Asymptotic stability for a nonlinear evolution equation

Zhang Hongwei, Chen Guowang

Abstract. We establish the asymptotic stability of solutions of the mixed problem for the nonlinear evolution equation $(|u_t|^{r-2}u_t)_t - \Delta u_{tt} - \Delta u - \delta\Delta u_t = f(u)$.

Keywords: nonlinear evolution equation, mixed problem, asymptotic stability of solutions

Classification: 35L35, 35L25

1. Introduction

This paper deals with asymptotic stability, as time tends to infinity, of solutions of the following mixed problem

$$(|u_t|^{r-2}u_t)_t - \Delta u_{tt} - \Delta u - \delta\Delta u_t = f(u), \quad x \in \Omega,\ t > 0, \tag{1.1}$$

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t \geq 0, \tag{1.2}$$

$$u(x,0) = u_0(x),\ u_t(x,0) = u_1(x),\ x \in \Omega, \tag{1.3}$$

where $\Omega \subset \mathbb{R}^n$ ($n \geq 1$ is a natural number) is a bounded open set with smooth boundary $\partial\Omega$, $r \geq 2$ and $\delta > 0$ are real number. Problems related to the equation

$$f(u_t)u_{tt} - \Delta u_{tt} - \Delta u = 0 \tag{1.4}$$

are interesting not only from the point of view of PDE general theory, but also due to its applications in mechanics. For instance, when the material density, $f(u_t)$, is equal to 1, Equation (1.4) describes the extensional vibrations of thin rods, see Love [1] for the physical details. When the material density $f(u_t)$ is not constant, we are dealing with a thin rod which possesses a rigid surface and whose interior is somehow permissive to slight deformations such that the material density varies according to the velocity, see [2], [3]. J. Ferreira and M.A. Rojas-Medar [2] have studied the existence of global weak solutions to the problem (1.1)–(1.3) with

This project is supported by the National Natural Science Foundation of China (Grant No.10371111) and partially by the Natural Science Foundation of Henan Province.

$\delta = 0$ in noncylindrical domain. Cavalcanti et al. [3] studied the existence and uniform decay of global weak solution to the following problem

$$(|u_t|^{r-2}u_t)_t - \Delta u_{tt} - \Delta u - \delta\Delta u_t + \int_0^t g(t-z)\Delta u(z)\,dz = 0$$

with initial and boundary condition, where $r > 2$ and $\delta > 0$ are constants, g represents the kernel of the memory term. However, no asymptotic stability result was presented in [2], [3] for the problem (1.1)–(1.3). In this paper, we study the asymptotic stability of solutions of the problem (1.1)–(1.3). Throughout this paper, we use the following notations. $(\cdot\ ,\ \cdot)$ denotes the inner product of $L^2(\Omega)$. $\|\cdot\|$, $\|\cdot\|_r$ and $\|\cdot\|_0$ denote the norms of the spaces $L^2(\Omega)$, $L^r(\Omega)$ and $H_0^1(\Omega)$ respectively.

2. Main theorem

We assume that the function $f(s)$ satisfies the following condition

(H) $|f(s)| \le a|s|^{p-1}$, $0 \le F(s) \le a|s|^p$,

where $F(s) = \int_0^s f(\rho)d\rho$ for $2 < p \le \infty$ if $n = 1, 2$ or for $2 < p \le \dfrac{2n}{n-2}$ if $n \ge 3$, and a is a positive constant. Furthermore, let $2 \le r \le p$.

Now, we define the energy associated with Equation (1.1) by

$$E(t) = \frac{r-1}{r}\|u_t\|_r^r + \frac{1}{2}\|\nabla u_t(t)\|^2 + J(u(t)), \qquad t \in \mathbb{R}^+ = [0, +\infty),$$

where

$$J(u) = J(u(t)) = \frac{1}{2}\|\nabla u(t)\|^2 - \int_\Omega F(u(t))\,dx.$$

We see that the energy has the so-called energy identity

$$E(t) + \delta\int_0^t \|\nabla u_t(s)\|^2 ds = E(0), \tag{2.1}$$

where $E(0) = \dfrac{r-1}{r}\|u_1\|_r^r + \dfrac{1}{2}\|\nabla u_1\|^2 + J(u_0)$ is the initial energy. Obviously, $E(t)$ is a non-increasing function in time.

Lemma 2.1. *Let $u_0 \in H_0^1(\Omega)$ and $u_1 \in H_0^1(\Omega)$. Then under the assumption* (H), *the problem* (1.1)–(1.3) *possesses at least one weak solution $u : \Omega \times \mathbb{R}^+ \to \mathbb{R}$ with*

$$u \in L^\infty(0,\infty; H_0^1(\Omega)), \quad u_t \in L^\infty(0,\infty; H_0^1(\Omega)), \quad u_{tt} \in L^2(0,\infty; H_0^1(\Omega)),$$

and for all $\eta \in C_0^\infty(0,T;H_0^1)$ *we have*

$$\Big[(|u_t(s)|^{r-2}u_t(s),\eta(s)) + (\nabla u_t(s),\nabla\eta(s))\Big]\Big|_{s=0}^{s=t}$$
$$= \int_0^t \Big[(|u_t(s)|^{r-2}u_t(s),\eta_t(s)) + (\nabla u_t(s),\nabla\eta_t(s)) - (\nabla u(s),\nabla\eta(s)) - \delta(\nabla u_t(s),\nabla\eta(s)) + (f(u(s)),\eta(s))\Big]\,ds.$$

The proof of Lemma 2.1 is omitted, since the proof of Lemma 2.1 is analogous to Theorem 3.1 in [2].

In order to get the asymptotic stability of the solution of the problem (1.1)–(1.3), we introduce the set

$$\Sigma = \Big\{(\lambda,E(0)) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 \le \lambda < \lambda_1, 0 \le \frac{1}{2}\lambda^2 - aC_0^p\lambda^p < E(0) < E_1\Big\},$$

where

$$\lambda_1 = \left(\frac{1}{paC_0^p}\right)^{\frac{1}{p-2}}, \qquad E_1 = \lambda_1^2\left(\frac{1}{2}-\frac{1}{p}\right)$$

and C_0 is the embedding constant (when H_0^1 is embedded into L^p).

Then our main theorem reads as follows:

Main theorem. *Under the assumptions of Lemma* 2.1, *if* $(\|\nabla u_0\|, E(0)) \in \Sigma$ *and* u *is a solution of the problem* (1.1)–(1.3), *then*

$$\lim_{t\to\infty} E(t) = 0. \tag{2.2}$$

We divide the proof into several steps.

Lemma 2.2. *Let* u *be a weak solution of the problem* (1.1)–(1.3). *If* $(\|\nabla u_0\|, E(0)) \in \Sigma$, *then for all* $t \in \mathbb{R}^+$,

(i) $(\|\nabla u(t)\|, E(t)) \in \Sigma$;

(ii) $E(t) \ge \dfrac{r-1}{r}\|u_t\|_r^r + \dfrac{1}{2}\|\nabla u_t\|^2$;

(iii) $\dfrac{1}{2}\|\nabla u\|^2 - \dfrac{1}{2}(f(u),u) \ge \dfrac{1}{4}\|\nabla u\|^2$.

Proof: By the definition of $E(t)$, (H) and embedding theorem, we have

$$E(t) \ge \frac{r-1}{r}\|u_t\|_r^r + \frac{1}{2}\|\nabla u_t\|^2 + \frac{1}{2}\|\nabla u\|^2 - aC_0^p\|\nabla u\|_p \ge G(\|\nabla u\|), \tag{2.3}$$

where $G(\lambda)=\frac{1}{2}\lambda^2-aC_0^p\lambda^p$. It is easy to see that $G(\lambda)$ attains its maximum E_1 for $\lambda=\lambda_1$, $G(\lambda)$ is strictly decreasing for $\lambda\geq\lambda_1$ and $G(\lambda)\to-\infty$ as $\lambda\to\infty$. Since $E(t)\leq E(0)<E_1$ for $t\in\mathbb{R}^+$ by (2.1), we have $\|\nabla u\|<\lambda_1$ for $t\in\mathbb{R}^+$. From (2.3) and $G(\|\nabla u\|)\geq 0$ for $0\leq\|\nabla u\|<\lambda_1$, we get $E(t)\geq G(\|\nabla u\|)\geq 0$, so (i) holds.

To obtain (ii), it remains to note that $G(\|\nabla u\|)\geq 0$ whenever $0\leq\|\nabla u\|<\lambda_1$ and to use (2.3) again, then (ii) follows at once.

By (H) and embedding theorem,we obtain

$$\frac{1}{2}\|\nabla u\|^2-\frac{1}{2}(f(u),u)\geq\frac{1}{4}\|\nabla u\|^2+\frac{1}{2}(\frac{1}{2}\|\nabla u\|^2-aC_0^p\|\nabla u\|^p).$$

Hence (iii) holds since $0\leq\|\nabla u(t)\|<\lambda_1$ for $t\in\mathbb{R}^+$ and $G(\|\nabla u\|)\geq 0$ for $0\leq\|\nabla u\|<\lambda_1$. The lemma is proved. □

Lemma 2.3. *Let $(\|\nabla u_0\|,E(0))\in\Sigma$ and $E(t)\geq\beta$, where $\beta>0$. Then there exists $\alpha=\alpha(\beta)>0$ such that*

$$\frac{r-1}{r}\|u_t\|_r^r+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla u\|^2-\frac{1}{2}(f(u),u)\geq\alpha,\quad\textit{for}\ \ t\in\mathbb{R}^+. \tag{2.4}$$

Proof: By the definition of $E(t)$, (H) and $E(t)\geq\beta$, we have

$$\frac{r-1}{r}\|u_t\|_r^r+\frac{1}{2}\|\nabla u_t\|^2+\frac{1}{2}\|\nabla u\|^2\geq\beta,\qquad t\in\mathbb{R}^+. \tag{2.5}$$

Now suppose that (2.4) does not hold. For Lemma 2.1(iii), there is a sequence $\{t_n\}\subset\mathbb{R}^+$ such that

$$\begin{aligned}&\frac{r-1}{r}\|u_t(t_n)\|_r^r+\frac{1}{2}\|\nabla u_t(t_n)\|^2+\frac{1}{2}\|\nabla u(t_n)\|^2-\frac{1}{2}(f(u(t_n)),u(t_n))\\ &\geq\frac{r-1}{r}\|u_t(t_n)\|_r^r+\frac{1}{2}\|\nabla u_t(t_n)\|^2+\frac{1}{4}\|\nabla u(t_n)\|^2\to 0,\qquad n\to\infty.\end{aligned}$$

Then we get

$$\frac{r-1}{r}\|u_t(t_n)\|_r^r+\frac{1}{2}\|\nabla u_t(t_n)\|^2\to 0,\qquad\|\nabla u(t_n)\|^2\to 0,\qquad n\to\infty.$$

This is contradiction with (2.5). The lemma is proved. □

Proof of main theorem: Suppose that (2.2) fails. Then there exists $\beta>0$ such that $E(t)\geq\beta$ for all $t\in\mathbb{R}^+$ since (2.1) and $E(t)\geq 0$ by Lemma 2.2 (i).

Multiplying both sides of (1.1) by u, integrating over $[T,t]$ $(0 < T \le t < \infty)$ and integrating by parts with respect to t, we obtain

(2.6)
$$\begin{aligned}
&\Big[(|u_t(s)|^{r-2}u_t(s), u(s)) + (\nabla u_t(s), \nabla u(s))\Big]\Big|_{s=T}^{t}\\
&= \int_T^t \Big\{\frac{3r-2}{r}\|u_t(s)\|_r^r + 2\|\nabla u_t(s)\|^2 - 2\big[\frac{r-1}{r}\|u_t(s)\|_r^r\\
&\quad + \frac{1}{2}\|\nabla u_t(s)\|^2 + \frac{1}{2}\|\nabla u(s)\|^2 - \frac{1}{2}(f(u(s)), u(s))\big] - \delta(\nabla u(s), \nabla u_t(s))\Big\}\, ds\\
&= \int_T^t (I_1 + I_2 + I_3)\, ds.
\end{aligned}$$

Using $H_0^1 \hookrightarrow L^r$, $E(t) \le E(0) < \infty$, Hölder inequality and $\|\nabla u_t\|^2 \in L^1(0,\infty)$, we have

$$\begin{aligned}
\int_T^t I_1\, ds &\le C_1 \int_T^t (\|\nabla u_t(s)\|^r + \|\nabla u_t(s)\|^2)\, ds\\
&\le C_2 (E^{\frac{r-1}{r}}(0) + E^{\frac{1}{2}}(0)) \int_T^t \|\nabla u_t(s)\|\, ds\\
&\le C_3 \left(\int_T^t \|\nabla u_t(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\int_T^t ds\right)^{\frac{1}{2}}\\
&\le C_4 \left(\int_T^t ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.7}$$

Here and in the following C_i $(i = 1, 2, \dots)$ denotes positive constants which do not depend on t and T. By virtue of Lemma 2.3, we have

$$\int_T^t I_2\, ds \le -2\alpha \int_T^t ds. \tag{2.8}$$

Furthermore, by use of $\|\nabla u\| \le \lambda_1$, $E(t) \ge 0$, Lemma 2.2, Hölder inequality and $\|\nabla u_t\|^2 \in L^1(0,\infty)$, we have

$$\begin{aligned}
\int_T^t I_3 &\le \delta \left(\int_T^t \|\nabla u_t(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\int_T^t \|\nabla u(s)\|^2\, ds\right)^{\frac{1}{2}}\\
&\le \lambda_1 \delta \left(\int_T^\infty \|\nabla u_t(s)\|^2\, ds\right)^{\frac{1}{2}} \left(\int_T^t ds\right)^{\frac{1}{2}} \le C_5 \left(\int_T^t ds\right)^{\frac{1}{2}}.
\end{aligned} \tag{2.9}$$

Then from (2.6)–(2.9) we know

$$\begin{aligned}
&\Big[(u_t(s)|^{r-2}u_t(s), u(s)) + (\nabla u_t(s), \nabla u(s))\Big]\Big|_{s=T}^{t}\\
&\qquad\qquad \le C_6 \left(\int_T^t ds\right)^{\frac{1}{2}} - 2\alpha \int_T^t ds.
\end{aligned} \tag{2.10}$$

On the other hand, from Young inequality, $H_0^1 \hookrightarrow L^r$, $\|\nabla u\| \le \lambda_1 < \infty$, $E(t) < E(0) < \infty$ and Lemma 2.2(i), we get

$$\begin{aligned}&\Big|(|u_t(t)|^{r-2}u_t(t), u(t)) + (\nabla u_t(t), \nabla u(t))\Big| \\ &\le C_7\left(\|u_t\|_r^r + \|\nabla u\|^r + \|\nabla u_t\|^2 + \|\nabla u\|^2\right) \le C_8 < \infty.\end{aligned}$$

In turn, we reach a contradiction with (2.10) for fixing T when $t \to \infty$. Hence we derive $\lim_{t\to\infty} E(t) = 0$. This completes the proof. □

Remark 1. If we take $f(s) = |s|^{p-2}s$ in (1.1), then $F(s) = \frac{1}{p}|s|^p$ and $\dfrac{1}{p}sf(s) = F(s)$, so (H) holds. By straightforward calculation we get

$$\lambda_1 = C_0^{-\frac{p}{p-2}}, \qquad E_1 = \left(\frac{1}{2} - \frac{1}{p}\right)\left(\frac{1}{C_0^p}\right)^{\frac{2}{p-2}}.$$

It is easy to see that E_1 is exactly the potential well depth corresponding to the problem (1.1)–(1.3) obtained by Payne and Sattinger [10], that is

$$E_1 = \inf_{u\in H_0^1\setminus\{0\}} \sup_{\lambda\in\mathbb{R}} J(\lambda u),$$

where $J(u) = \dfrac{1}{2}\|\nabla u\|^2 - \dfrac{1}{p}\|u\|_p^p$.

Remark 2. If the initial point $(\|u_0\|, E(0))$ lies in set

$$\begin{aligned}\Sigma_0 = \Bigg\{&(\lambda, E(0)) \in \mathbb{R}^+ \times \mathbb{R}^+, 0 \le \lambda < \lambda_2 = \left(\frac{1}{2pcC_0^p}\right)^{\frac{1}{p-2}}, \\ &0 \le \frac{1}{4}\lambda^2 - aC_0^p\lambda^p < E(0) < E_2 = \frac{1}{2}\lambda_1^2\left(\frac{1}{2} - \frac{1}{p}\right)\Bigg\},\end{aligned}$$

which is smaller than Σ, we can prove (2.2) and moreover,

$$\lim_{t\to\infty} \|\nabla u(t)\|^2 = 0.$$

References

[1] Love A.H., *A Treatise on Mathematical Theory of Elasticity*, Dover, New York, 1944.

[2] Ferreira J., Rojas-Medar M., *On global weak solutions of a nonlinear evolution equation in noncylindrical domain*, in Proceedings of the 9th International Colloquium on Differential Equations, D. Bainov (Ed.), VSP, 1999, pp. 155–162.

[3] Cavalcanti M.M., Domingos Cavalcanti V.N., Ferreira J., *Existence and uniform decay for a nonlinear viscoelastic equation with strong damping*, Math. Meth. Appl. Sci. **24** (2001), 1043–1053.

[4] Nakao M., Ono K., *Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations*, Math. Z. **214** (1993), 325–342.

[5] Ono K., *On global solutions and blow-up solutions of nonlinear Kirchhoff strings with nonlinear dissipation*, J. Math. Anal. Appl. **216** (1997), 321–342.

[6] Park J.Y., Bae J.J., *On solutions of quasilinear wave equations with nonlinear damping terms*, Czechoslovak Math. J. **50** (2000), 565–585.

[7] Levine H.A., Pucci P., Serrin J., *Some remarks on global nonexistence for nonautonomous abstract evolution equations*, Contemporary Mathematics **208** (1997), 253–263.

[8] Pucci P., Serrin J., *Stability for abstract evolution equations*, in Partial Differential Equation and Applications, P. Marcellimi, et al. (Eds.), Marcel Dekker, 1996, pp. 279–288.

[9] Pucci P., Serrin J., *Asymptotic stability for nonautonomous wave equation*, Comm. Pure Appl. Math. **XLXX** (1996), 177–216.

[10] Payne L.E., Sattinger D.H., *Saddle points and unstability of nonlinear hyperbolic equations*, Israel J. Math. **22** (1975), 273–303.

Department of Mathematics-Physics, Zhengzhou Institute of Technology, and Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, P.R. China

Department of Mathematics, Zhengzhou University, Zhengzhou, 450052, P.R. China

(*Received* March 24, 2003, *revised* September 7, 2003)