Non-autonomous implicit integral equations with discontinuous right-hand side

Giovanni Anello, Paolo Cubiotti

Abstract. We deal with the implicit integral equation

$$h(u(t)) = f(\,t\,, \int_I g(t,z)\,u(z)\,dz) \;\text{ for a.a. }\; t \in I,$$

where $I := [0,1]$ and where $f : I \times [0,\lambda] \to \mathbb{R}$, $g : I \times I \to [0,+\infty[$ and $h :]\,0,+\infty\,[\to \mathbb{R}$. We prove an existence theorem for solutions $u \in L^s(I)$ where the contituity of f with respect to the second variable is not assumed.

Keywords: implicit integral equations, discontinuity, lower semicontinuous multifunctions, operator inclusions, selections

Classification: 45P05, 47G10

1. Introduction

Let $I := [0,1]$ and $J := [0,\lambda]$, with $\lambda > 0$. Let us first consider the implicit integral equation

$$h(u(t)) = f\Big(\int_I g(t,z)\,u(z)\,dz\Big) \quad \text{for a.a.} \quad t \in I, \tag{1}$$

where $f : J \to \mathbb{R}$, $g : I \times I \to [0,+\infty\,[$ and $h :]0,+\infty\,[\to \mathbb{R}$. Recently, in [4], an existence theorem for solutions $u \in L^\infty(I)$ of equation (1) has been proved, where, unlike other recent results in the field, the continuity of the function f is not assumed. More precisely, f is assumed to be a.e. equal to a function $f^* : J \to \mathbb{R}$ such that the set

$$\{x \in J \,:\, f^* \;\text{ is discontinuous at }\; x\}$$

has null Lebesgue measure. It is immediate to check that such a function f can be discontinuous at each point of the set J.

For the special case where h is the identity mapping, the latter result has been later extended to the non-autonomous version of problem (1), that is to the equation

$$u(t) = f\big(\,t\,, \int_I g(t,z)\,u(z)\,dz\big) \quad \text{for a.a.} \quad t \in I, \tag{2}$$

where $f : I \times J \to \mathbb{R}$ (see Theorem 1 of [6]). For this latter problem, the above assumption (which specifies what kind of discontinuity is allowed for f) has the following form: there exists a function $f^* : I \times J \to \mathbb{R}$ and a set $E \subseteq J$, with null Lebesgue measure, such that $f(\,\cdot\,, x)$ is measurable for each x in a countable dense subset of J and, for a.a. $t \in I$, one has

$$\text{(3)}\quad \{\, x \in J : f^*(t, \cdot\,) \text{ is discontinuous at } x \,\} \cup \{\, x \in J : f^*(t,x) \neq f(t,x) \,\} \subseteq E.$$

It was also proved that none of the two sets on the left hand side of (3) can depend on t.

At this point, it is natural to consider the implicit non-autonomous integral equation

$$h(u(t)) = f\Big(t\,, \int_I g(t,z)\, u(z)\, dz\Big) \quad \text{for a.a.} \quad t \in I, \tag{4}$$

(which contains equations (1) and (2) as special cases), and to ask whether it is possible to extend to this latter problem the existence results of [4] and [6]. Our effort in this paper goes exactly in such a direction. Indeed, our aim is to prove the following result (where m denotes the Lebesgue measure on the real line and "int" stands for "interior").

Theorem 1. *Let $I := [0,1]$ and $J := [0, \lambda]$, with $\lambda > 0$. Let $s \in\,]1, +\infty]$, $A \subseteq\,]0, +\infty[$ an interval, $h : A \to \mathbb{R}$ a continuous functions. Let $f : I \times J \to \mathbb{R}$, $g : I \times I \to [0, +\infty[$, $\beta \in L^s(I)$, $\phi_0 \in L^j(I)$, with $j \geq s'$ and $j > 1$, $\phi_1 \in L^{s'}(I)$, and let P be a countable dense subset of J. Assume that:*

- (i) *there exist a function $f^* : I \times J \to \mathbb{R}$ and two sets $E_1, E_2 \subseteq J$, with E_2 closed and $m(E_1 \cup E_2) = 0$, such that for each $x \in P$ the function $f^*(\,\cdot\,, x)$ is measurable and for a.a. $t \in I$ one has*

$$\big\{x \in J \;:\; f^*(t,x) \neq f(t,x)\big\} \subseteq E_1 \tag{5}$$

and

$$\big\{x \in J \;:\; f^*(t, \cdot\,) \text{ is discontinuous at } x\big\} \subseteq E_2\,; \tag{6}$$

- (ii) $\operatorname{int} h^{-1}(z) = \emptyset$ *for all* $z \in \operatorname{int} h(A)$;
- (iii) *if one puts*

$$v(t) := \operatorname{ess\,inf}_{x \in J} f(t,x), \qquad z(t) := \operatorname{ess\,sup}_{x \in J} f(t,x),$$

then for a.a. $t \in I$ one has

$$[v(t), z(t)] \subseteq h(A) \quad \text{and} \quad \sup\, h^{-1}([v(t), z(t)]) \;\leq \beta(t); \tag{7}$$

(iv) *one has*

$$0 < \|\phi_0\|_{L^{s'}(I)} \le \frac{\lambda}{\|\beta\|_{L^s(I)}} \ ;$$

(v) *for each* $t \in I$*, the function* $g(t, \cdot)$ *is measurable;*

(vi) *for a.a.* $z \in I$*, the function* $g(\cdot\,, z)$ *is continuous in* I*, differentiable in* $]0, 1[$ *and*

$$g(t, z) \le \phi_0(z), \quad 0 < \frac{\partial g}{\partial t}(t, z) \le \phi_1(z) \quad \textit{for all} \quad t \in]0, 1[\,.$$

Then there exists a solution $\hat{u} \in L^s(I)$ *to equation* (4).

Theorem 1 partially extends the main results of [4] and [6] to problem (4). Such an extension is not full since it is assumed, in addition, that the set E_2 is closed. The reader can easily check that such a function f can be discontinuous (with respect to the second variable) at each point $x \in J$. In particular, our assumption is weaker than the usual Carathéodory condition assumed in the literature (in this connection, the reader can see for instance [3], [7], [8], [10] and the references therein; in particular, we refer to [10] and to the references therein for motivations for studying equation (4)). The proof of Theorem 1 will be given in Section 3, while in Section 2 we shall fix some notations and give some preliminary technical results.

2. Notations and preliminary results

As before, m denotes the usual Lebesgue measure over the real line $\mathbb{R}$. Moreover, we denote by $\mathcal{L}(A)$ (resp., $\mathcal{B}(A)$) the family of all Lebesgue (resp., Borel) measurable subsets of the set A. In the sequel, the word "measurable" will stand for "Lebesgue measurable". Also, we denote by $\overline{A}$ and $\overline{\mathrm{co}}\, A$ the closure and the closed convex hull of the set A, respectively.

If $p \in [1, +\infty]$, we denote by p' the conjugate exponent of p. As usual, we denote by $L^p(I)$ the space of all (equivalence classes of) measurable functions $u : I \to \mathbb{R}$ such that

$$\begin{aligned} &\int_I |u(t)|^p \, dt < +\infty &&\text{if} \quad p < +\infty, \\ &\operatorname{ess\,sup}_{t\in I} |u(t)| < +\infty &&\text{if} \quad p = +\infty, \end{aligned}$$

with the usual norm

$$\begin{aligned} \|u\|_{L^p(I)} &:= \left(\int_I |u(t)|^p \, dt \right)^{\frac{1}{p}} &&\text{if} \quad p < +\infty, \\ \|u\|_{L^\infty(I)} &:= \operatorname{ess\,sup}_{t\in I} |u(t)| &&\text{if} \quad p = +\infty. \end{aligned}$$

Moreover, we denote by $C^0(I)$ the space of all continuous functions $v : I \to \mathbb{R}$.

From now on, we denote by X the space $\{0,1\}^{\mathbb{N}}$ endowed with the product topology, and we put

$$D := \Big\{ \{a_n\} \in X : \ a_n = 0 \ \text{ for infinitely many } n \Big\} \cup \Big\{ \{1_n\} \Big\}$$

($\{1_n\}$ denoting the sequence which has each term equal to 1),

$$C := \Big\{ \{a_n\} \in X : \quad \{a_{2n}\} \in D \quad \text{and} \quad \{a_{2n-1}\} \in D \Big\},$$
$$H := \Big\{ s \in [0,1] : \quad s = \frac{p}{2^m}, \quad \text{with} \quad p, m \in \mathbb{N} \quad \text{and} \quad p \le 2^m \Big\} \cup \{ 0 \},$$
$$\Omega := (I \setminus H) \times (J \setminus \lambda H).$$

Finally, let $\varphi : X \to I \times J$ be the function defined by putting, for each $\{a_n\} \in X$,

$$\varphi(\{a_n\}) = \Big(\sum_{n=1}^{\infty} \frac{a_{2n}}{2^n} \, , \ \lambda \sum_{n=1}^{\infty} \frac{a_{2n-1}}{2^n} \Big).$$

The following lemma follows easily by well-known facts and can be checked directly by the reader.

Lemma 2. *The function φ is continuous in X and its restriction $\varphi|_C : C \to I \times J$ is a bijection. Moreover, the function $(\varphi|_C)^{-1} : I \times J \to C$ is continuous at each point $(t,x) \in \Omega$.*

For the definitions and the basic facts about multifunctions, we refer the reader to [2], [14]. Here we only recall that if Y and S are nonempty sets and $F : Y \to 2^S$ is a multifunction, then a function $f : Y \to S$ is called a *selection* of F if $f(x) \in F(x)$ for all $x \in Y$. The following result comes directly from the proof of Lemma 2 of [19] (for the definition and the basic properties of 0-dimensional spaces, the reader is referred to [9]).

Lemma 3. *Let Y and S be two metric spaces, and assume that Y is 0-dimensional. Let $G : Y \to 2^S$ be a multifunction with nonempty and complete values, and let $M \subseteq Y$ a given set. If G is lower semicontinuous at each point of $Y \setminus M$, then there exists a selection $s : Y \to S$ of G which is continuous at each point of $Y \setminus M$.*

Lemma 4. *Let S be a metric space, let $V \subseteq I \times J$ and $B \subseteq I \times J$ be two given sets (with $B \neq \emptyset$), and $F : B \to 2^S$ be a multifunction with nonempty and complete values. Assume that F is lower semicontinuous at each point of $B \setminus V$.*

Then there exists a selection $g : B \to S$ of F which is continuous at each point of the set $(B \cap \Omega) \setminus V$.

Proof: Let us put for simplicity $\varphi_C := \varphi|_C$, and let $Y := \varphi_C^{-1}(B)$. Then the space Y is 0-dimensional. Let $G : Y \to 2^S$ be the multifunction defined by putting, for each $\{a_n\} \in Y$,

$$G(\{a_n\}) = F(\varphi(\{a_n\})).$$

Since φ is continuous in X, G is lower semicontinuous at each point of $Y \setminus \varphi^{-1}(V)$. By Lemma 3, there exists a selection $s : Y \to S$ of G which is continuous at each point of $Y \setminus \varphi^{-1}(V)$. For each $(t,x) \in B$, let us put

$$g(t,x) := s\,(\varphi_C^{-1}(t,x)).$$

At this point, it is immediate to check that g satisfies the conclusion. $\square$

The following lemma follows at once from the proof of Lemma 2.3 of [1].

Lemma 5. *Let Y and S be metric spaces, with S separable, $F : Y \to 2^S$ a multifunction with nonempty values, $\{u_n\}$ a dense sequence in S, and $y_0 \in Y$. Let d denotes the distance in S. Then one has:*

(a) *if F is lower semicontinuous at y_0, then for each $u \in S$ the function $y \in Y \to d(u, F(y))$ is upper semicontinuous at y_0;*
(b) *if for each $n \in \mathbb{N}$ the function $y \in Y \to d(u_n, F(y))$ is upper semicontinuous at y_0, then F is lower semicontinuous at y_0.*

Lemma 6. *Let $T \in \mathcal{L}(I)$, let $f : T \times J \to \mathbb{R}$ be a function and $E \subseteq J$ a given set. Assume that:*

(i) *f is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable;*
(ii) *for each $t \in T$ one has*

$$\Big\{x \in J : \ f(t,\cdot\,) \text{ is not lower semicontinuous at } x\Big\} \subseteq E\,;$$

(iii) $\inf_{T\times J} f > -\infty$.

Then, for each $\varepsilon > 0$ there exists $K \in \mathcal{L}(T)$ such that $m(T \setminus K) \leq \varepsilon$ and the function $f|_{K\times J}$ is lower semicontinuous at each point $(t,x) \in K \times (J \setminus E)$.

Proof: Without loss of generality we can assume that $f(t,x) \geq 0$ for all $(t,x) \in T \times J$. For each $n \in \mathbb{N}$, let $f_n : T \times J \to [0, +\infty[$ be the function defined by putting, for each $(t,x) \in T \times J$,

$$f_n(t,x) := \inf_{y\in J} \Big[n\,|x-y| + f(t,y)\Big].$$

Of course, for each $n \in \mathbb{N}$ and each $(t,x) \in T \times J$ one has $f_n(t,x) \le f(t,x)$. Consequently, the function $f^* : T \times J \to [0,+\infty[$ defined by

$$f^*(t,x) := \sup_{n\in\mathbb{N}} f_n(t,x)$$

satisfies the inequality

$$f^*(t,x) \le f(t,x) \quad \text{for all} \quad (t,x) \in T \times J. \tag{8}$$

Now, let us observe the following facts.

(a) For each $n \in \mathbb{N}$ and each $x \in J$, the function $f_n(\,\cdot\,,x)$ is measurable. This follows from Lemma III.39 of [5], since the function

$$(t,y) \to n\,|x-y| + f(t,y)$$

is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable.

(b) For each $n \in \mathbb{N}$ and each $t \in T$, the function $f_n(t,\cdot\,)$ is n-Lipschitzian over J. Indeed, for each $x, z \in J$ one has

$$\begin{aligned} f_n(t,x) &\le \inf_{y\in J} \Big[n\,|x-z| + n\,|z-y| + f(t,y) \Big] \\ &= n\,|x-z| + f_n(t,z), \end{aligned}$$

hence the claim follows easily.

(c) One has

$$f^*(t,x) = f(t,x) \quad \text{for all} \quad (t,x) \in T \times (J \setminus E). \tag{9}$$

To see this, choose any $(t,x) \in T \times (J \setminus E)$ and $\eta > 0$. Since the function $f(t,\cdot\,)$ is lower semicontinuous at x, there exists $\delta > 0$ such that for each $y \in J$ with $|x-y| < \delta$ one has

$$f(t,y) > \beta := f(t,x) - \eta.$$

Fix $n^* > \beta/\delta$. Then, for each $y \in J$ one has

$$\begin{cases} n^*\,|x-y| + f(t,y) \ge f(t,y) > \beta & \text{if } \ |x-y| < \delta \\ n^*\,|x-y| + f(t,y) \ge n^*\,\delta + f(t,y) > \beta + f(t,y) \ge \beta & \text{if } \ |x-y| \ge \delta. \end{cases}$$

It follows that $f_{n^*}(t,x) \ge \beta$, hence the claim follows.

Now, choose any $\varepsilon > 0$. By Theorem 2 of [15], for each $n \in \mathbb{N}$ there exists a set $K_n \in \mathcal{L}(T)$ such that

$$m(T \setminus K_n) \le \frac{\varepsilon}{2^n}$$

and the function $f_n|_{K_n \times J}$ is continuous. If we put $K := \bigcap_{n\in\mathbb{N}} K_n$, then $K \in \mathcal{L}(T)$, $m(T \setminus K) \le \varepsilon$ and the function $f^*|_{K\times J}$ is lower semicontinuous. Fix any point $(t^*, x^*) \in K \times (J \setminus E)$, and let us show that the function $f|_{K\times J}$ is lower semicontinuous at (t^*, x^*). To this aim, let $\gamma > 0$. By the lower semicontinuity of $f^*|_{K\times J}$, there exists a neighborhood U of (t^*, x^*) in $K \times J$ such that

$$f^*(t^*, x^*) - \gamma < f^*(t, x) \quad \text{for all} \quad (t, x) \in U.$$

By (8) and (9), it follows that

$$f(t, x) \ge f^*(t, x) > f^*(t^*, x^*) - \gamma = f(t^*, x^*) - \gamma \quad \text{for all} \quad (t, x) \in U,$$

as desired. □

Lemma 7. *Let $T \in \mathcal{L}(I)$, let S be a separable metric space, $F : T \times J \to 2^S$ a multifunction with nonempty values and $E \subseteq J$ a given set. Assume that:*

(i) *F is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable;*
(ii) *for each $t \in T$ one has*

$$\big\{x \in J : \; F(t, \cdot\,) \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Then, for each $\varepsilon > 0$ there exists a set $K \in \mathcal{L}(T)$ such that $m(T \setminus K) \le \varepsilon$ and the multifunction $F|_{K\times J}$ is lower semicontinuous at each point $(t, x) \in K \times (J \setminus E)$.

PROOF: Let ρ be an equivalent distance over S such that $\rho \le 1$, and let $\{y_n\}$ be a dense sequence in S. By Proposition 13.2.2 of [14], for each $y \in S$ the function $\rho(y, F(\cdot,\cdot))$ is $\mathcal{L}(T) \otimes \mathcal{B}(J)$-measurable. Moreover, by Lemma 5, for each $t \in T$ and each $y \in S$ one has that

$$\big\{x \in J : \; \rho(y, F(t, \cdot\,)) \;\text{ is not upper semicontinuous at } x\big\} \subseteq E.$$

Fix $\varepsilon > 0$. For each $n \in \mathbb{N}$, applying Lemma 6 to the function $-\rho(y_n, F(\cdot,\cdot))$, we have that there exists $K_n \in \mathcal{L}(T)$ such that

$$m(T \setminus K_n) \le \frac{\varepsilon}{2^n}$$

and the function

$$\rho(y_n, F(\cdot,\cdot))|_{K_n \times J}$$

is upper semicontinuous at each point $(t, x) \in K_n \times (J \setminus E)$. Putting $K := \bigcap_{n\in\mathbb{N}} K_n$, we have that $m(T \setminus K) \le \varepsilon$ and for each $n \in \mathbb{N}$ the function

$$\rho(y_n, F(\cdot,\cdot))|_{K\times J}$$

is upper semicontinuous at each point $(t, x) \in K \times (J \setminus E)$. By Lemma 5 our claim follows. □

Lemma 8. *Let S be a separable metric space, $F : I \times J \to 2^S$ a multifunction with nonempty complete values, $E \subseteq J$ a given set. Assume that:*

(i) *F is $\mathcal{L}(I) \otimes \mathcal{B}(J)$-measurable;*
(ii) *for each $t \in I$ one has*

$$\big\{x \in J : \ F(t,\cdot\,) \ \text{ is not lower semicontinuous at } x\big\} \subseteq E.$$

Then, there exists a selection $\phi : I \times J \to S$ of F such that:

(a) *for a.a. $t \in I$, one has*

$$\big\{\, x \in J : \ \phi(t,\cdot\,) \ \text{ is discontinuous at } x \,\big\} \subseteq E \cup \lambda H;$$

(b) *for each $x \in J \setminus (E \cup \lambda H)$, the function $\phi(\,\cdot\,, x)$ is measurable.*

Proof: By Lemma 7, the interval I can be partitioned into a sequence of measurable sets $\{K_n\}$ and in one negligible set Y such that for each $n \in \mathbb{N}$ the multifunction $F|_{K_n \times J}$ is lower semicontinuous at each point $(t,x) \in K_n \times (J \setminus E)$. By Lemma 4, for each $n \in \mathbb{N}$ there exists a function $g_n : K_n \times J \to S$ such that

$$g_n(t,x) \in F(t,x) \quad \text{for all} \quad (t,x) \in K_n \times J$$

and g_n is continuous at each point $(t,x) \in \big[K_n \times (J \setminus E)\big] \cap \Omega$. For each $t \in Y$, let $h_t : J \to S$ be any selection of the multifunction $F(t,\cdot\,)$. Now, let the function $\phi : I \times J \to S$ be defined by putting, for each $(t,x) \in I \times J$,

$$\phi(t,x) = \begin{cases} g_n(t,x) & \text{if } \ t \in K_n \\ h_t(x) & \text{if } \ t \in Y. \end{cases}$$

Of course, ϕ is a selection of F. To show conclusion (a), choose $t^* \in I \setminus (Y \cup H)$, and let $n \in \mathbb{N}$ be such that $t^* \in K_n$. Since $t^* \notin H$, we have that $g_n : K_n \times J \to S$ is continuous at each point (t^*, x) with $x \in J \setminus (E \cup \lambda H)$. Hence, we have that

$$\big\{x \in J : \ g_n(t^*,\cdot\,) \ \text{ is discontinuous at } x\big\} \subseteq E \cup \lambda H.$$

Since one has $\phi(t^*,\cdot\,) = g_n(t^*,\cdot\,)$, (a) follows. To show (b), fix $\hat{x} \in J \setminus (\,E \cup \lambda H\,)$. Observe that for each $n \in \mathbb{N}$ the function $g_n : K_n \times J \to S$ is continuous at each point $(t, \hat{x})$ such that $t \in K_n \setminus H$. It follows that $g_n(\,\cdot\,, \hat{x}) : K_n \to S$ is continuous at each point $t \in K_n \setminus H$, hence the function $g_n(\,\cdot\,, \hat{x})|_{K_n \setminus H}$, being continuous, is measurable. Since H and Y are negligible, the conclusion follows. □

3. Proof of Theorem 1

Without loss of generality we can assume that (5), (6) and (7) hold for all $t \in I$. Moreover, we can assume $j < +\infty$.

Firstly, let us show that $v(t)$ and $z(t)$ are measurable in I. Indeed, by assumption (i) it is not difficult to check that for each $t \in I$ one has

$$v(t) = \inf_{x \in J \setminus E_2} f^*(t,x), \qquad z(t) = \sup_{x \in J \setminus E_2} f^*(t,x). \tag{10}$$

Again by (i), the set $P \cap (J \setminus E_2)$ is dense in $J \setminus E_2$ and countable. Hence, the function $f^*|_{I \times (J \setminus E_2)}$ is $\mathcal{L}(I) \otimes \mathcal{B}(J \setminus E_2)$-measurable by the Lemma at p. 198 of [15]. By Lemma III.39 of [5] our claim follows.

Let $l : I \to \mathbb{R}$ be any measurable function such that

$$v(t) \le l(t) \le z(t) \quad \text{for all} \quad t \in I, \tag{11}$$

and let $\hat{f} : I \times J \to \mathbb{R}$ be defined by

$$\hat{f}(t,x) = \begin{cases} f^*(t,x) & \text{if } x \notin E_2 \\ l(t) & \text{if } x \in E_2. \end{cases}$$

Since E_2 is closed, (6) implies that for each $t \in I$ one has

$$\{x \in J \,:\, \hat{f}(t,\cdot) \text{ is discontinuous at } x\} \subseteq E_2. \tag{12}$$

Moreover, the function $\hat{f}$ is $\mathcal{L}(I) \otimes \mathcal{B}(J)$-measurable and by (10) and (11), one has

$$v(t) \le \hat{f}(t,x) \le z(t) \quad \text{for all} \quad (t,x) \in I \times J. \tag{13}$$

Now, observe that by (ii) and by Theorem 2.4 of [18] the function h is inductively open. That is, there exists a set $Y \in \mathcal{B}(A)$ such that $h|_Y$ is open and $h(Y) = h(A)$. It follows that the multifunction $T : h(A) \to 2^Y$ defined by

$$T(s) = h^{-1}(s) \cap Y$$

is lower semicontinuous in $h(A)$ with nonempty values. Let $G : I \times J \to 2^Y$ be defined by

$$G(t,x) = T(\hat{f}(t,x)) = h^{-1}(\hat{f}(t,x)) \cap Y$$

(G is well defined by (7) and (13)). Then G is $\mathcal{L}(I) \otimes \mathcal{B}(J)$-measurable and, by (12), for all $t \in I$ one has

$$\{x \in J \,:\, G(t,\cdot) \text{ is not lower semicontinuous at } x\} \subseteq E_2.$$

Consequently, the multifunction

$$(t,x) \in I \times J \to \overline{G(t,x)} \tag{14}$$

is $\mathcal{L}(I) \otimes \mathcal{B}(J)$-measurable and for each $t \in I$ one has

$$\big\{x \in J \,:\, \overline{G(t,\cdot\,)} \text{ is not lower semicontinuous at } x\big\} \subseteq E_2.$$

By Lemma 8, there exists a selection $k : I \times J \to \mathbb{R}$ of the multifunction (14) such that for a.a. $t \in I$ one has

$$\big\{x \in J \,:\, k(t,\cdot\,) \text{ is discontinuous at } x\big\} \subseteq E_2 \cup \lambda H, \tag{15}$$

and for each $x \in J \setminus (E_2 \cup \lambda H)$ the function $k(\,\cdot\,,x)$ is measurable. For each $t \in I$, let us put

$$\alpha(t) := \inf\ h^{-1}([v(t), z(t)]).$$

By the continuity of h and by (7) and (13) we get

$$k(t,x) \in h^{-1}(\hat{f}(t,x)) \quad \text{for all} \quad (t,x) \in I \times J \tag{16}$$

and

$$0 < \alpha(t) \le k(t,x) \le \beta(t) \quad \text{for all} \quad (t,x) \in I \times J.$$

Let $T_1 \subseteq I$ be such that $m(T_1) = 0$ and (15) holds for all $t \in I \setminus T_1$. Let $\psi : I \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\psi(t,x) = \begin{cases} k(t,x) & \text{if } (t,x) \in (I \setminus T_1) \times (J \setminus E_2) \\ \beta(t) & \text{otherwise.} \end{cases}$$

Then, for each $t \in I \setminus T_1$ one has

$$\big\{x \in \mathbb{R} \,:\, \psi(t,\cdot\,) \text{ is discontinuous at } x\big\} \subseteq E_2 \cup \lambda H. \tag{17}$$

Let $P' := \lambda\,((\mathbb{Q} \cap I) \setminus H)$ (where $\mathbb{Q}$ denotes the set of rational real numbers). Then P' is countable and dense in J. If P'' is any countable dense subset of $\mathbb{R} \setminus J$, then the set $P^* := P' \cup P''$ is countable and dense in $\mathbb{R}$, and by the above construction the function $\psi(\cdot, x)$ is measurable for all $x \in P^*$.

Thus, all the assumptions of Proposition 2 of [6] are satisfied. Consequently, the multifunction $F : I \times \mathbb{R} \to \mathbb{R}$ defined by

$$F(t,x) := \bigcap_{m \in \mathbb{N}} \overline{\overline{\text{co}}\Big(\bigcup_{\substack{y \in P'' \\ |y-x| \le \frac{1}{m}}} \{\psi(t,y)\} \Big)}$$

satisfies the conclusion of the same proposition. Moreover, by the above construction it follows that

$$F(t,x) \subseteq [\alpha(t), \beta(t)] \quad \text{for all} \quad (t,x) \in I \times \mathbb{R}. \tag{18}$$

Now we want to apply Theorem 1 of [17], with $T = I$, $X = Y = \mathbb{R}$, $p = s$, $q = j'$, $V = L^s(I)$, $\Psi(u) = u$, $r = \|\beta\|_{L^s(I)}$, $\varphi \equiv +\infty$,

$$\Phi(u)(t) = \int_I g(t,z)\, u(z)\, dz,$$

and $F : I \times \mathbb{R} \to 2^{\mathbb{R}}$ as defined above. To this aim, we argue as in [6] and observe the following facts.

(a) $\Phi(L^s(I)) \subseteq C^0(I)$. This follows from our assumptions (v) and (vi) and the Lebesgue's dominated convergence theorem.

(b) If $v \in L^s(I)$ and $\{v^k\}$ is a sequence in $L^s(I)$, weakly convergent to v in $L^{j'}(I)$, then the sequence $\{\Phi(v^k)\}$ converges to $\Phi(v)$ strongly in $L^1(I)$. This follows by Theorem 2 at p. 359 of [13], since g is j-th power summable in $I \times I$ (note that g is measurable on $I \times I$ by the classical Scorza-Dragoni's theorem; see [20] or also [12]).

(c) By (18), the function

$$\omega : t \in I \to \sup_{x \in \mathbb{R}} d(0, F(t,x))$$

belongs to $L^s(I)$ and $\|\omega\|_{L^s(I)} \leq \|\beta\|_{L^s(I)}$ (for what concerns the measurability of ω, we refer to [17]).

Thus, all the assumptions of Theorem 1 of [17] are satisfied. Consequently there exist $\hat{u} \in L^s(I)$ and a set $T_2 \subseteq I$, with $m(T_2) = 0$, such that

$$\hat{u}(t) \in F(\,t\,, \Phi(\hat{u})(t)) \quad \text{for all} \quad t \in I \setminus T_2. \tag{19}$$

We now want to prove that $\hat{u}(t)$ is a solution of equation (4). To this aim, we argue as in [6]. Firstly, let us observe that by (18) we have

$$\hat{u}(t) \in [\alpha(t), \beta(t)] \quad \text{for all} \quad t \in I \setminus (T_1 \cup T_2). \tag{20}$$

For each $t \in I$, put

$$\gamma(t) := \Phi(\hat{u})(t) = \int_I g(t,z)\, \hat{u}(z)\, dz.$$

By assumptions (iv) and (v), taking into account (20), for each $t \in I$ we get

$$0 \le \gamma(t) \le \|\phi_0\|_{L^{s'}(I)} \cdot \|\hat{u}\|_{L^s(I)} \le \frac{\lambda}{\|\beta\|_{L^s(I)}} \cdot \|\beta\|_{L^s(I)} = \lambda,$$

hence $\gamma(I) \subseteq J$. By assumptions (v) and (vi), by (20) and by Lemma 2.2 at p. 226 of [16], we get

$$\gamma'(t) = \int_I \frac{\partial g}{\partial t}(t,z)\, \hat{u}(z)\, dz > 0 \quad \text{for all} \quad t \in]0,1[\,.$$

In particular, the continuous function γ is strictly increasing in I. Hence, by Theorem 2 of [21] the function γ^{-1} is absolutely continuous. Let us put

$$S := \gamma^{-1}\big[(E_1 \cup E_2 \cup \lambda H) \cap \gamma(I)\big].$$

By assumption (i) and by Theorem 18.25 of [11] we have that $m(S) = 0$. Let

$$S^* := S \cup T_1 \cup T_2.$$

For each $t \in I \setminus S^*$, since $\gamma(t) \in J \setminus (E_1 \cup E_2 \cup \lambda H)$ and taking into account (17), (19) and Proposition 2 of [6], we get

$$\hat{u}(t) \in F(\,t\,, \gamma(t)) = \{\psi(\,t\,, \gamma(t))\} = \{k(\,t\,, \gamma(t))\}.$$

Consequently, taking into account (5) and (16), for each $t \in I \setminus S^*$ we get

$$h(\hat{u}(t)) = \hat{f}(\,t\,, \gamma(t)) = f^*(\,t\,, \gamma(t)) = f(\,t\,, \gamma(t)) = f(\,t\,, \int_I g(t,z)\, \hat{u}(z)\, dz).$$

This ends our proof. □

Remark. The example at p. 245 of [4] shows that in the assumption (vi) of Theorem 1 one cannot assume that

$$0 \le \frac{\partial g}{\partial t}(t,z) \le \phi_1(z).$$

Moreover, the Example at the end of [6] shows that none of the sets E_1, E_2 in the statement of Theorem 1 can depend on t.

References

[1] Artstein Z., Prikry K., *Caratheodory selections and the Scorza Dragoni property*, J. Math. Anal. Appl. **127** (1987), 540–547.
[2] Aubin J.P., Frankowska H., *Set-Valued Analysis*, Birkhäuser, Boston, 1990.
[3] Banas J., Knap Z., *Integrable solutions of a functional-integral equation*, Rev. Mat. Univ. Complut. Madrid **2** (1989), 31–38.
[4] Cammaroto F., Cubiotti P., *Implicit integral equations with discontinuous right-hand side*, Comment. Math. Univ. Carolinae **38** (1997), 241–246.
[5] Castaing C., Valadier M., *Convex Analysis and Measurable Multifunctions*, Springer-Verlag, Berlin, 1977.
[6] Cubiotti P., *Non-autonomous vector integral equations with discontinuous right-hand side*, Comment. Math. Univ. Carolinae **42** (2001), 319–329.
[7] Emmanuele G., *About the existence of integrable solutions of a functional-integral equation*, Rev. Mat. Univ. Complut. Madrid **4** (1991), 65–69.
[8] Emmanuele G., *Integrable solutions of a functional-integral equation*, J. Integral Equations Appl. **4** (1992), 89–94.
[9] Engelking R., *Theory of Dimensions, Finite and Infinite*, Heldermann-Verlag, 1995.
[10] Fečkan M., *Nonnegative solutions of nonlinear integral equations*, Comment. Math. Univ. Carolinae **36** (1995), 615–627.
[11] Hewitt E., Stromberg K., *Real and Abstract Analysis*, Springer-Verlag, Berlin, 1965.
[12] Himmelberg C.J., Van Vleck F.S., *Lipschitzian generalized differential equations*, Rend. Sem. Mat. Univ. Padova **48** (1973), 159–169.
[13] Kantorovich L.V., Akilov G.P., *Functional Analysis in Normed Spaces*, Pergamon Press, Oxford, 1964.
[14] Klein E., Thompson A.C., *Theory of Correspondences*, John Wiley and Sons, New York, 1984.
[15] Kucia A., *Scorza Dragoni type theorems*, Fund. Math. **138** (1991), 197–203.
[16] Lang S., *Real and Functional Analysis*, Springer-Verlag, New York, 1993.
[17] Naselli Ricceri O., Ricceri B., *An existence theorem for inclusions of the type* $\Psi(u)(t) \in F(t, \Phi(u)(t))$ *and application to a multivalued boundary value problem*, Appl. Anal. **38** (1990), 259–270.
[18] Ricceri B., *Sur la semi-continuitè infèrieure de certaines multifonctions*, C.R. Acad. Sci. Paris **294** (1982), 265–267.
[19] Saint Raymond J., *Riemann-measurable selections*, Set Valued Anal. **2** (1994), 481–485.
[20] Scorza Dragoni G., *Un teorema sulle funzioni continue rispetto ad una e misurabili rispetto ad un'altra variabile*, Rend. Sem. Mat. Univ. Padova **17** (1948), 102–106.
[21] Villani A., *On Lusin's condition for the inverse function*, Rend. Circ. Mat. Palermo **33** (1984), 331–335.

Department of Mathematics, University of Messina, Contrada Papardo, Salita Sperone 31, 98166 Messina, Italy

(*Received* July 3, 2003, *revised* December 17, 2003)