A class of Bol loops with a subgroup of index two

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Abstract. Let G be a finite group and C_2 the cyclic group of order 2. Consider the 8 multiplicative operations $(x, y) \mapsto (x^i y^j)^k$, where i, j, $k \in \{-1, 1\}$. Define a new multiplication on $G \times C_2$ by assigning one of the above 8 multiplications to each quarter $(G \times \{i\}) \times (G \times \{j\})$, for $i, j \in C_2$. We describe all situations in which the resulting quasigroup is a Bol loop. This paper also corrects an error in P. Vojtěchovský: On the uniqueness of loops $M(G, 2)$.

Keywords: Moufang loops, loops $M(G, 2)$, inverse property loops, Bol loops

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1. Introduction

Let G be a finite group. Consider the two maps $\theta_{yx}, \theta_{xy^{-1}} : G\times G \to G\times G$ defined by $\theta_{yx}(a, b) = (b, a)$, $\theta_{xy^{-1}}(a, b) = (a, b^{-1})$. The group $\Theta = (\Theta, \circ)$ generated by θ_{yx} and $\theta_{xy^{-1}}$ consists of eight maps, and is isomorphic to the quaternion group. We will denote elements of Θ suggestively by θ_{xy}, $\theta_{xy^{-1}}$, $\theta_{x^{-1}y}$, $\theta_{x^{-1}y^{-1}}$, θ_{yx}, $\theta_{yx^{-1}}$, $\theta_{y^{-1}x}$ and $\theta_{y^{-1}x^{-1}}$. For instance, $\theta_{y^{-1}x}$ is the map defined by $\theta_{y^{-1}x}(a, b) = (b^{-1}, a)$.

Let us identify $\theta_{uv} \in \Theta$ with $\Delta\theta_{uv}$, where $\Delta : G \times G \to G$ is given by $\Delta(a, b) = ab$. Thanks to this double perspective, each $\theta_{uv} \in \Theta$ determines a multiplication on G, yet it is possible to compose the multiplications.

Let $\overline{G} = \{\overline{g};\ g \in G\}$ be a copy of G. Given four maps $\alpha, \beta, \gamma, \delta \in \Theta$, define multiplication $*$ on $G \cup \overline{G}$ by

$$g * h = \alpha(g, h), \quad g * \overline{h} = \overline{\beta(g, h)}, \quad \overline{g} * h = \overline{\gamma(g, h)}, \quad \overline{g} * \overline{h} = \delta(g, h),$$

or, more precisely,

$$g * h = \Delta\alpha(g, h), \quad g * \overline{h} = \overline{\Delta\beta(g, h)}, \quad \overline{g} * h = \overline{\Delta\gamma(g, h)}, \quad \overline{g} * \overline{h} = \Delta\delta(g, h),$$

where $g, h \in G$. Note that $G * G = G = \overline{G} * \overline{G}$, $G * \overline{G} = \overline{G} = \overline{G} * G$. Hence $G(\alpha, \beta, \gamma, \delta) = (G \cup \overline{G}, *)$ is a quasigroup with normal subgroup G.

Chein proved in [2] that

$$M(G, 2) = G(\theta_{xy}, \theta_{yx}, \theta_{xy^{-1}}, \theta_{y^{-1}x})$$

is a Moufang loop for every group G, and that it is associative if and only if G is abelian. Many small Moufang loops are of this kind, e.g. the smallest nonassociative Moufang loop $M(S_3, 2)$ (see [3], [7], [12]).

It is natural to ask if there are other constructions $G(\alpha, \beta, \gamma, \delta)$ besides $M(G, 2)$ that produce Moufang loops. I have obtained the following result in [11]:

Theorem 1.1. *Let G be a nonabelian group, and let $\alpha, \beta, \gamma, \delta \in \Theta$ be as above. Then $G(\alpha, \beta, \gamma, \delta)$ is a Moufang loop if and only if it is among*

$$\begin{array}{ll}
G(\theta_{xy}, \theta_{xy}, \theta_{xy}, \theta_{xy}), & G(\theta_{yx}, \theta_{yx}, \theta_{yx}, \theta_{yx}),\\
G(\theta_{xy}, \theta_{yx^{-1}}, \theta_{y^{-1}x}, \theta_{x^{-1}y^{-1}}), & G(\theta_{yx}, \theta_{x^{-1}y}, \theta_{xy^{-1}}, \theta_{y^{-1}x^{-1}}),\\
G(\theta_{xy}, \theta_{yx}, \theta_{xy^{-1}}, \theta_{y^{-1}x}), & G(\theta_{yx}, \theta_{yx^{-1}}, \theta_{xy}, \theta_{x^{-1}y}),\\
G(\theta_{xy}, \theta_{x^{-1}y}, \theta_{yx}, \theta_{yx^{-1}}), & G(\theta_{yx}, \theta_{xy}, \theta_{y^{-1}x}, \theta_{xy^{-1}}).
\end{array}$$

The four loops in the first two rows are associative and isomorphic to the direct product of G with the 2-element cyclic group. The remaining four loops are not associative and are isomorphic to the loop $M(G, 2)$.

2. The mistake

Hence Chein's construction $M(G, 2)$ is the “unique” construction $G(\alpha, \beta, \gamma, \delta)$ that produces nonassociative Moufang loops. I claim in [11] that $M(G, 2)$ is the “unique” construction $G(\alpha, \beta, \gamma, \delta)$ that produces nonassociative Bol loops, too. Unfortunately, this is not correct, as we shall see.

The mistake (pointed out to me by Michael Kinyon) is in Lemma 2 [11], where I claim that any loop of the form $G(\alpha, \beta, \gamma, \delta)$ is an inverse property loop. But I only prove in Lemma 2 [11] that such a loop has two-sided inverses. Hence the conclusion that any Bol loop $G(\alpha, \beta, \gamma, \delta)$ is automatically Moufang was not justified in [11], and is in fact not true.

This note can be considered an erratum for [11]. However, in dealing with the Bol case, I had to develop additional techniques not found in [11]. Moreover, several constructions obtained in this way appear to be new, and should be of interest in the classification of small Bol loops.

3. Reductions and assumptions

Our goal is to describe all Bol loops of the form $G(\alpha, \beta, \gamma, \delta)$.

Note that $|\Delta\Theta| = 1$ when G is an elementary abelian 2-group or when $|G| = 1$. In such a case, any $G(\alpha, \beta, \gamma, \delta)$ is equal to $G(\theta_{xy}, \theta_{xy}, \theta_{xy}, \theta_{xy})$, and is therefore isomorphic to $G \times C_2$. We thus lose nothing by making this assumption:

Assumption 1. $|G| > 1$ *and G is not an elementary abelian 2-group.*

Although $G(\alpha, \beta, \gamma, \delta)$ does not have to be a loop, it is not hard to determine when it is (cf. Lemma 1 [11]):

Lemma 3.1. *The quasigroup* $(M,*) = G(\alpha,\beta,\gamma,\delta)$ *is a loop if and only if*

$$(1)\quad \alpha\in\{\theta_{xy},\theta_{yx}\},\ \beta\in\{\theta_{xy},\theta_{yx},\theta_{yx^{-1}},\theta_{x^{-1}y}\},\ \gamma\in\{\theta_{xy},\theta_{yx},\theta_{y^{-1}x},\theta_{xy^{-1}}\}.$$

When $(M,*)$ *is a loop, its neutral element coincides with the neutral element of* G.

Proof: We first show that if $(M,*)$ is a loop, its neutral element e coincides with the neutral element 1 of G. We have $1*1=\alpha(1,1)=1=e*1$, no matter what $\alpha\in\Theta$ is. Since $(M,*)$ is a quasigroup, $1=e$ follows.

The equation $y=1*y$ holds for every $y\in G$ if and only if $y=\alpha(1,y)$ holds for every $y\in G$, which happens if and only if α does not invert its second argument. (We will use this trick many times. Note how Assumption 1 is used.) Thus $y=1*y$ holds for every $y\in G$ if and only if $\alpha\in\{\theta_{xy},\theta_{x^{-1}y},\theta_{yx},\theta_{yx^{-1}}\}$.

Similarly, the equation $y=y*1$ holds for every $y\in G$ if and only if $\alpha\in\{\theta_{xy},\ \theta_{xy^{-1}},\theta_{yx},\theta_{y^{-1}x}\}$. Altogether, $y=y*1=1*y$ holds for every $y\in G$ if and only if $\alpha\in\{\theta_{xy},\theta_{yx}\}$.

Following a similar strategy, $\overline{y}=1*\overline{y}$ holds for every $y\in G$ if and only if $\beta\in\{\theta_{xy},\theta_{yx},\theta_{yx^{-1}},\theta_{x^{-1}y}\}$, and $\overline{y}=\overline{y}*1$ holds for every $y\in G$ if and only if $\gamma\in\{\theta_{xy},\theta_{yx},\theta_{y^{-1}x},\theta_{xy^{-1}}\}$. □

Since we are only interested in loops here, we assume:

Assumption 2. *The maps* α, β, γ *are as in* (1).

For a groupoid $(A,\circ)$, let $A^{\mathrm{op}}=(A,\circ^{\mathrm{op}})$ be the *opposite* of A, defined by $x\circ^{\mathrm{op}}y=y\circ x$. Then

$$(2)\qquad G(\alpha,\beta,\gamma,\delta)^{\mathrm{op}}=G(\theta_{yx}\alpha,\theta_{yx}\gamma,\theta_{yx}\beta,\theta_{yx}\delta),$$

$$(3)\qquad G(\alpha,\beta,\gamma,\delta)=G^{\mathrm{op}}(\theta_{yx}\alpha,\theta_{yx}\beta,\theta_{yx}\gamma,\theta_{yx}\delta).$$

It is not necessarily true that $G(\alpha,\beta,\gamma,\delta)^{\mathrm{op}}$ is isomorphic to $G(\alpha,\beta,\gamma,\delta)$. However, by (3), any loop $G(\theta_{yx},\beta,\gamma,\delta)$ can also be obtained as $G^{\mathrm{op}}(\theta_{xy},\beta',\gamma',\delta')$, for some β', γ', δ'. As G is isomorphic to G^{op} (via $x\mapsto x^{-1}$), we postulate:

Assumption 3. $\alpha=\theta_{xy}$.

Our last reduction concerns the maps β and γ.

Lemma 3.2. *The loop* $G(\alpha,\beta,\gamma,\delta)$ *is isomorphic to* $G(\alpha,\beta',\gamma',\theta_{x^{-1}y^{-1}}\delta)$ *if*

$$(\beta,\beta')\in\{(\theta_{xy},\theta_{yx^{-1}}),(\theta_{yx},\theta_{x^{-1}y}),(\theta_{x^{-1}y},\theta_{yx}),(\theta_{yx^{-1}},\theta_{xy})\},$$
$$(\gamma,\gamma')\in\{(\theta_{xy},\theta_{y^{-1}x}),(\theta_{yx},\theta_{xy^{-1}}),(\theta_{xy^{-1}},\theta_{yx}),(\theta_{y^{-1}x},\theta_{xy})\}.$$

Proof: Let $(M,*)=G(\alpha,\beta,\gamma,\delta)$, and $(M,\circ)=G(\alpha,\beta',\gamma',\theta_{x^{-1}y^{-1}}\delta)$. Consider the permutation $f:M=G\cup\overline{G}\to M$ defined by $f(x)=x$, $f(\overline{x})=\overline{x^{-1}}$, for $x\in G$.

We show that f is an isomorphism of $(M,*)$ onto $(M,\circ)$ if (and only if)

$$(\Delta\beta(x,y))^{-1} = \Delta\beta'(x,y^{-1}), \quad (\Delta\gamma(x,y))^{-1} = \Delta\gamma'(x^{-1},y). \tag{4}$$

Once we establish this fact, the proof is finished by checking that the pairs (β,β'), (γ,γ') in the statement of the Lemma satisfy (4).

In the following computation, we emphasize by Δ the multiplication in M. Let $x, y \in G$. Then

$$\begin{aligned}
f(x*y) &= f(\Delta\alpha(x,y)) = \Delta\alpha(x,y),\\
f(x*\overline{y}) &= f(\overline{\Delta\beta(x,y)}) = \overline{(\Delta\beta(x,y))^{-1}},\\
f(\overline{x}*y) &= f(\overline{\Delta\gamma(x,y)}) = \overline{(\Delta\gamma(x,y))^{-1}},\\
f(\overline{x}*\overline{y}) &= f(\Delta\delta(x,y)) = \Delta\delta(x,y),
\end{aligned}$$

while

$$\begin{aligned}
f(x)\circ f(y) &= x\circ y = \Delta\alpha(x,y),\\
f(x)\circ f(\overline{y}) &= x\circ\overline{y^{-1}} = \overline{\Delta\beta'(x,y^{-1})},\\
f(\overline{x})\circ f(y) &= \overline{x^{-1}}\circ y = \overline{\Delta\gamma'(x^{-1},y)},\\
f(\overline{x})\circ f(\overline{y}) &= \overline{x^{-1}}\circ\overline{y^{-1}} = \Delta\theta_{x^{-1}y^{-1}}\delta(x^{-1},y^{-1}).
\end{aligned}$$

We see that $f(x*y) = f(x)\circ f(y)$, $f(\overline{x}*\overline{y}) = f(\overline{x})\circ f(\overline{y})$ always hold, and that $f(x*\overline{y}) = f(x)\circ f(\overline{y})$, $f(\overline{x}*y) = f(\overline{x})\circ f(y)$ hold if (β,β'), (γ,γ') satisfy (4). □

Observe that for any admissible value of β, γ and δ, Lemma 3.2 provides an isomorphism of $G(\alpha,\beta,\gamma,\delta)$ that leaves α intact. Furthermore, if $\gamma = \theta_{xy^{-1}}$, the corresponding γ' is equal to θ_{yx}, and if $\gamma = \theta_{y^{-1}x}$, we have $\gamma' = \theta_{xy}$. We can therefore assume:

Assumption 4. $\gamma \in \{\theta_{xy}, \theta_{yx}\}$.

We could have reduced β instead of γ, but the choice we have made will be more convenient later on.

4. The technique

Suppose we want to check if $(M,*) = G(\alpha,\beta,\gamma,\delta)$ satisfies a given identity. Since the product $a*b$ in M depends on whether the elements a, b belong to G or $\overline{G}$, it is natural to treat the cases separately. For instance, the *left alternative law* $a*(a*b) = (a*a)*b$ for M leads to four identities $x*(x*y) = (x*x)*y$, $x*(x*\overline{y}) = (x*x)*\overline{y}$, $\overline{x}*(\overline{x}*y) = (\overline{x}*\overline{x})*y$, and $\overline{x}*(\overline{x}*\overline{y}) = (\overline{x}*\overline{x})*\overline{y}$, where $x, y \in G$. In turn, each of these identities can be rewritten as an identity for G, using the maps α, β, γ, δ. Here are the four left alternative identities together with their translations:

$$x * (x * y) = (x * x) * y \qquad \alpha(x, \alpha(x, y)) = \alpha(\alpha(x, x), y), \tag{5}$$

$$x * (x * \overline{y}) = (x * x) * \overline{y} \qquad \beta(x, \beta(x, y)) = \beta(\alpha(x, x), y), \tag{6}$$

$$\overline{x} * (\overline{x} * y) = (\overline{x} * \overline{x}) * y \qquad \delta(x, \gamma(x, y)) = \alpha(\delta(x, x), y), \tag{7}$$

$$\overline{x} * (\overline{x} * \overline{y}) = (\overline{x} * \overline{x}) * \overline{y} \qquad \gamma(x, \delta(x, y)) = \beta(\delta(x, x), y). \tag{8}$$

In case we need to prove that $(M, *)$ does not satisfy a given identity, it suffices to show that any of the translated identities does not hold for G. For such purposes, it is often advantageous to look at an identity that does not involve many different maps (i.e., (6) is preferable to (8), say).

More importantly, since we know nothing about G besides the fact that it satisfies Assumption 1, how do we decide if a given identity is true or false in G? Well, if we treat the identity as an identity in a free group, and if the identity reduces to $x = x^{-1}$, it must be false in G. More identities reduce to $x = x^{-1}$ if we assume that the free group is abelian. Instead of making this distinction for each particular identity, we will treat the abelian and nonabelian cases separately from the start.

When G is a group and m a positive integer, we let

$$G^m = \{g^m;\ g \in G\}, \quad G_m = \{g \in G;\ g^m = 1\}.$$

Note that, in the literature, G_m often denotes the set of all elements whose order is a power of m. In our case, G_m consists of all elements of exponent m.

Because of the nature of Bol identities (see below), we will often come across group identities involving squares. Two conditions for G will then help us characterize the groups in which such identities hold. Namely, $G^2 \subseteq Z(G)$, and $G^4 = 1$. The former assumption says that $G/Z(G)$ is an elementary abelian 2-group. The latter assumption is equivalent to $G_4 = G$.

We are now ready to start the search for all Bol loops $G(\alpha, \beta, \gamma, \delta)$. Recall that a loop is *left Bol* if it satisfies the identity

$$x(y(xz)) = (x(yx))z, \tag{9}$$

and it is *right Bol* if it satisfies the identity $((zx)y)x = z((xy)x)$. (See [9].) Hence left Bol loops are opposites of right Bol loops, and thus all right Bol loops of the form $G(\alpha, \beta, \gamma, \delta)$ can be obtained from the left Bol ones via (2).

We therefore restrict our search to left Bol loops.

5. The abelian case

In this section we suppose that G is a finite abelian group. Then $|\Delta\Theta| = 4$, and it suffices to consider multiplications θ_{xy}, $\theta_{x^{-1}y}$, $\theta_{xy^{-1}}$, $\theta_{x^{-1}y^{-1}}$.

Assumptions 1–4 are now equivalent to: *G is abelian, $|G| > 1$, G is not an elementary abelian 2-group, and*

$$\alpha = \theta_{xy},\ \beta \in \{\theta_{xy}, \theta_{x^{-1}y}\},\ \gamma = \theta_{xy},\ \delta \in \{\theta_{xy}, \theta_{x^{-1}y}, \theta_{xy^{-1}}, \theta_{x^{-1}y^{-1}}\}.$$

Left Bol loops are left alternative, as is immediately obvious upon substituting $y = 1$ into (9). Let us therefore first describe all left alternative loops $G(\alpha, \beta, \gamma, \delta)$.

Lemma 5.1. *Suppose that G is an abelian group and that Assumptions 1–4 are satisfied. Then $G(\alpha, \beta, \gamma, \delta)$ is a left alternative loop if and only if one of the following conditions is satisfied:*

(i) $(\beta, \delta) \in \{(\theta_{xy}, \theta_{xy}), (\theta_{xy}, \theta_{x^{-1}y}), (\theta_{x^{-1}y}, \theta_{x^{-1}y})\}$,

(ii) $G^4 = 1$ *and* $(\beta, \delta) = (\theta_{x^{-1}y}, \theta_{xy})$.

Proof: Identity (5) holds since $\alpha = \theta_{xy}$. Identity (6) clearly holds when $\beta = \theta_{xy}$. When $\beta = \theta_{x^{-1}y}$, it becomes $x^{-1}x^{-1}y = (xx)^{-1}y$; again true.

When $\beta = \theta_{xy}$, identity (8) becomes $x\delta(x, y) = \delta(x, x)y$. Note that this identity cannot hold if δ inverts its second argument. (Since then there is y^{-1} on the left, y on the right, and with $x = 1$ the identity becomes $y = y^{-1}$.) On the other hand, the identity $x\delta(x, y) = \delta(x, x)y$ holds for $\delta = \theta_{xy}$ and $\delta = \theta_{x^{-1}y}$.

When $\beta = \theta_{x^{-1}y}$, (8) becomes $x\delta(x, y) = \delta(x, x)^{-1}y$. Again, this identity cannot hold if δ inverts its second argument. When $\delta = \theta_{x^{-1}y}$, it becomes $y = y$ (true). When $\delta = \theta_{xy}$, it becomes $xxy = (xx)^{-1}y$, which holds if and only if $G^4 = 1$.

Finally, (7) holds for $\delta \in \{\theta_{xy},\ \theta_{x^{-1}y}\}$. □

Theorem 5.2. *Assume that G is an abelian group and that Assumptions 1–4 are satisfied. Then $(M, *) = G(\alpha, \beta, \gamma, \delta)$ is a group if and only if it is among*

$$G(\theta_{xy}, \theta_{xy}, \theta_{xy}, \theta_{xy}), \tag{10}$$

$$G(\theta_{xy}, \theta_{x^{-1}y}, \theta_{xy}, \theta_{x^{-1}y}). \tag{11}$$

The group (10) is the direct product $G \times C_2$, and the group (11) is isomorphic to the Chein loop $M(G, 2)$.

Furthermore, $(M, *)$ is a nonassociative left Bol loop if and only if $(M, *)$ is a left Bol loop that is not Moufang if and only if $G^4 = 1$ and $(M, *)$ is among

$$G(\theta_{xy}, \theta_{xy}, \theta_{xy}, \theta_{x^{-1}y}), \tag{12}$$

$$G(\theta_{xy}, \theta_{x^{-1}y}, \theta_{xy}, \theta_{xy}). \tag{13}$$

For a given G, the two loops (12), (13) are not isomorphic.

Proof: There cannot be more than 4 left Bol loops $G(\alpha, \beta, \gamma, \delta)$ satisfying Assumptions 1–4, since there are only 4 such left alternative loops, by Lemma 5.1.

Clearly, (10) is the direct product $G \times C_2$. We claim that the loop (11) is isomorphic to $M(G, 2)$. To see that, write $M(G, 2)$ as $G(\theta_{xy}, \theta_{xy}, \theta_{xy^{-1}}, \theta_{xy^{-1}})$, and apply a suitable isomorphism of Lemma 3.2 to it. Since G is abelian, (11) is associative by [2].

Consider one of the flexible identities $\overline{x} * (\overline{y} * \overline{x}) = (\overline{x} * \overline{y}) * \overline{x}$. It translates into $\gamma(x, \delta(y, x)) = \beta(\delta(x, y), x)$. This becomes $xy^{-1}x = x^{-1}yx$ for (12), and $xyx = y^{-1}x^{-1}x$ for (13); both false (let $y = 1$).

Thus neither of the loops (12), (13) is flexible, and hence neither is a Moufang loop. We must now check that (12), (13) are left Bol. This follows by straightforward calculation:

The left Bol identity for $(M, *)$ translates into 8 identities for G. Here they are:

(14)	$x*(y*(x*z)) = (x*(y*x))*z$	$\alpha(x,\alpha(y,\alpha(x,z))) = \alpha(\alpha(x,\alpha(y,x)),z)$
(15)	$x*(y*(x*\overline{z})) = (x*(y*x))*\overline{z}$	$\beta(x,\beta(y,\beta(x,z))) = \beta(\alpha(x,\alpha(y,x)),z)$
(16)	$x*(\overline{y}*(x*z)) = (x*(\overline{y}*x))*z$	$\beta(x,\gamma(y,\alpha(x,z))) = \gamma(\beta(x,\gamma(y,x)),z)$
(17)	$\overline{x}*(y*(\overline{x}*z)) = (\overline{x}*(y*\overline{x}))*z$	$\delta(x,\beta(y,\gamma(x,z))) = \alpha(\delta(x,\beta(y,x)),z)$
(18)	$x*(\overline{y}*(x*\overline{z})) = (x*(\overline{y}*x))*\overline{z}$	$\alpha(x,\delta(y,\beta(x,z))) = \delta(\beta(x,\gamma(y,x)),z)$
(19)	$\overline{x}*(y*(\overline{x}*\overline{z})) = (\overline{x}*(y*\overline{x}))*\overline{z}$	$\gamma(x,\alpha(y,\delta(x,z))) = \beta(\delta(x,\beta(y,x)),z)$
(20)	$\overline{x}*(\overline{y}*(\overline{x}*z)) = (\overline{x}*(\overline{y}*\overline{x}))*z$	$\gamma(x,\delta(y,\gamma(x,z))) = \gamma(\gamma(x,\delta(y,x)),z)$
(21)	$\overline{x}*(\overline{y}*(\overline{x}*\overline{z})) = (\overline{x}*(\overline{y}*\overline{x}))*\overline{z}$	$\delta(x,\gamma(y,\delta(x,z))) = \delta(\gamma(x,\delta(y,x)),z)$

Since the only nontrivial multiplication in (12) is δ, it suffices to verify identities (17)–(21) for (12). We obtain $x^{-1}yxz = x^{-1}yxz$ (true), $xy^{-1}xz = (xyx)^{-1}z$ (true if and only if $G^4 = 1$), $xyx^{-1}z = x^{-1}yxz$ (true), $xy^{-1}xz = xy^{-1}xz$ (true), and $x^{-1}yx^{-1}z = (xy^{-1}x)^{-1}z$ (true), respectively.

Since the only nontrivial multiplication in (13) is β, it suffices to verify identities (15)–(19) for (13). We obtain $x^{-1}y^{-1}x^{-1}z = (xyx)^{-1}z$ (true), $x^{-1}yxz = x^{-1}yxz$ (true), $xy^{-1}xz = xy^{-1}xz$ (true), $xyx^{-1}z = x^{-1}yxz$ (true), $xyxz = (xy^{-1}x)^{-1}z$ (true if and only if $G^4 = 1$), respectively.

It remains to check that (12) is not isomorphic to (13). To see that, notice that all elements $\overline{x}$ are of order 2 in (12) (since $\delta = \theta_{xy^{-1}}$), while the order of $\overline{x}$ is bigger or equal to the order of x in (13) (since the nth power of $\overline{x}$ in $(M, *)$ can be written as $\overline{x} * (\overline{x} * (\dots))$, and $\gamma = \delta = \theta_{xy}$). □

Remark 5.3. By the Fundamental theorem for finite abelian groups [10], the groups required for (12), (13) are exactly the groups of the form $(C_4)^m \times (C_2)^n$, where $m > 0$ and $n \geq 0$.

Remark 5.4. The smallest left Bol loops that are not Moufang are of order 8. In fact, there are 6 such loops, up to isomorphism [1]. Five of them contain a subgroup isomorphic to C_4. (See [1], or use the LOOPS package [8] for GAP, where the six Bol loops can be obtained via `BolLoop(8,i)`, with $1 \leq$i≤ 6.) Our constructions (12) and (13) yield two of these five loops. We can of course write down their multiplication tables easily:

1	2	3	4	5	6	7	8
2	3	4	1	6	7	8	5
3	4	1	2	7	8	5	6
4	1	2	3	8	5	6	7
5	6	7	8	1	2	3	4
6	7	8	5	4	1	2	3
7	8	5	6	3	4	1	2
8	5	6	7	2	3	4	1

1	2	3	4	5	6	7	8
2	3	4	1	8	5	6	7
3	4	1	2	7	8	5	6
4	1	2	3	6	7	8	5
5	6	7	8	1	2	3	4
6	7	8	5	2	3	4	1
7	8	5	6	3	4	1	2
8	5	6	7	4	1	2	3

Remark 5.5. Drápal showed in [4] that proximity of group multiplication tables implies proximity of algebraic properties. More precisely, he defined the *distance* $d(*, \circ)$ of two groups $(G, *)$, $(G, \circ)$ with the same underlying set G as the cardinality of $\{(g, h) \in G \times G;\ g * h \neq g \circ h\}$, and showed that if $d(*, \circ) < |G|^2/9$ then $(G, *)$ is isomorphic to $(G, \circ)$. If $(G, *)$ (and thus $(G, \circ)$) is a 2-group, the isomorphism follows already from $d(*, \circ) < |G|^2/4$ (cf. [5]). Drápal and the author conjecture in [6] that the same is true for Moufang 2-loops, i.e., $(G, *) \cong (G, \circ)$ if $(G, *)$, $(G, \circ)$ are Moufang 2-loops satisfying $d(*, \circ) < |G|^2/4$.

Note that the distance of any of the two nonassociative left Bol loops in Remark 5.4 from the canonical multiplication table of the direct product $C_4 \times C_2$ is 8, i.e., only $1/8 \cdot 8^2$. Hence the conjecture cannot be generalized from Moufang to Bol loops.

6. The nonabelian case

In this section we suppose that G is a finite nonabelian group. This assumption has some consequences on the validity of identities. For instance, while $xy^{-1} = y^{-1}x$ holds in the abelian case, it is *always* false in this section, no matter what G is.

We stick to the same strategy as in the abelian case.

Lemma 6.1. *Assume that G is a nonabelian group, and that Assumptions* 1–4 *are satisfied. Then $G(\alpha, \beta, \gamma, \delta)$ is a left alternative loop if and only if one of the following conditions is satisfied:*

(i) $(\gamma, \delta) \in \{(\theta_{xy}, \theta_{x^{-1}y}), (\theta_{yx}, \theta_{yx^{-1}})\}$,

(ii) $(\beta, \gamma, \delta) = (\theta_{xy}, \theta_{xy}, \theta_{xy})$,

(iii) $G^2 \subseteq Z(G)$ *and* $(\beta, \gamma, \delta) \in \{(\theta_{yx}, \theta_{xy}, \theta_{xy}), (\theta_{xy}, \theta_{yx}, \theta_{yx}), (\theta_{yx}, \theta_{yx}, \theta_{yx})\}$,

(iv) $G^4 = 1$ *and* $(\beta, \gamma, \delta) = (\theta_{x^{-1}y}, \theta_{xy}, \theta_{xy})$,

(v) $G^2 \subseteq Z(G)$, $G^4 = 1$ *and* $(\beta, \gamma, \delta) \in \{(\theta_{yx^{-1}}, \theta_{xy}, \theta_{xy}), (\theta_{yx^{-1}}, \theta_{yx}, \theta_{yx}), (\theta_{x^{-1}y}, \theta_{yx}, \theta_{yx})\}$.

Proof: We must check for which admissible values of α, β, γ and δ the identities (5)–(8) hold.

The identities (5), (6) always hold.

Denote by $I(\delta)$ the identity (7), i.e., $\delta(x,\gamma(x,y)) = \delta(x,x)y$. First observe that if δ inverts its second argument then $I(\delta)$ does not have a solution; otherwise γ would have to invert its second argument, too, and that is not allowed by Assumption 4. Now, $I(\theta_{xy})$ is $x\gamma(x,y) = xxy$, which has a unique solution $\gamma = \theta_{xy}$; $I(\theta_{x^{-1}y})$ is $x^{-1}\gamma(x,y) = y$ with solution $\gamma = \theta_{xy}$; $I(\theta_{yx})$ is $\gamma(x,y)x = x^2y$ with solution $\gamma = \theta_{yx}$, but only if $G^2 \subseteq Z(G)$; and $I(\theta_{yx^{-1}})$ is $\gamma(x,y)x^{-1} = y$ with solution $\gamma = \theta_{yx}$.

We now consider the pairs $(\gamma,\delta) = (\theta_{xy},\theta_{xy})$, $(\theta_{xy},\theta_{x^{-1}y})$, $(\theta_{yx},\theta_{yx})$, $(\theta_{yx},\theta_{yx^{-1}})$, and test for which values of β they satisfy (8). It turns out that every such triple (β,γ,δ) satisfies (8), but additional assumptions on G are sometimes needed. Here is the calculation:

Assume $(\gamma,\delta) = (\theta_{xy},\theta_{xy})$, and denote by $J(\beta)$ the identity (8), i.e., $xxy = \beta(xx,y)$. Then $J(\theta_{xy})$ holds, $J(\theta_{yx})$ holds if $G^2 \subseteq Z(G)$, $J(\theta_{yx^{-1}})$ is $x^2y = yx^{-2}$, which holds if and only if $G^4 = 1$ and $G^2 \subseteq Z(G)$, and $J(\theta_{x^{-1}y})$ holds when $G^4 = 1$.

Assume $(\gamma,\delta) = (\theta_{xy},\theta_{x^{-1}y})$ and denote by $J(\beta)$ the identity (8), i.e., $y = \beta(1,y)$. Since β never inverts its second argument, $J(\beta)$ always holds.

Assume $(\gamma,\delta) = (\theta_{yx},\theta_{yx})$, $G^2 \subseteq Z(G)$, and denote by $J(\beta)$ the identity (8), i.e., $yx^2 = \beta(x^2,y)$. Then $J(\theta_{xy})$ holds, $J(\theta_{yx})$ holds, $J(\theta_{yx^{-1}})$ holds if $G^4 = 1$, and $J(\theta_{x^{-1}y})$ holds if $G^4 = 1$.

Assume $(\gamma,\delta) = (\theta_{yx},\theta_{yx^{-1}})$ and denote by $J(\beta)$ the identity (8), i.e., $y = \beta(1,y)$. Using our usual trick with the second argument, we see that $J(\beta)$ always holds. □

Theorem 6.2. *Suppose that G is a nonabelian group and that Assumptions 1–4 are satisfied. Then $(M,*) = G(\alpha,\beta,\gamma,\delta)$ is a group if and only if $(M,*)$ is equal to*

$$G(\theta_{xy},\theta_{xy},\theta_{xy},\theta_{xy}). \tag{22}$$

The loop $(M,*)$ is a nonassociative Moufang loop if and only if it is equal to

$$G(\theta_{xy},\theta_{x^{-1}y},\theta_{yx},\theta_{yx^{-1}}), \tag{23}$$

and then it is isomorphic to the Chein loop $M(G,2)$.

The loop $(M,*)$ is a left Bol loop that is not Moufang if and only if: either $G^2 \subseteq Z(G)$ and $(M,*)$ is among

$$G(\theta_{xy},\theta_{x^{-1}y},\theta_{xy},\theta_{x^{-1}y}), \tag{24}$$

$$G(\theta_{xy},\theta_{xy},\theta_{yx},\theta_{yx}); \tag{25}$$

or $G^2 \subseteq Z(G)$, $G^4 = 1$ *and* $(M, *)$ *is among*

$$G(\theta_{xy}, \theta_{xy}, \theta_{xy}, \theta_{x^{-1}y}), \tag{26}$$

$$G(\theta_{xy}, \theta_{xy}, \theta_{yx}, \theta_{yx^{-1}}), \tag{27}$$

$$G(\theta_{xy}, \theta_{x^{-1}y}, \theta_{xy}, \theta_{xy}), \tag{28}$$

$$G(\theta_{xy}, \theta_{x^{-1}y}, \theta_{yx}, \theta_{yx}). \tag{29}$$

Proof: According to Lemma 6.1, there are 16 left alternative loops in the nonabelian case. We now use the left Bol identity (16) to eliminate 8 of them.

When $\gamma = \theta_{xy}$, (16) becomes $\beta(x, yxz) = \beta(x, yx)z$, and it is not satisfied when $\beta = \theta_{yx}$, $\beta = \theta_{yx^{-1}}$. When $\gamma = \theta_{yx}$, (16) becomes $\beta(x, xzy) = z\beta(x, xy)$, and it is not satisfied when $\beta = \theta_{yx}$, $\beta = \theta_{yx^{-1}}$.

We claim that the remaining 8 loops $G(\alpha, \beta, \gamma, \delta)$ are all left Bol. It is clear for (22). The loop (23) is isomorphic to $M(G, 2)$ via Lemma 3.2. Since G is nonabelian, (23) is nonassociative [2].

A straightforward verification of identities (14)–(21) shows that the loops (24)–(29) are left Bol, however, additional assumptions on G are needed. We will indicate when the assumptions are needed, and leave the verification of (14)–(21) to the reader.

First of all, $G^2 \subseteq Z(G)$ is needed for (25) already in Lemma 6.1, and $G^2 \subseteq Z(G)$, $G^4 = 1$ is needed for (29) by the same lemma. The condition $G^2 \subseteq Z(G)$ is required for (24) in (18). The conditions $G^2 \subseteq Z(G)$, $G^4 = 1$ are required for (26) and (27) in (18), and for (28) in (19).

The loops (24)–(29) are not Moufang, by Theorem 1.1. □

Remark 6.3. There are nonabelian groups satisfying $G^2 \subseteq Z(G)$ and $G^4 = 1$. For instance the 8-element dihedral group D_8 and the 8-element quaternion group Q_8 have this property. So do the groups $(D_8)^a \times (Q_8)^b \times (C_4)^c \times (C_2)^d$ with $a+b > 0$.

7. Questions

Our methods certainly do not yield all Bol loops with a subgroup of index 2. This is apparent already from Remark 5.4. Is there some way of determining all such Bol loops?

Is it true that the 6 constructions (24)–(29) of Theorem 6.2 produce 6 pairwise nonisomorphic loops when G is fixed? Note that $\overline{x} \in \overline{G}$ is of order 2 if and only if $\delta(x, x) = 1$. Hence there are $|G_2| + |G|$ elements of exponent 2 in (24), (26), (27), and $2|G_2| < |G| + |G_2|$ elements of exponent 2 in the remaining three loops. The following lemma allows us to distinguish more loops:

Lemma 7.1. *Let* (25), (28), (29) *be as in Theorem* 6.2. *Then* (28) $\not\cong$ (25) $\not\cong$ (29).

Proof: Let $(M, *) = G(\alpha, \beta, \gamma, \delta)$. Let us count the cardinality of $Z(M) = \{x \in M;\ x * y = y * x \text{ for every } y \in M\}$ for the three loops in question.

Assume $x \in G$. When $y \in G$, we have $x * y = y * x$ if and only if $x \in Z(G)$. When $\overline{y} \in \overline{G}$, we have $x * \overline{y} = \overline{y} * x$ if and only if $\beta(x, y) = \gamma(y, x)$. This is always true for (25).

Assume $\overline{x} \in \overline{G}$. When $\overline{y} \in \overline{G}$, we have $\overline{x} * \overline{y} = \overline{y} * \overline{x}$ if and only if $\delta(x, y) = \delta(y, x)$. This is true (in all three cases) if and only if $x \in Z(G)$. When $x \in Z(G)$ and $y \in G$, we have $\overline{x} * y = y * \overline{x}$ if and only if $\gamma(x, y) = \beta(y, x)$. This reduced to $yx = yx$ (always true) for (25), to $xy = y^{-1}x$ (always false, because $xy = xy^{-1}$ holds for every y if and only if $G_2 = G$) for (28), and to $yx = y^{-1}x$ (always false) for (29).

Altogether, $|Z(M)| = 2|Z(G)|$ for (25), while $|Z(M)| \leq |Z(G)|$ for the other two loops. □

Even if the 6 loops are pairwise nonisomorphic for a given G, it is possible that $G(\alpha, \beta, \gamma, \delta) \cong H(\alpha', \beta', \gamma', \delta')$ holds for some nonisomorphic groups G, H to which Theorem 6.2 is applied. Can you determine these “exceptional” isomorphisms? Are they exceptional or common?

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