Note on countable unions of Corson countably compact spaces

Ondřej F.K. Kalenda

Abstract. We show that a compact space K has a dense set of G_δ points if it can be covered by countably many Corson countably compact spaces. If these Corson countably compact spaces may be chosen to be dense in K, then K is even Corson.

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Introduction

Corson compact and countably compact spaces, i.e. compact and countably compact subsets of the space

$$\Sigma(\Gamma) = \{x \in \mathbb{R}^\Gamma \colon \operatorname{supp} x = \{\gamma \in \Gamma \colon x(\gamma) \neq 0\} \text{ is countable}\}$$

equipped with the pointwise topology play an important role in theory of nonseparable Banach spaces. Functional analytic properties of Corson compacta were studied for example in [AM], [AMN], [O] or [V1]. In the present paper we study certain topological properties of these classes of spaces.

It is well known that any Corson compact space has a dense set of G_δ points (an easy proof is given in [K4, Theorem 3.3]). As Corson compact spaces are stable to continuous images (by [MR] or [G]), the same obviously holds for continuous images of Corson compact spaces. On the other hand, there are Corson countably compact spaces with no G_δ points (take for example $[0,1]^\Gamma \cap \Sigma(\Gamma)$ for Γ uncountable). Corson countably compact spaces are stable to quotient images by [G] but not to general continuous images (an easy example is the space $[0,\omega_1]$ which is a continuous image of the Corson countably compact space $[0,\omega_1)$ but is not itself Corson — for a more general formulation see [K5, Theorem 2.5]). Hence the following problem [K5, Question 1] seems to be quite natural.

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Problem 1. *Let K be a compact space which is a continuous image of a Corson countably compact space. Does K have a dense set of G_δ points?*

We do not know the general answer to this problem but our Theorem 1 gives a partial positive answer for compact spaces which are finite unions of Corson countably compact spaces. The stated question is natural in itself but moreover it is related to problems concerning the structure of open continuous images of Valdivia compacta. Let us recall the definition of Valdivia compacta.

Let K be a compact space and $A \subset K$. We say that A is a Σ-subset of K if there is a homeomorphic injection $h : K \to \mathbb{R}^\Gamma$ for a set Γ with $A = h^{-1}(\Sigma(\Gamma))$. A compact K is Valdivia if it has a dense Σ-subset. Recall also that a dense set $A \subset K$ is a Σ-subset if and only if A is a Corson countably compact space and K is the Čech-Stone compactification of A (i.e., $K = \beta A$), see [K4, Proposition 1.9]. Valdivia compact spaces are not stable to continuous images (see [V2], [K1] or [K4, Section 3.3]). However, the following question remained open until recently.

Problem 2. *Let $\varphi : K \to L$ be an open continuous surjection between compact spaces. Suppose K is Valdivia. Is L Valdivia as well?*

A counterexample was recently found in [KU]. A partial positive answer is given in [K2, Theorem 4.5] (see also [K4, Theorem 3.24]). It is proved there that the answer is positive provided L has a dense set of G_δ points. In fact, it is proved slightly more — if L has a dense set of G_δ points and $A \subset K$ is a dense Σ-subset of K, then $\varphi(A)$ is a dense Σ-subset of L. This more general result does not hold without the assumption on L (an easy example is given in Remark after Theorem 4.5 in [K2], see also [K4, Remark 3.25]). It follows that the positive answer to Problem 1 would give positive answer to the following question.

Problem 3. *Let $\varphi : K \to L$ be an open continuous surjection between compact spaces. Suppose A is a dense Σ-subset of K such that $\varphi(A) = L$. Is L Corson?*

This problem may look a bit artificially in itself but the answer to it may help to better understand Problem 2. (Note that the counterexample of [KU] uses cohomology theory.)

Main results

Our main results are the following two theorems.

Theorem 1. *Let K be a compact space such that $K = \bigcup_{i=1}^\infty B_i$ where B_i is a Corson countably compact space for each $i \in \mathbb{N}$. Then K has a dense set of G_δ points.*

Theorem 2. *Let K be a compact space such that $K = \bigcup_{i=1}^\infty B_i$ where B_i is a Corson countably compact space dense in K for each $i \in \mathbb{N}$. Then K is a Corson compact space.*

Note that Theorem 1 yields a partial positive answer to Problem 1. Compact spaces which are finite unions of Corson countably compact spaces form a subclass of continuous images of Corson countably compact spaces (as finite union is a continuous image of a finite topological sum; cf. [K5, Lemma 2.2]). It follows from [S] that a compact space from Theorem 1 need not be a continuous image of Corson countably compact space. (In fact, the quoted paper contains an example of a compact space which is a countable union of Corson (even Eberlein) compact subspaces without being Corson. It is easy to check that this space is not even a continuous image of a Valdivia compact space.)

The author knows no example of a concrete compact space for which Theorem 1 would yield a non-trivial result. The assumptions are satisfied for example by Corson compact spaces, by the spaces $[0, \alpha]$ for $\alpha < \omega_1 \cdot \omega_1$ or by the space from [S]. In all these examples it is easy to check they have a dense set of G_δ points. However, the real meaning of Theorem 1 is different. Its purpose is to show that there are no non-trivial spaces satisfying its assumptions.

Remark also that Theorem 2 gives a *positive answer to Problem* 3 *for the case $K = L \times C$ where C is a countable compact space and φ being the natural projection of K onto L.* (The answer to Problem 2 in this case is trivial as L is homemomorphic to a clopen subset of K.)

Another consequence of Theorem 2 is the following. If K is a super-Valdivia compact space (i.e. the family of dense Σ-subsets of K covers K), then either K is Corson or K cannot be covered by countably many Corson countably compact subspaces.

Theorems 1 and 2 are easily seen to be equivalent. Indeed, suppose Theorem 1 holds and let K and B_i, $i \in \mathbb{N}$, be as in Theorem 2. Then K has, by Theorem 1, a dense set of G_δ points. Each B_i contains all G_δ points of K (by Lemma 3 below) and so for any pair $i, j \in \mathbb{N}$ the set $B_i \cap B_j$ is dense in K, and hence $B_i = B_j$ (by Lemma 2). Hence $K = B_1 = B_2 = \dots$ and thus K is Corson.

Conversely, suppose that Theorem 2 holds and let K and B_i, $i \in \mathbb{N}$, be as in Theorem 1. Let $U \subset K$ be a nonempty open set. We can construct by induction nonempty open sets V_i, $i \in \mathbb{N}$ such that

$$U \supset \overline{V_1} \supset V_1 \supset \overline{V_2} \supset \dots$$

and that for each $i \in \mathbb{N}$ the set $V_i \cap B_i$ is either empty or dense in V_i. Further, put $H = \bigcap_{i \in \mathbb{N}} V_i$. This is clearly a nonempty closed G_δ subset of K. Moreover, $H \cap B_i$ is dense in H whenever $H \cap B_i \neq \emptyset$ (by Lemma 3). Hence H is Corson by Theorem 2. It follows that H has a G_δ point. Such a point is also a G_δ point of K. This completes the proof.

Auxiliary results

In this section we collect some auxiliary results needed to prove Theorems 1 and 2. Most of them are known but we recall their formulation.

Lemma 1. *Let X be a Corson countably compact space. Then the following holds.*

(a) *X is Fréchet-Urysohn, i.e. whenever $x \in X$ and $A \subset X$ are such that $x \in \overline{A}$ there is a sequence $x_n \in A$ with $x_n \to x$.*
(b) *$\overline{C}$ is compact for any $C \subset X$ countable. In particular, if Y is a space containing X, then $\overline{C} \subset X$ whenever $C \subset X$ is countable, i.e. X is countably closed in Y.*

The point (a) follows from [N, Theorem 2.1], see also [K4, Lemma 1.6]. The point (b) is obvious.

Lemma 2. *Let K be a space, A and B two Corson countably compact subsets of K and M be any subset of K. If $M \cap A \cap B$ is dense in M, then $M \cap A = M \cap B$.*

This is an easy consequence of Lemma 1. It was observed in [K2, Lemma 2.15].

Lemma 3. *Let K be a regular space and $A \subset K$ a dense countably compact subset. Then $G \cap A$ is dense in G for each G_δ set $G \subset K$.*

This well-known result is proved for example in [K4, Lemma 1.11].

Lemma 4. *Let K be a regular space, G a G_δ subset of K and $x \in G$. Then there is a closed G_δ subset $H \subset K$ such that $x \in H \subset G$.*

This lemma is an easy consequence of the definition of regular spaces.

Lemma 5. *Let K be a compact space containing no copy of $[0, \omega_2]$ and $B \subset K$ a dense subset which is a continuous image of a Corson countably compact space. Then for each $x \in K$ there is $C \subset B$ with cardinality at most $\aleph_1$ such that $x \in \overline{C}$.*

Proof: We will use ideas of the proof of [K3, Theorem 1]. Let $f\colon A \to B$ be a continuous surjection where $A \subset \Sigma(\Gamma)$ is a Corson countably compact space. As, due to [K4, Lemma 1.8 and Proposition 1.9], $\overline{A} = \beta A$ (the closure is taken in $\mathbb{R}^\Gamma$), there is a continuous extension $g : \overline{A} \to K$ of f. For any uncountable cardinal number κ put

$$A_\kappa = \{x \in \overline{A}\colon \operatorname{card}\operatorname{supp} x < \kappa\}.$$

Then $A_{\aleph_1} = A$. We will prove that $g(A_{\aleph_2}) = K$.

Suppose on the contrary that $g(A_{\aleph_2}) \subsetneqq K$. Then we can put

$$\tilde{\tau} = \min\{\kappa\colon g(A_{\aleph_2}) \subsetneqq g(A_\kappa)\}.$$

Obviously such a $\tilde{\tau}$ exists, $\tilde{\tau} > \aleph_2$ and $\tilde{\tau}$ is not a limit cardinal. Hence $\tilde{\tau} = \tau^+$ for some cardinal $\tau \geq \aleph_2$. Let $x \in g(A_{\tau^+}) \setminus g(A_{\aleph_2})$ and $y \in A_{\tau^+}$ satisfy $g(y) = x$. Then clearly $y \in A_{\tau^+} \setminus A_\tau$, i.e. $\operatorname{card}\operatorname{supp} y = \tau$. Let $\varphi\colon [0, \tau] \to \overline{A}$ be a continuous injection satisfying conditions

(i) $\operatorname{card}\operatorname{supp} \varphi(\alpha) \leq \max\{\operatorname{card} \alpha, \aleph_0\}$ for all $\alpha \leq \tau$;
(ii) $\varphi(\tau) = y$.

Such a mapping exists due to [K3, Theorem 2].

If τ is singular, then there is an infinite cardinal $\lambda < \tau$ and cardinals $(\theta_\gamma)_{\gamma<\lambda}$ with $\theta_\gamma < \tau$ for $\gamma < \lambda$ and $\tau = \sup_{\gamma<\lambda} \theta_\gamma$. By the definition of τ we have $g(\varphi(\theta_\gamma)) \in g(A_{\aleph_2})$. Hence there are $z_\gamma \in A_{\aleph_2}$, $\gamma < \lambda$ with $g(z_\gamma) = g(\varphi(\theta_\gamma))$. Then clearly

$$\overline{\{z_\gamma \colon \gamma < \lambda\}} \subset A_{\lambda^+}.$$

The image by g of the set on the left-hand side is compact and hence

$$x \in \overline{\{g(\varphi(\theta_\gamma)) \colon \gamma < \lambda\}} \subset g\left(\overline{\{z_\gamma \colon \gamma < \lambda\}}\right) \subset g(A_{\lambda^+}).$$

Moreover, $g(A_{\lambda^+}) = g(A_{\aleph_2})$, hence $x \in g(A_{\aleph_2})$, a contradiction.

Hence τ is a regular cardinal. Put $h = g \circ \varphi$. At first let us note that $x = h(\tau) \notin h([0,\tau))$. Hence $h^{-1}(k)$ is bounded in $[0,\tau)$ for each $k \in h([0,\tau))$. Therefore, by regularity of τ, the set $h^{-1}(h([0,\eta]))$ is bounded in $[0,\tau)$ for each $\eta < \tau$. So we can choose by transfinite induction ordinals $\eta_\alpha < \tau$ for $\alpha < \tau$ such that

(a) $\eta_{\alpha+1} > \sup h^{-1}(h([0,\eta_\alpha]))$;
(b) $\eta_\alpha = \sup_{\beta<\alpha} \eta_\beta$ for $\alpha < \tau$ limit.

Now it is clear that K contains a homeomorphic copy of $[0,\tau]$ and thus also that of $[0,\omega_2]$. But this contradicts our assumptions.

Therefore $g(A_{\aleph_2}) = K$. Choose $x \in K$ arbitrarily and find $y \in A_{\aleph_2}$ with $g(y) = x$. If $\operatorname{supp} y$ is countable then $y \in A$, thus $x \in B$ and we can take $C = \{x\}$. If $\operatorname{card}\operatorname{supp} y = \aleph_1$ then there is (by [K1, Proposition 2.7] or also by [K3, Theorem 2]) a continuous injection $\phi : [0,\omega_1] \to \overline{A}$ with $\phi([0,\omega_1)) \subset A$ and $\phi(\omega_1) = y$. Therefore we can take $C = g(\phi([0,\omega_1]))$. $\square$

Proof of Theorem 2

Let K be a compact space such that $K = \bigcup_{i=1}^{\infty} B_i$ where B_i is, for each $i \in \mathbb{N}$, a dense subset of K which is a Corson countably compact space. If all the B_i's are equal, K is Corson by definition. Hence suppose that at least two of them are different, say $B_1 \neq B_2$. The proof will be done is several steps.

Step 1. *K contains no copy of* $[0,\omega_2]$.

Suppose, on the contrary, that there is a homeomorphic copy L of $[0,\omega_2]$ such that $L \subset K$. As $K = \bigcup_{i=1}^{\infty} B_i$, there is some i with the cardinality of $B_i \cap L$ being $\aleph_2$. But $B_i \cap L$ is homeomorphic to a Corson countably compact subspace of $[0,\omega_2]$ and hence, by [K5, Theorem 2.5], it has cardinality at most $\aleph_1$, a contradiction. (Recall the basic idea of the proof of the quoted theorem: Suppose that $M \subset [0,\omega_2]$ is a Corson countably compact subset of cardinality $\aleph_2$. Then it is easy to show that M contains a closed subset homeomorphic to $M' = \{\alpha < \omega_2;\ \alpha$ is either isolated or of countable cofinality$\}$. Hence M' is Corson. But this can be easily led to a contradiction.)

Step 2. *We can suppose without loss of generality that K has weight $\aleph_1$.*

Put $M_1 = \{x\}$, where $x \in B_2 \setminus B_1$. As K contains no copy of $[0, \omega_2]$, we can, due to Lemma 5, construct by induction a sequence of sets $M_k \subset K$ such that each M_k has cardinality at most $\aleph_1$ and $M_k \subset \overline{B_i \cap M_{k+1}}$ for all $i, k \in \mathbb{N}$. Put $H = \overline{\bigcup_{k=1}^{\infty} M_k}$. Then $H \cap B_i$ is dense in H for all $i \in \mathbb{N}$ and $H \cap B_1 \neq H \cap B_2$. Moreover, H has weight $\aleph_1$. Indeed, $H \cap B_1$ is a Corson countably compact space of density at most $\aleph_1$. Hence it is homeomorphic to some $C \subset \Sigma([0, \omega_1)) \cap [0,1]^{[0,\omega_1)}$. Then clearly $\overline{C}$ has weight at most $\aleph_1$. Further $\overline{C} = \beta C$ by [K4, Proposition 1.9]. Therefore H is a continuous image of $\overline{C}$ and thus the weight of H is at most $\aleph_1$. On the other hand, the weight of H cannot be countable, otherwise H would be metrizable and hence $H \cap B_1 = H \cap B_2 = H$.

Step 3. *We can suppose without loss of generality that for any pair $i, j \in \mathbb{N}$ either $B_i = B_j$ or $B_i \cap B_j = \emptyset$.*

Let $\eta = (\eta_1, \eta_2) \colon \mathbb{N} \to \mathbb{N}^2$ be a bijection. Due to Lemma 2 we can construct by induction nonempty open sets V_k, $k \in \mathbb{N}$ such that

- $\overline{V_{k+1}} \subset V_k$ for all $k \in \mathbb{N}$;
- $B_1 \cap B_2 \cap V_1 = \emptyset$;
- $B_{\eta_1(k)} \cap V_k = B_{\eta_2(k)} \cap V_k$ or $B_{\eta_1(k)} \cap B_{\eta_2(k)} \cap V_k = \emptyset$ for all $k \in \mathbb{N}$.

As $B_1 \neq B_2$, it follows from Lemma 2 that $B_1 \cap B_2$ is not dense in K. Hence we can choose V_1 satisfying the appropriate condition. We continue by the obvious induction using Lemma 2.

Put $H = \bigcap_{k \in \mathbb{N}} V_k$. Then H is clearly a nonempty closed G_δ subset of K. Then $B_i \cap H$ is dense in H for every $i \in \mathbb{N}$ (by Lemma 3). Moreover, H is the union of these Corson countably compact spaces, any two of them are either identical or disjoint and at least two of them are different.

As $H \subset K$, the weight of H is at most $\aleph_1$. As $B_1 \cap B_2 \cap H = \emptyset$, H cannot be metrizable and hence the weight of H is equal to $\aleph_1$.

Step 4. *Assume that K has weight $\aleph_1$ and $B_1, B_2, \ldots$ are pairwise disjoint dense Corson countably compact spaces. Then $K \setminus \bigcup_{i=1}^{\infty} B_i \neq \emptyset$.*

For $i \in \mathbb{N}$ let $A_i \subset \Sigma([0, \omega_1))$ be homeomorphic to B_i and let L_i denote the closure of A_i in $\mathbb{R}^\Gamma$. Then L_i is compact and $L_i = \beta A_i$ (see [K4, Lemma 1.8 and Proposition 1.9]). So there is a continuous surjection $g_i : L_i \to K$ such that $g_i \restriction A_i$ is a homeomorphism of A_i onto B_i. Then it clearly holds $g_i(A_i) = B_i$ and $g_i(L_i \setminus A_i) = K \setminus B_i$.

We put $F_0 = K$. Further we will construct points $x_\gamma^i \in A_i$, nonempty closed G_δ sets $G_\gamma^i \subset L_i$, $H_\gamma^i \subset K$ and $F_\gamma \subset K$ for $1 \leq \gamma < \omega_1$ and $i \in \mathbb{N}$ and points

$y^i_\gamma \in L_i$ for $1 \le \gamma < \omega_1$ isolated and $i \in \mathbb{N}$ in the following way.

(i) $y^1_{\gamma+1} \in g_1^{-1}(F_\gamma) \setminus A_1$ for all $\gamma < \omega_1$.

(ii) $G^1_{\gamma+1} = \{x \in g_1^{-1}(F_\gamma) \colon x \restriction [0,\gamma] = y^1_{\gamma+1} \restriction [0,\gamma]$ $\& \ \forall \delta \in (0,\gamma] \colon x \restriction \operatorname{supp} x^1_\delta = y^1_{\gamma+1} \restriction \operatorname{supp} x^1_\delta\}$ for all $\gamma < \omega_1$.

(iii) $x^i_{\gamma+1} \in G^i_{\gamma+1} \cap A_i \setminus \{x^i_\delta \colon 0 < \delta \le \gamma\}$ for all $\gamma < \omega_1$ and $i \in \mathbb{N}$.

(iv) $g_i(x^i_{\gamma+1}) \in H^i_{\gamma+1} \subset K \setminus g_i(L_i \setminus G^i_{\gamma+1})$ for all $\gamma < \omega_1$ and $i \in \mathbb{N}$.

(v) $y^{i+1}_{\gamma+1} \in g_{i+1}^{-1}(g_i(x^i_{\gamma+1}))$ for all $\gamma < \omega_1$ and $i \in \mathbb{N}$.

(vi) $G^{i+1}_{\gamma+1} = \{x \in g_{i+1}^{-1}(H^i_{\gamma+1}) \colon x \restriction [0,\gamma] = y^{i+1}_{\gamma+1} \restriction [0,\gamma] \ \& \ \forall \delta \in (0,\gamma] :$ $x \restriction \operatorname{supp} x^{i+1}_\delta = y^{i+1}_{\gamma+1} \restriction \operatorname{supp} x^{i+1}_\delta\}$ for all $\gamma < \omega_1$ and $i \in \mathbb{N}$.

(vii) $F_{\gamma+1} = \bigcap_{i=1}^\infty H^i_{\gamma+1}$ for all $\gamma < \omega_1$.

(viii) $x^i_\lambda = \lim_{\gamma<\lambda} x^i_\gamma$ for all $\lambda < \omega_1$ limit and $i \in \mathbb{N}$.

(ix) $G^i_\lambda = \bigcap_{\gamma<\lambda} G^i_\gamma$ for all $\lambda < \omega_1$ limit and $i \in \mathbb{N}$.

(x) $F_\lambda = H^i_\lambda = \bigcap_{\gamma<\lambda} H^i_\gamma = \bigcap_{\gamma<\lambda} F_\gamma$ for all $\lambda < \omega_1$ limit and $i \in \mathbb{N}$.

Let us show that the construction may be done. We can surely put $F_0 = K$. This is a nonempty closed G_δ subset of K.

Suppose that $\gamma < \omega_1$ and we have already constructed x^i_δ, G^i_δ, H^i_δ and F_δ for $\delta \in (0,\gamma]$ and $i \in \mathbb{N}$; y^i_δ for $\delta \in (0,\gamma]$ isolated and $i \in \mathbb{N}$.

Choose $y^1_{\gamma+1}$ as in (i). This is possible, as F_γ is a nonempty G_δ set and hence (by Lemma 3) we have $F_\gamma \cap B_2 \ne \emptyset$, so $F_\gamma \setminus B_1 \ne \emptyset$ and thus $g_1^{-1}(F_\gamma) \setminus A_1 \ne \emptyset$.

Define $G^1_{\gamma+1}$ as in (ii). Then $G^1_{\gamma+1}$ is closed and G_δ in L_1 as it is of the form $\{x \in g_1^{-1}(F_\gamma) \colon x \restriction C = y^1_{\gamma+1} \restriction C\}$ for a countable set C and $g_1^{-1}(F_\gamma)$ is closed and G_δ.

Further suppose that $k \in \mathbb{N}$ and we have constructed $x^i_{\gamma+1}$ and $H^i_{\gamma+1}$ for $1 \le i < k$ and $y^i_{\gamma+1}$ and $G^i_{\gamma+1}$ for $1 \le i \le k$.

By Lemma 3 the set $G^k_{\gamma+1} \cap A_k$ is dense in $G^k_{\gamma+1}$. If this intersection were countable, we would have $G^k_{\gamma+1} \subset A_k$ (by Lemma 1(b)). But $y^k_{\gamma+1} \in G^k_{\gamma+1} \setminus A_k$ (this follows from (i) in case $k = 1$ and from (v) in case $k > 1$). Therefore $G^k_{\gamma+1} \cap A_k$ is uncountable and we can choose $x^k_{\gamma+1}$ as in (iii).

As $G^k_{\gamma+1}$ is G_δ and L_k compact, the set $K \setminus g_k(L_k \setminus G^k_{\gamma+1})$ is a G_δ subset of K. Moreover, this set contains $g(x^k_{\gamma+1})$ as $x^k_{\gamma+1} \in G^k_{\gamma+1} \cap A_k$. Hence, due to Lemma 4, we can choose a closed G_δ set $H^k_{\gamma+1}$ satisfying (iv). Further, choose $y^{k+1}_{\gamma+1}$ as in (v) and define $G^{k+1}_{\gamma+1}$ by (vi). Again $G^{k+1}_{\gamma+1}$ is a closed G_δ subset of L_{k+1} containing $y^{k+1}_{\gamma+1}$.

We have already constructed $x^k_{\gamma+1}$, $y^k_{\gamma+1}$, $G^k_{\gamma+1}$ and $H^k_{\gamma+1}$ for all $k \in \mathbb{N}$.

Remark that we have

$$(*)\qquad F_\gamma \supset H^1_{\gamma+1} \supset H^2_{\gamma+1} \supset \dots .$$

Next define $F_{\gamma+1}$ according to (vii). This is clearly a nonempty closed G_δ subset of K.

Next suppose that $\lambda < \omega_1$ is a limit ordinal and we have constructed x^i_γ, G^i_γ, H^i_γ and F_γ for all $\gamma \in (0,\lambda)$ and $i \in n$ and y^i_γ for $\gamma \in (0,\lambda)$ isolated and $i \in \mathbb{N}$.

Fix $i \in \mathbb{N}$. Let us show that the net $\{x^i_\gamma \colon \gamma < \lambda\}$ converges. Let $\delta < \omega_1$ be arbitrary. If $\delta \notin \bigcup_{0<\gamma<\lambda} \operatorname{supp} x^i_\gamma$, then $x^i_\gamma(\delta) = 0$ for $\gamma \in (0,\lambda)$ and hence the net $x^i_\gamma(\delta)$ converges to 0. Next suppose $\delta \in \operatorname{supp} x^i_{\gamma_0}$ for some $\gamma_0 \in (0,\lambda)$. Then for each $\gamma \in (\gamma_0,\lambda)$ we have $x^i_\gamma \in G^i_\gamma \subset G^i_{\gamma_0+1}$, and hence $x^i_\gamma(\delta) = y^k_{\gamma_0+1}(\delta)$. Thus the net $x^i_\gamma(\delta)$, $\gamma \in (\gamma_0,\lambda)$ is constant and so convergent. This completes the proof that (viii) can be fulfilled. Moreover, $x^i_\lambda \in A_i$ as A_i is countably closed in L_i (Lemma 1(b)). Further, define G^i_λ as in (ix). It is obviously a closed G_δ set. Further, as $x^i_\gamma \in G^i_\gamma$ for all $\gamma \in (0,\lambda)$ and the family $(G^i_\gamma \colon \gamma \in (0,\lambda))$ is a decreasing family of closed sets, we get $x^i_\lambda \in G^i_\lambda$.

Finally define H^i_λ and F_λ as in (x). The definition is correct due to (*). Moreover, we have

$$g_i^{-1}(F_\lambda) = \bigcap_{\gamma<\lambda} g_i^{-1}(H^i_\gamma) \subset \bigcap_{\gamma<\lambda} G^i_\gamma = G^i_\lambda$$

for all $i \in \mathbb{N}$.

This completes the construction.

Fix $i \in \mathbb{N}$. In the same way as we proved the existence of x^i_λ for λ limit, we can prove that the net x^i_γ, $\gamma < \omega_1$ converges to some $x^i \in L_i$. This time we have $x_i \notin A_i$. Indeed, the mapping $\phi : [1,\omega_1] \to L_i$ defined by $\phi(\alpha) = x^i_\alpha$ for $\alpha \in [1,\omega_1)$ and $\phi(\omega_1) = x^i$ is continuous and $\phi \restriction [1,\omega_1)$ is one-to-one due to condition (iii). Thus, there is $\alpha < \omega_1$ such that ϕ is one-to-one on $[\alpha,\omega_1]$. Then $\phi([\alpha,\omega_1])$ is homeomorphic to $[0,\omega_1]$ and therefore it cannot be contained in A_i. As $\phi([\alpha,\omega_1)) \subset A_i$, we get $x^i = \phi(\omega_1) \notin A_i$.

Let us show that the net y^i_γ, $\gamma \in (0,\omega_1)$ isolated, converges also to x^i. Fix arbitrary $\delta < \omega_1$. For any $\gamma \geq \delta$ we have $x^i_{\gamma+1} \in G^i_{\gamma+1}$ and hence $x^i_{\gamma+1}(\delta) = y^i_{\gamma+1}(\delta)$. Therefore $x^i(\delta) = \lim_\gamma x^i_{\gamma+1}(\delta) = \lim_\gamma y^i_{\gamma+1}(\delta)$.

Further, we claim that $g_1(x^1) = g_2(x^2) = \dots$. Indeed, let $i \in \mathbb{N}$ be arbitrary. Then

$$g_{i+1}(x^{i+1}) = \lim_\gamma g_{i+1}(y^{i+1}_{\gamma+1}) = \lim_\gamma g_i(x^i_\gamma) = g_i(x^i).$$

We conclude by noting that $g_i(x^i) \notin B_i$ (as $x^i \notin A_i$) for all $i \in \mathbb{N}$.

Step 5. *Assume that K has weight $\aleph_1$, $n > 1$ and $B_1, \dots, B_n$ are pairwise disjoint dense Corson countably compact spaces. Then $K \setminus (B_1 \cup \dots \cup B_n) \neq \emptyset$.*

We can perform the same construction as in Step 4 with the obvious changes — we replace the set $\mathbb{N}$ by $\{1, \dots, n\}$ (in (iii), (iv), (vii)–(x)) or by $\{1, \dots, n-1\}$ (in (v) and (vi)). In this way we obtain the same result.

The proof is now completed. By Step 2 and Step 3 we can suppose that K has weight $\aleph_1$, $B_1 \neq B_2$ and that for any $i, j \in \mathbb{N}$ we have $B_i \cap B_j = \emptyset$ or $B_i = B_j$. Step 5 shows that K cannot be covered by finitely many of B_i's. Therefore we can suppose that all the B_i's are pairwise disjoint. Step 4 then yields a contradiction.

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Department of Mathematical Analysis, Faculty of Mathematics and Physics, Charles University, Sokolovská 83, 186 75 Praha 8, Czech Republic

E-mail: kalenda@karlin.mff.cuni.cz

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