly positive C^∞-bounded density $\rho(x)$ such that $d\mu = \rho(x)dx^1dx^2 \dots dx^n$.

Let E be a Hermitian vector bundle over M. We will assume that E is a bundle of bounded geometry (i.e. it is supplied by an additional structure: trivializations of E on every canonical coordinate neighborhood U such that the corresponding matrix transition functions $h_{U,U'}$ on all intersections $U \cap U'$ of such neighborhoods are C^∞-bounded, i.e. all derivatives $\partial_y^\alpha h_{U,U'}(y)$, where α is a multiindex, with respect to canonical coordinates are bounded with bounds C_α which do not depend on the chosen pair U, U').

We denote by $L^2(E)$ the Hilbert space of square integrable sections of E with respect to the scalar product

$$(u, v) = \int_M \langle u(x), v(x)\rangle \, d\mu(x). \tag{1.1}$$

Here $\langle \cdot, \cdot \rangle$ denotes the fiberwise inner product in E_x.

In what follows, $C^\infty(E)$ denotes smooth sections of E, and $C_c^\infty(E)$ denotes smooth compactly supported sections of E.

Let

$$\nabla: C^\infty(E) \to C^\infty(T^*M \otimes E)$$

be a Hermitian connection on E which is C^∞-bounded as a linear differential operator, i.e. in any canonical coordinate system U (with the chosen trivializations of $E|_U$ and $(T^*M \otimes E)|_U$), ∇ is written in the form

$$\nabla = \sum_{|\alpha|\leq 1} a_\alpha(y)\partial_y^\alpha,$$

where α is a multiindex, and the coefficients $a_\alpha(y)$ are matrix functions whose derivatives $\partial_y^\beta a_\alpha(y)$ for any multiindex β are bounded by a constant C_β which does not depend on the chosen canonical neighborhood.

We will consider a Schrödinger type differential expression of the form

$$H_V = \nabla^*\nabla + V,$$

where V is a measurable section of the bundle $\operatorname{End} E$ of endomorphisms of E. Here

$$\nabla^*: C^\infty(T^*M \otimes E) \to C^\infty(E)$$

is a differential operator which is formally adjoint to ∇ with respect to the scalar product (1.1).

If we take $\nabla = d$, where $d: C^\infty(M) \to \Omega^1(M)$ is the standard differential, then $d^*d: C^\infty(M) \to C^\infty(M)$ is called the scalar Laplacian and will be denoted by Δ_M.

In what follows, we use the notations

$$(1.2)\quad (\operatorname{Re} V)(x) := \frac{V(x) + (V(x))^*}{2}, \quad (\operatorname{Im} V)(x) := \frac{V(x) - (V(x))^*}{2i}, \quad x \in M,$$

where $i = \sqrt{-1}$ and $(V(x))^*$ denotes the adjoint of the linear operator $V(x): E_x \to E_x$ (in the sense of linear algebra).

By (1.2), for all $x \in M$, $(\operatorname{Re} V)(x)$ and $(\operatorname{Im} V)(x)$ are self-adjoint linear operators $E_x \to E_x$, and we have the following decomposition:

$$V(x) = (\operatorname{Re} V)(x) + i(\operatorname{Im} V)(x).$$

For every $x \in M$, we have the following decomposition:

$$(1.3)\qquad (\operatorname{Re} V)(x) = (\operatorname{Re} V)^+(x) - (\operatorname{Re} V)^-(x).$$

Here $(\operatorname{Re} V)^+(x) = P_+(x)(\operatorname{Re} V)(x)$, where $P_+(x) := \chi_{[0,+\infty)}((\operatorname{Re} V)(x))$, and $(\operatorname{Re} V)^-(x) = -P_-(x)(\operatorname{Re} V)(x)$, where $P_-(x) := \chi_{(-\infty,0)}((\operatorname{Re} V)(x))$. Here χ_A denotes the characteristic function of the set A.

We make the following assumption on V.

Assumption A.

(i) $(\operatorname{Re} V)^+ \in L^1_{\mathrm{loc}}(\operatorname{End} E)$, $(\operatorname{Re} V)^- \in L^1_{\mathrm{loc}}(\operatorname{End} E)$ and $(\operatorname{Im} V) \in L^1_{\mathrm{loc}}(\operatorname{End} E)$.

(ii) There exists a constant $L > 0$ such that for all $u \in L^2(E)$ and all $x \in M$,

$$|(\operatorname{Im} V)(x)|\,|u(x)|^2 \leq L\langle(\operatorname{Re} V)^+(x)u(x), u(x)\rangle, \tag{1.4}$$

where $|(\operatorname{Im} V)(x)|$ denotes the norm of the operator $(\operatorname{Im} V)(x)\colon E_x \to E_x$, $|u(x)|$ denotes the norm in the fiber E_x and $\langle\cdot,\cdot\rangle$ denotes the inner product in E_x.

1.2 Sobolev space $W^{1,2}(E)$. By $W^{1,2}(E)$ we will denote the completion of the space $C_c^\infty(E)$ with respect to the norm $\|\cdot\|_1$ defined by the scalar product

$$(u, v)_1 \;:=\; (u, v) \;+\; (\nabla u, \nabla v) \qquad u, v \in C_c^\infty(E).$$

By $W^{-1,2}(E)$ we will denote the dual of $W^{1,2}(E)$.

1.3 Quadratic forms. In what follows, all quadratic forms are considered in the Hilbert space $L^2(E)$.

1. By h_0 we denote the quadratic form

$$h_0(u) = \int |\nabla u|^2\, d\mu \tag{1.5}$$

with the domain $\mathrm{D}(h_0) = W^{1,2}(E) \subset L^2(E)$. The quadratic form h_0 is non-negative, densely defined (since $C_c^\infty(E) \subset \mathrm{D}(h_0)$) and closed (see Section 1.2).

2. By h_1 we denote the quadratic form

$$h_1(u) = \int \langle(\operatorname{Re} V)^+ u, u\rangle + i\langle(\operatorname{Im} V)u, u\rangle\, d\mu \tag{1.6}$$

with the domain

$$\mathrm{D}(h_1) = \Big\{u \in L^2(E) : \int \big|\langle(\operatorname{Re} V)^+ u, u\rangle + i\langle(\operatorname{Im} V)u, u\rangle\big|\, d\mu < +\infty\Big\}. \tag{1.7}$$

Here $\langle\cdot,\cdot\rangle$ denotes the fiberwise inner product in E_x.

In what follows, we will denote by $h_1(\cdot,\cdot)$ the corresponding sesquilinear form obtained via polarization identity from h_1.

The quadratic form h_1 is sectorial. Indeed, by the inequalities

$$|\langle(\operatorname{Im} V)u(x), u(x)\rangle| \leq |(\operatorname{Im} V)(x)u(x)||u(x)| \leq |(\operatorname{Im} V)(x)||u(x)|^2 \tag{1.8}$$

and by (1.4), for all $u \in \mathrm{D}(h_1)$, the values of $h_1(u)$ lie in a sector of $\mathbb{C}$ with vertex $\gamma = 0$. The form h_1 is densely defined since, by (i) of Assumption A, we have

$C_c^\infty(E) \subset \mathrm{D}(h_1)$. The form h_1 is closed. Indeed, by Theorem VI.1.11 in [6], it suffices to show that the pre-Hilbert space $\mathrm{D}(h_1)$ with the inner product

$$(u,v)_{h_1} = (\operatorname{Re} h_1)(u,v) + (u,v) = \int \langle (\operatorname{Re} V)^+ u, v\rangle \, d\mu + (u,v),$$

is complete. Here $(\cdot,\cdot)$ denotes the inner product in $L^2(E)$ and $(\operatorname{Re} h_1)(\cdot,\cdot)$ denotes the real part of the sesquilinear form $h_1(\cdot,\cdot)$ (see the definition below the equation (1.9) in Section VI.1.1 of [6]).

By (1.7), (1.8) and (1.4), it follows that $\mathrm{D}(h_1)$ is the set of all $u \in L^2(E)$ such that $\|u\|_{h_1}^2 < +\infty$, where $\|\cdot\|_{h_1}$ denotes the norm corresponding to the inner product $(\cdot,\cdot)_{h_1}$. By Example VI.1.15 in [6], it follows that $\mathrm{D}(h_1)$ is complete.

3. By h_2 we denote the quadratic form

$$h_2(u) = \int \langle -(\operatorname{Re} V)^- u, u\rangle \, d\mu \tag{1.9}$$

with the domain

$$\mathrm{D}(h_2) = \Big\{u \in L^2(E) : \int \langle (\operatorname{Re} V)^- u, u\rangle \, d\mu < +\infty\Big\}, \tag{1.10}$$

where $\langle\cdot,\cdot\rangle$ denotes the fiberwise inner product in E_x.

The form h_2 is densely defined because, by (i) of Assumption A, we have $C_c^\infty(E) \subset \mathrm{D}(h_2)$. Moreover, since for all $x \in M$, the operator $(\operatorname{Re} V)^-(x)\colon E_x \to E_x$ is self-adjoint, it follows that the quadratic form h_2 is symmetric.

We make the following assumption on h_2.

Assumption B. Assume that h_2 is h_0-bounded with relative bound $b < 1$, i.e.

(i) $\mathrm{D}(h_2) \supset \mathrm{D}(h_0)$,

(ii) there exist constants $a \geq 0$ and $0 \leq b < 1$ such that

$$|h_2(u)| \leq a\|u\|^2 + b|h_0(u)|, \quad \text{for all } \ u \in \mathrm{D}(h_0), \tag{1.11}$$

where $\|\cdot\|$ denotes the norm in $L^2(E)$.

Remark 1.4. If (M,g) is a manifold of bounded geometry, Assumption B holds if $(\operatorname{Re} V)^- \in L^p(\operatorname{End} E)$, where $p = n/2$ for $n \geq 3$, $p > 1$ for $n = 2$, and $p = 1$ for $n = 1$. For the proof, see, for example, the proof of Remark 2.1 in [7].

1.5 A realization of H_V in $L^2(E)$. We define an operator S in $L^2(E)$ by the formula $Su = H_V u$ on the domain

$$\Big\{u \in W^{1,2}(E) : \int \big|\langle (\operatorname{Re} V)^+ u, u\rangle + i\langle (\operatorname{Im} V) u, u\rangle\big| \, d\mu < +\infty \text{ and } H_V u \in L^2(E)\Big\}. \tag{1.12}$$

We will denote the set in (1.12) by $\mathrm{Dom}(S)$.

Remark 1.6. For all $u \in \mathrm{D}(h_0) = W^{1,2}(E)$ we have $\nabla^*\nabla u \in W^{-1,2}(E)$. From Corollary 2.11 below it follows that for all $u \in W^{1,2}(E) \cap \mathrm{D}(h_1)$, we have $Vu \in L^1_{\mathrm{loc}}(E)$. Thus $H_V u$ in (1.12) is a distributional section of E, and the condition $H_V u \in L^2(E)$ makes sense.

Remark 1.7. By (1.4) and by (1.8), the set $\mathrm{Dom}(S)$ in (1.12) is equal to

$$\Big\{u \in W^{1,2}(E) : \int \langle (\mathrm{Re}\, V)^+ u, u\rangle \, d\mu < +\infty \text{ and } H_V u \in L^2(E)\Big\}.$$

We now state the main result.

Theorem 1.8. *Assume that (M, g) is a manifold of bounded geometry with positive C^∞-bounded measure $d\mu$, E is a Hermitian vector bundle of bounded geometry over M, and ∇ is a C^∞-bounded Hermitian connection on E. Suppose that Assumptions A and B hold. Then the operator S is m-sectorial.*

Remark 1.9. Theorem 1.8 extends a result of T. Kato; see Theorem VI.4.6(a) in [6] (see also Remark 5(a) in [5]) which was proven for the operator $-\Delta + V$, where Δ is the standard Laplacian on $\mathbb{R}^n$ with the standard metric and measure, and $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ is as in Assumptions A and B above (with $\mathrm{Im}\, V = 0$). Theorem 1.8 also extends the result in [7] which establishes the self-adjointness of a realization in $L^2(E)$ of $H_V = \nabla^*\nabla + V$ on manifold (M, g) with $d\mu$, E, and ∇ as in the hypotheses of Theorem 1.8, and $V = V_1 + V_2$, where $0 \leq V_1 \in L^1_{\mathrm{loc}}(\mathrm{End}\, E)$ and $0 \geq V_2 \in L^1_{\mathrm{loc}}(\mathrm{End}\, E)$ are linear self-adjoint bundle endomorphisms satisfying Assumptions A and B (with $\mathrm{Im}\, V = 0$).

2. Proof of Theorem 1.8

We adopt the arguments from Section VI.4.3 in [6] to our setting with the help of a more general version of Kato's inequality (2.1).

2.1 Kato's inequality. We begin with the following variant of Kato's inequality for Bochner Laplacian (for the proof, see Theorem 5.7 in [2]).

Lemma 2.2. *Assume that (M, g) is a Riemannian manifold. Assume that E is a Hermitian vector bundle over M and ∇ is a Hermitian connection on E. Assume that $w \in L^1_{\mathrm{loc}}(E)$ and $\nabla^*\nabla w \in L^1_{\mathrm{loc}}(E)$. Then*

$$\Delta_M |w| \;\leq\; \mathrm{Re}\langle \nabla^*\nabla w, \mathrm{sign}\, w\rangle, \tag{2.1}$$

where

$$\mathrm{sign}\, w(x) \;=\; \begin{cases} \frac{w(x)}{|w(x)|} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 2.3. The original version of Kato's inequality was proven in Kato [3].

2.4 Positivity. In what follows, we will use the following lemma whose proof is given in Appendix B of [2].

Lemma 2.5. *Assume that (M, g) is a manifold of bounded geometry with a smooth positive measure $d\mu$. Assume that*

$$\big(b + \Delta_M\big)\, u \; = \; \nu \; \geq \; 0, \quad u \in L^2(M),$$

where $b > 0$, $\Delta_M = d^*d$ is the scalar Laplacian on M, and the inequality $\nu \geq 0$ means that ν is a positive distribution on M, i.e. $(\nu, \phi) \geq 0$ for any $0 \leq \phi \in C_c^\infty(M)$.

Then $u \geq 0$ (almost everywhere or, equivalently, as a distribution).

Remark 2.6. It is not known whether Lemma 2.5 holds if M is an arbitrary complete Riemannian manifold. For more details about difficulties in the case of arbitrary complete Riemannian manifolds, see Appendix B of [2].

Lemma 2.7. *The quadratic form $h := (h_0 + h_1) + h_2$ is densely defined, sectorial and closed.*

Proof: Since h_0 and h_1 are sectorial and closed, it follows by Theorem VI.1.31 from [6] that $h_0 + h_1$ is sectorial and closed. By (i) of Assumption B it follows that $\mathrm{D}(h_2) \supset \mathrm{D}(h_0) \cap \mathrm{D}(h_1)$, and by (1.11), (1.5), and (1.6), the following inequality holds:

$$|h_2(u)| \leq a\|u\|^2 + b|h_0(u) + h_1(u)|, \quad \text{for all } \; u \in \mathrm{D}(h_0) \cap \mathrm{D}(h_1),$$

where $\|\cdot\|$ denotes the norm in $L^2(E)$, and $a \geq 0$ and $0 \leq b < 1$ are as in (1.11). Thus the quadratic form h_2 is $(h_0 + h_1)$-bounded with relative bound $b < 1$. Since $h_0 + h_1$ is a closed sectorial form, by Theorem VI.1.33 from [6], it follows that $h = (h_0 + h_1) + h_2$ is a closed sectorial form. Since $C_c^\infty(E) \subset \mathrm{D}(h_0) \cap \mathrm{D}(h_1) \subset \mathrm{D}(h_2)$, it follows that h is densely defined. $\square$

In what follows, $h(\cdot, \cdot)$ will denote the corresponding sesquilinear form obtained from h via polarization identity.

2.8 m-sectorial operator H associated with h. Since h is a densely defined, closed and sectorial form in $L^2(E)$, by Theorem VI.2.1 from [6] there exists an m-sectorial operator H in $L^2(E)$ such that

(i) $\mathrm{Dom}(H) \subset \mathrm{D}(h)$ and

$$h(u, v) = (Hu, v), \quad \text{for all } u \in \mathrm{Dom}(H) \text{ and } v \in \mathrm{D}(h),$$

(ii) $\mathrm{Dom}(H)$ is a core of h,

(iii) if $u \in \mathrm{D}(h)$, $w \in L^2(E)$, and

$$h(u, v) = (w, v)$$

holds for every v belonging to a core of h, then $u \in \mathrm{Dom}(H)$ and $Hu = w$.

The operator H is uniquely determined by condition (i).

We will also use the following lemma.

Lemma 2.9. *Assume that* $0 \leq T \in L^1_{\text{loc}}(\operatorname{End} E)$ *is a linear self-adjoint bundle map. Assume also that* $u \in Q(T)$, *where* $Q(T) = \{u \in L^2(E) \colon \langle Tu, u\rangle \in L^1(M)\}$. *Then* $Tu \in L^1_{\text{loc}}(E)$.

Proof: By adding a constant we can assume that $T \geq 1$ (in operator sense).

Assume that $u \in Q(T)$. We choose (in a measurable way) an orthogonal basis in each fiber E_x and diagonalize $1 \leq T(x) \in \operatorname{End}(E_x)$ to get $T(x) = \operatorname{diag}(c_1(x), c_2(x), \dots, c_m(x))$, where $0 < c_j \in L^1_{\text{loc}}(M)$, $j = 1, 2, \dots, m$ and $m = \dim E_x$.

Let $u_j(x)$ ($j = 1, 2, \dots, m$) be the components of $u(x) \in E_x$ with respect to the chosen orthogonal basis of E_x. Then for all $x \in M$

$$\langle Tu, u\rangle = \sum_{j=1}^{m} c_j(x)|u_j(x)|^2.$$

Since $u \in Q(T)$, we know that $0 < \int \langle Tu, u\rangle \, d\mu < +\infty$. Since $c_j > 0$, it follows that $c_j|u_j|^2 \in L^1(M)$, for all $j = 1, 2, \dots, m$.

Now, for all $x \in M$ and $j = 1, 2, \cdots, m$

$$2|c_j u_j| = 2|c_j||u_j| \leq |c_j| + |c_j||u_j|^2. \tag{2.2}$$

The right hand side of (2.2) is clearly in $L^1_{\text{loc}}(M)$. Therefore $c_j u_j \in L^1_{\text{loc}}(M)$.

But $(Tu)(x)$ has components $c_j(x)u_j(x)$ ($j = 1, 2, \dots, m$) with respect to chosen bases of E_x. Therefore $Tu \in L^1_{\text{loc}}(E)$, and the lemma is proven. □

Corollary 2.10. *If* $u \in D(h_1)$, *then* $((\operatorname{Re} V)^+ + i(\operatorname{Im} V))u \in L^1_{\text{loc}}(E)$.

Proof: Let $u \in \mathrm{D}(h_1)$. Then $\langle (\operatorname{Re} V)^+ u, u\rangle \in L^1(M)$, and, hence, by Lemma 2.9 we get $(\operatorname{Re} V)^+ u \in L^1_{\text{loc}}(E)$. By (1.4) we obtain $|(\operatorname{Im} V)||u|^2 \in L^1(M)$. Since for all $x \in M$ we have

$$2|(\operatorname{Im} V)(x)u(x)| \leq 2|(\operatorname{Im} V)(x)||u(x)| \leq |(\operatorname{Im} V)(x)| + |(\operatorname{Im} V)(x)||u(x)|^2,$$

and since, by Assumption A, $|(\operatorname{Im} V)| \in L^1_{\text{loc}}(M)$, it follows that $(\operatorname{Im} V)u \in L^1_{\text{loc}}(E)$, and the corollary is proven. □

Corollary 2.11. *If* $u \in D(h)$, *then* $Vu \in L^1_{\text{loc}}(E)$.

Proof: Let $u \in \mathrm{D}(h) = \mathrm{D}(h_0) \cap \mathrm{D}(h_1)$. By Corollary 2.10 it follows that $((\operatorname{Re} V)^+ + i(\operatorname{Im} V))u \in L^1_{\text{loc}}(E)$. Since $\mathrm{D}(h) \subset \mathrm{D}(h_2)$ and since $(\operatorname{Re} V)^-(x) \geq 0$ as an operator $E_x \to E_x$, by Lemma 2.9 we have $(\operatorname{Re} V)^- u \in L^1_{\text{loc}}(E)$. Thus $Vu \in L^1_{\text{loc}}(E)$, and the corollary is proven. □

Lemma 2.12. *The following operator relation holds: $H \subset S$.*

Proof: We will show that for all $u \in \mathrm{Dom}(H)$, we have $Hu = H_V u$.

Let $u \in \mathrm{Dom}(H)$. By property (i) of Section 2.8 we have $u \in \mathrm{D}(h)$; hence, by Corollary 2.11 we get $Vu \in L^1_{\mathrm{loc}}(E)$. Then, for any $v \in C_c^\infty(E)$, we have

$$(Hu, v) = h(u, v) = (\nabla u, \nabla v) + \int \langle Vu, v \rangle \, d\mu, \tag{2.3}$$

where $(\cdot, \cdot)$ denotes the L^2-inner product.

The first equality in (2.3) holds by property (i) from Section 2.8, and the second equality holds by definition of h.

Hence, using integration by parts in the first term on the right hand side of the second equality in (2.3) (see, for example Lemma 8.8 from [2]), we get

$$(u, \nabla^* \nabla v) = \int \langle Hu - Vu, v \rangle \, d\mu, \qquad \text{for all } \; v \in C_c^\infty(E). \tag{2.4}$$

Since $Vu \in L^1_{\mathrm{loc}}(E)$ and $Hu \in L^2(E)$, it follows that $(Hu - Vu) \in L^1_{\mathrm{loc}}(E)$, and (2.4) implies $\nabla^* \nabla u = Hu - Vu$ (as distributional sections of E). Therefore,

$$\nabla^* \nabla u + Vu = Hu,$$

and this shows that $Hu = H_V u$ for all $u \in \mathrm{Dom}(H)$.

Now by definition of S it follows that $\mathrm{Dom}(H) \subset \mathrm{Dom}(S)$ and $Hu = Su$ for all $u \in \mathrm{Dom}(H)$. Therefore $H \subset S$, and the lemma is proven. □

Lemma 2.13. *$C_c^\infty(E)$ is a core of the quadratic form $h_0 + h_1$.*

Proof: It suffices to show (see Theorem VI.1.21 in [6] and the paragraph above the equation (1.31) in Section VI.1.3 of [6]) that $C_c^\infty(E)$ is dense in the Hilbert space $\mathrm{D}(h_0 + h_1) = \mathrm{D}(h_0) \cap \mathrm{D}(h_1)$ with the inner product

$$(u, v)_{h_0 + h_1} := h_0(u, v) + (\operatorname{Re} h_1)(u, v) + (u, v),$$

where $h_0(\cdot, \cdot)$ denotes the sesquilinear form corresponding to h_0 via polarization identity and $(\operatorname{Re} h_1)$ denotes the real part of the sesquilinear form $h_1(\cdot, \cdot)$.

Let $u \in \mathrm{D}(h_0 + h_1)$ and $(u, v)_{h_0 + h_1} = 0$ for all $v \in C_c^\infty(E)$. We will show that $u = 0$.

We have

$$\begin{aligned} 0 &= h_0(u, v) + (\operatorname{Re} h_1)(u, v) + (u, v) \\ &= (u, \nabla^* \nabla v) + \int \langle (\operatorname{Re} V)^+ u, v \rangle \, d\mu + (u, v). \end{aligned} \tag{2.5}$$

Here we used integration by parts in the first term on the right hand side of the second equality.

Since $u \in \mathrm{D}(h_0 + h_1) \subset \mathrm{D}(h_1)$, it follows that $|\langle(\operatorname{Re} V)^+ u, u\rangle + i\langle(\operatorname{Im} V)u, u\rangle| \in L^1(M)$. Hence $\langle(\operatorname{Re} V)^+ u, u\rangle \in L^1(M)$. By Lemma 2.9 we get $(\operatorname{Re} V)^+ u \in L^1_{\mathrm{loc}}(E)$. From (2.5) we get the following distributional equality:

$$\nabla^*\nabla u = (-(\operatorname{Re} V)^+ - 1)u. \tag{2.6}$$

From (2.6) we have $\nabla^*\nabla u \in L^1_{\mathrm{loc}}(E)$. By Lemma 2.2 and by (2.6), we obtain

$$\Delta_M |u| \le \operatorname{Re}\langle \nabla^*\nabla u, \operatorname{sign} u\rangle = \langle -((\operatorname{Re} V)^+ + 1)u, \operatorname{sign} u\rangle \le -|u|. \tag{2.7}$$

The last inequality in (2.7) holds since $(\operatorname{Re} V)^+(x) \ge 0$ as an operator $E_x \to E_x$.

Therefore,

$$(\Delta_M + 1)|u| \le 0. \tag{2.8}$$

By Lemma 2.5, it follows that $|u| \le 0$. So $u = 0$, and the lemma is proven. □

Lemma 2.14. *$C_c^\infty(E)$ is a core of the quadratic form $h = (h_0 + h_1) + h_2$.*

Proof: Since the quadratic form h_2 is $(h_0 + h_1)$-bounded, the lemma follows immediately from Lemma 2.13. □

3. Proof of Theorem 1.8

By Lemma 2.12 we have $H \subset S$, so it is enough to show that $\mathrm{Dom}(S) \subset \mathrm{Dom}(H)$.

Let $u \in \mathrm{Dom}(S)$. By definition of $\mathrm{Dom}(S)$ in Section 1.5, we have $u \in \mathrm{D}(h_0) \subset \mathrm{D}(h_2)$ and $u \in \mathrm{D}(h_1)$. Hence $u \in \mathrm{D}(h)$.

For all $v \in C_c^\infty(E)$ we have

$$h(u, v) = h_0(u, v) + h_1(u, v) + h_2(u, v) = (u, \nabla^*\nabla v) + \int \langle Vu, v\rangle \, d\mu = (H_V u, v).$$

The last equality holds since $H_V u = Su \in L^2(E)$. By Lemma 2.14 it follows that $C_c^\infty(E)$ is a form core of h. Now from property (iii) of Section 2.8 we have $u \in \mathrm{Dom}(H)$ with $Hu = H_V u$. This concludes the proof of the theorem. □

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