A maximum principle for linear elliptic systems with discontinuous coefficients

S. Leonardi

To the memory of Jindřich Nečas

Abstract. We prove a maximum principle for linear second order elliptic systems in divergence form with discontinuous coefficients under a suitable condition on the dispersion of the eigenvalues of the coefficients matrix.

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1. Introduction

In $\mathbb{R}^n$ $(n \geq 2)$, with generic point $x = (x_1, x_2, \dots, x_n)$, we shall denote by Ω a bounded open nonempty set with diameter d_Ω.

If $u : \Omega \to \mathbb{R}^N$ $(N \geq 2)$, we set

$$D_i \equiv \frac{\partial}{\partial x_i}, \quad Du = (D_i u_s)\,.$$

Moreover, by

$$H^1(\Omega, \mathbb{R}^N),\ H^1_o(\Omega, \mathbb{R}^N),\ L^{2,\lambda}(\Omega, \mathbb{R}^N),\ \mathcal{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$$

we will denote respectively the usual Sobolev, Morrey and Campanato spaces.

We will prove the following results:

Theorem 1.1. *Let Ω be of class C^2 and let $u \in H^1(\Omega, \mathbb{R}^N)$ be the weak solution of the Dirichlet problem*

$$\begin{aligned} &u - u_o \in H^1_o(\Omega, \mathbb{R}^N), \\ &D_i(A_{ij}(x) D_j u) = 0 \ \text{ in } \Omega. \end{aligned} \tag{1}$$ [1]

[1] Einstein's convention will be used throughout the paper.

Suppose that the following structural conditions hold:

(2)
$$A_{ij}(x) = \left\{A_{ij}^{rs}(x)\right\} \in L^\infty(\Omega, \mathbb{R}^{N^2}),$$
$$A_{ij}^{rs}(x) = A_{ji}^{sr}(x) \quad \textit{for a.a.} \quad x \in \Omega,$$

and there exist two positive constants Λ_1 *and* Λ_2 *such that*

(3)
$$\Lambda_2\,|\xi|^2 \geq A_{ij}(x)\xi_i\xi_j \geq \Lambda_1\,|\xi|^2 \quad \textit{for a.a.} \quad x \in \Omega,\ \forall\,\xi = (\xi_i) \in \mathbb{R}^{nN}.$$

Moreover, set

(4)
$$\gamma = (n-1)\left[1 - \left(\frac{\Lambda_2 - \Lambda_1}{\Lambda_2 + \Lambda_1}\right)^2\right]$$

and assume that

(5)
$$u_o \in \left\{u \in H^1 \cap L^\infty(\Omega, \mathbb{R}^N) : Du \in L^{2,\lambda}(\Omega, \mathbb{R}^{nN})\right\}$$

with $\lambda \in [0, \gamma[$.

Then

$$u \in \mathcal{L}^{2,\lambda+2}(\Omega, \mathbb{R}^N), \quad Du \in L^{2,\lambda}(\Omega, \mathbb{R}^N)$$

and

(6)
$$[u]_{\mathcal{L}^{2,\lambda+2}(\Omega)} + \|Du\|_{L^{2,\lambda}(\Omega)} \leq c(n, \Lambda_1, \Lambda_2, \partial\Omega)\|Du_o\|_{L^{2,\lambda}(\Omega)}.$$

In particular, if

(7)
$$\frac{\Lambda_1}{\Lambda_2} > \frac{\sqrt{n-1} - 1}{\sqrt{n-1} + 1}$$

and

(8)
$$\lambda \in\ ]n-2, \gamma[\ ,$$

then $u \in C^{0,\mu}(\overline{\Omega}, \mathbb{R}^N)$, *with* $\mu = 1 - \frac{n-\lambda}{2}$, *and the inequality*

(9)
$$[u]_{C^{0,\mu}(\overline{\Omega})} \leq c(n, \Lambda_1, \Lambda_2, \partial\Omega)\|Du_o\|_{L^{2,\lambda}(\Omega)}$$

holds.

Theorem 1.2 (Maximum principle)**.** *Let Ω be of class C^2 and convex. Let $u \in H^1(\Omega, \mathbb{R}^N)$ be the weak solution of the Dirichlet problem* (1)*. Suppose that assumptions* (2), (3), (7) *hold true and that*

$$u_o \in \left\{ u \in H^1 \cap L^\infty(\Omega, \mathbb{R}^N) : Du \in L^{2,n-2}(\Omega, \mathbb{R}^{nN}) \right\}, \qquad (10)$$
$$\|Du_o\|_{L^{2,n-2}(\Omega)} \leq c_1 \|u_o\|_{L^\infty(\Omega)}.$$

Then $u \in L^\infty(\Omega, \mathbb{R}^N)$ and

$$\|u\|_{L^\infty(\Omega)} \leq c_2(c_1, n, \Lambda_1, \Lambda_2, \partial\Omega) \|u_o\|_{L^\infty(\Omega)}.$$

The value of $\frac{\Lambda_1}{\Lambda_2}$ expresses in some way the measure of the dispersion of the eigenvalues of the matrix $A = (A_{ij})$.

It is worth recalling that, due to De Giorgi's counterexample [7] (see also [17] and [15]), a system like $(1)_2$, satisfying only the structural hypotheses (2) and (3), can have in general both non-bounded and discontinuous weak solutions when $n \geq 3$. On the other hand, if $n = 2$ any weak solution of such a system is Hölder continuous (see [18] and also [3, p. 85] or [8, p. 143]).

That is, in case of $n \geq 3$ we cannot expect any maximum principle for strongly elliptic linear systems with discontinuous coefficients unless we do not add further restrictions like e.g. (7).

It has to be also pointed out that, as far as the author is aware, even for $n = 2$ the above stated maximum principle seems to be new.

Instead, as far as it concerns the linear elliptic systems with constant (or continuous) coefficients under Legendre-Hadamard ellipticity condition, it is proved in [5] that the assertion of Theorems 1.1 and 1.2 can be verified without any additional assumption on the dispersion of the eigenvalues of the matrix A.

The same kind of maximum principle holds, under the restriction $2 \leq n \leq 4$, for nonlinear elliptic systems (see [4]) of the type:

$$D_i a_i(Du) = 0$$

with structural hypotheses

$$a_i(p) \in C^1(\mathbb{R}^{nN}, \mathbb{R}^N),$$
$$a_i(0) = 0,$$
$$\left(\sum_{i,j=1}^{n} \sum_{h,k=1}^{N} \left| \frac{\partial a_i^h(p)}{\partial p_j^k} \right|^2 \right)^{1/2} \leq M, \quad \forall\, p = (p_i) \in \mathbb{R}^{nN},$$
$$\frac{\partial a_i^h(p)}{\partial p_j^k} \xi_i^h \xi_j^k \geq \nu |\xi|^2, \quad \forall\, p = (p_i), \xi = (\xi_i) \in \mathbb{R}^{nN},$$

where M and ν are suitable positive constants.

As far as it concerns the relationship between Hölder continuity of the solution and the dispersion of the eigenvalues of the coefficients matrix of system (1) the reader can refer e.g. to [21], [22], [11], [12], [13], [10] and [16].

2. Further notations and function spaces

For $\rho > 0$ and $x_o \in \mathbb{R}^n$ we define

$$B(x_o, \rho) = \{x \in \mathbb{R}^n : |x - x_o| < \rho\},$$
$$\Omega(x_o, \rho) = \Omega \cap B(x_o, \rho).$$

If $y_o = (y_{o1}, \ldots, y_{on-1}, 0)$ we define

$$B^+(y_o, \rho) = \{x \in B(y_o, \rho) : x_n > 0\},$$
$$\Gamma(y_o, \rho) = \{x \in B(y_o, \rho) : x_n = 0\}.$$

If $v \in L^1(S)$, S being a bounded open nonempty set of $\mathbb{R}^n$, then we will set

$$v_S = \frac{1}{\mathrm{meas}(S)} \int_S v(x)\, dx$$

where meas(S) is the n-dimensional Lebesgue measure of S.

Definition 2.1. Let $0 < \mu \leq 1$. By $C^{0,\mu}(\overline{\Omega}, \mathbb{R}^N)$ we denote the linear space of vector-functions $u : \overline{\Omega} \to \mathbb{R}^N$ such that

$$\|u\|_{C^{0,\mu}(\overline{\Omega})} = \sup_{\overline{\Omega}} |u| + [u]_{C^{0,\mu}(\overline{\Omega})} < +\infty \tag{11}$$

where

$$[u]_{C^{0,\mu}(\overline{\Omega})} = \sup_{x,y \in \overline{\Omega}} \frac{|u(x) - u(y)|}{|x - y|^{\mu}}\,.$$

$C^{0,\mu}(\overline{\Omega}, \mathbb{R}^N)$ equipped with the norm (11) is a Banach space.

Definition 2.2. By $H^1(\Omega, \mathbb{R}^N)$ [resp. $H^1_o(\Omega, \mathbb{R}^N)$] we denote the closure of $C^\infty(\Omega, \mathbb{R}^N)$ [resp. $C^\infty_o(\Omega, \mathbb{R}^N)$] with respect to the norm

$$\|u\|_{H^1(\Omega)} = \|u\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}.$$

Definition 2.3 (Morrey's space)**.** Let $0 \le \lambda \le n$. By $L^{2,\lambda}(\Omega, \mathbb{R}^N)$ we denote the linear space formed by the vector-functions $u \in L^2(\Omega, \mathbb{R}^N)$ for which

$$\|u\|_{L^{2,\lambda}(\Omega)} = \sup_{x_o \in \Omega,\, 0<\rho\le d_\Omega} \left\{ \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u(x)|^2 dx \right\}^{1/2} < +\infty.$$

$L^{2,\lambda}(\Omega, \mathbb{R}^N)$ equipped with the above norm is a Banach space.

Definition 2.4. Let $0 \le \lambda \le n$. By $H^{1,(\lambda)}(\Omega, \mathbb{R}^N)$ we denote the linear space of vector-functions $u \in H^1(\Omega, \mathbb{R}^N)$ such that $Du \in L^{2,\lambda}(\Omega, \mathbb{R}^{nN})$.

$H^{1,(\lambda)}(\Omega, \mathbb{R}^N)$ equipped with the norm

$$\|u\|_{H^{1,(\lambda)}(\Omega)} = \|u\|_{L^2(\Omega)} + \|Du\|_{L^{2,\lambda}(\Omega)}$$

is a Banach space.

Definition 2.5 (Campanato's space)**.** Let $0 \le \lambda \le n+2$. By $\mathcal{L}^{2,\lambda}(\Omega, \mathbb{R}^N)$ we denote the linear space of vector-functions $u \in L^2(\Omega, \mathbb{R}^N)$ such that

$$[u]_{\mathcal{L}^{2,\lambda}(\Omega)} = \left\{ \sup_{x_o \in \Omega, 0<\rho\le d_\Omega} \rho^{-\lambda} \int_{\Omega(x_0,\rho)} |u - u_{\Omega(x_o,\rho)}|^2 dx \right\}^{1/2} < +\infty.$$

3. Auxiliary results

Let us consider the linear system, in the unknown $u(x) = (u_s(x))$, $s = 1, 2, \dots, N$,

$$D_i(A_{ij}(x) D_j u) = 0 \ \text{ in } \Omega \tag{12}$$

with the structural conditions (2) and (3) [(2)].

Definition 3.1. A vector-function $u \in H^1(\Omega, \mathbb{R}^N)$ is a weak solution of the system (12) if

$$\int_\Omega A_{ij}(x) D_j u D_i \varphi \, dx = 0, \quad \forall \varphi \in H^1_o(\Omega, \mathbb{R}^N).$$

Analogously, let us take into account the system

(2) Assumptions (2) and (3) will always be implicitly used although not stated.

$$
(13)\qquad \begin{aligned} D_i(B_{ij}(x)D_ju) = 0, &\quad \text{in } \ B^+(y_o,R), \\ u = 0 &\quad \text{on } \ \Gamma(y_o,R); \end{aligned}
$$

under the structural assumptions

$$
(14)\qquad \begin{aligned} B_{ij}(x) = \left\{B_{ij}^{rs}\right\} \in L^\infty(B^+(y_o,R),\mathbb{R}^{N^2}), \\ B_{ij}^{rs}(x) = B_{ji}^{sr}(x) \ \text{ a.a. } x \in B^+(y_o,R), \end{aligned}
$$

there exist two positive constants Λ_1' and Λ_2' such that

$$
(15)\qquad \begin{gathered} \Lambda_2' \, |\xi|^2 \geq B_{ij}(x)\xi_i\xi_j \geq \Lambda_1' \, |\xi|^2 \\ \text{for a.a. } x \in B^+(y_o,R), \ \forall\, \xi = (\xi_i) \in \mathbb{R}^{nN}. \end{gathered}
$$

Assumptions (14) and (15) will always be implicitly used although not stated.

Definition 3.2. A vector-function $u \in H^1(B^+(y_o,R),\mathbb{R}^N)$ is a weak solution of the system (13) if

$$
\begin{cases} \displaystyle\int_{B^+(y_o,R)} B_{ij}(x)D_juD_i\varphi\,dx = 0, & \forall\,\varphi \in H_o^1(B^+(y_o,R),\mathbb{R}^N) \\ u = 0 & \text{on } \ \Gamma(y_o,R). \end{cases}
$$

Proposition 3.1. *Let $v \in H^1(B(x_o,R),\mathbb{R}^N)$ be a solution of the system*

$$
\Delta v = 0 \ \text{ in } \ B(x_o,R).
$$

Then, if $\int_{\partial B(x_0,R)} |D_T v|^2\,d\sigma < +\infty$,

$$
(16)\qquad \int_{B(x_o,R)} |Dv|^2\,dx \leq \frac{R}{n-1}\int_{\partial B(x_o,R)} |D_T v|^2\,d\sigma,
$$

where $D_T v$ denotes the tangential gradient of v on $\partial B(x_o,R)$.

Proof: The proof can be carried out by means of expansion in spherical harmonics and exploiting the properties of the Laplace-Beltrami operator (see e.g. [21, p. 31] or [22, p. 162] or [9, p. 391] or [13, p. 19]). □

Theorem 3.1. *Let $u \in H^1(\Omega, \mathbb{R}^N)$ be a solution of the system* (12). *Then, for every ball $B(x_o, \rho) \subset \Omega$ and $\forall\, t \in \,]0,1[\,$, we have*

$$\|Du\|^2_{L^2(B(x_o,t\rho))} \le t^\gamma \|Du\|^2_{L^2(B(x_o,\rho))} \text{ (3)}. \tag{17}$$

Proof: We will follow the idea of Kottas [12], [13] (see also [22, p. 163]) that is, we will prove that the function

$$f(r) = r^{-\gamma} \int_{B(x_o,r)} |Du|^2\, dx$$

is nondecreasing in $]0,\rho]$.

Let us fix $r \in \,]0,\rho\,[$ and let us set $w = u - v$ where $v \in H^1(B(x_o,r), \mathbb{R}^N)$ is the solution of the problem

$$\begin{aligned} &v - u \in H^1_0(B(x_o,r), \mathbb{R}^N) \\ &\Delta v = 0 \ \text{ in } \ B(x_o,r). \end{aligned}$$

It turns out that

$$\int_{B(x_o,r)} A_{ij} D_i u\, D_j w\, dx = 0$$

and so we get

$$\int_{B(x_o,r)} D_i u\, D_i w\, dx = \int_{B(x_o,r)} (\delta_{ij} - \Lambda\, A_{ij}) D_i u\, D_j w\, dx \tag{18}$$

where we have chosen

$$\Lambda = \frac{2}{\Lambda_1 + \Lambda_2}\,.$$

On the other hand, since $w \in H^1_0(B(x_o,r), \mathbb{R}^N)$ we obtain

$$\int_{B(x_o,r)} D_i u\, D_i w\, dx = \int_{B(x_o,r)} |Dw|^2\, dx. \tag{19}$$

From (18), (19) and Hölder inequality we deduce

$$\int_{B(x_o,r)} |Dw|^2\, dx \le \int_{B(x_o,r)} |(\delta_{ij} - \Lambda\, A_{ij}) D_i u|^2\, dx\,. \tag{20}$$

(3) The same γ defined in (4).

On the other hand, by the results in [12] we achieve

$$\int_{B(x_o,r)} |Dw|^2\,dx = \int_{B(x_o,r)} |Du|^2\,dx - \int_{B(x_o,r)} |Dv|^2\,dx. \tag{21}$$

Inequalities (20), (21), Lemma 8.I from [3, p. 87] and Cauchy-Schwartz inequality finally yield

$$\int_{B(x_o,r)} |Du|^2\,dx - \int_{B(x_o,r)} |Dv|^2\,dx \le (1-\Lambda\,\Lambda_1)^2 \int_{B(x_o,r)} |Du|^2\,dx$$

whence, exploiting inequality (16), we have

$$\begin{aligned}\int_{B(x_o,r)} |Du|^2\,dx &\le \frac{r}{(n-1)\left[1-(1-\Lambda\,\Lambda_1)^2\right]} \int_{\partial B(x_o,r)} |Du|^2\,d\sigma \\ &= \frac{r}{\gamma} \int_{\partial B(x_o,r)} |Du|^2\,d\sigma.\end{aligned} \tag{22}$$

Calculating

$$\begin{aligned}f'(r) = &-\gamma r^{-\gamma-1} \int_{B(x_o,r)} |Du|^2\,dx \\ &+ r^{-\gamma} \int_{\partial B(x_o,r)} |Du|^2\,d\sigma,\end{aligned}$$

we obtain from (22) that $f'(r) \ge 0$ for all $r \in \,]0,\rho\,[\,$. □

Proposition 3.2. *Let $v \in H^1(B^+(0,R),\mathbb{R}^N)$ be a solution of the system*

$$\begin{aligned}\Delta v &= 0 \quad \text{in } \ B^+(0,R), \\ v &= 0 \quad \text{on } \ \Gamma(0,R).\end{aligned}$$

Then, if $\int_{\partial B^+(0,R)} |D_T v|^2\,d\sigma < +\infty$,

$$\int_{B^+(0,R)} |Dv|^2\,dx \le \frac{R}{n-1} \int_{\partial B^+(0,R)} |D_T v|^2\,d\sigma,$$

where $D_T v$ denotes the tangential gradient of v on $\partial B^+(0,R)$.

PROOF: By odd reflection (see [9, p. 238]) v may be extended to a harmonic function on $B(0,R)$. Let

$$V(x_1,x_2,\dots,x_n) = \begin{cases} v(x_1,x_2,\dots,x_n) & \text{if } \ x_n \ge 0 \\ -v(x_1,x_2,\dots,-x_n) & \text{if } \ x_n < 0 \end{cases}$$

the harmonic extension of v.

The conclusion follows from (16) and the identities

$$\int_{B(0,R)} |DV|^2\,dx = 2\int_{B^+(0,R)} |Dv|^2\,dx,$$
$$\int_{\partial B(0,R)} |D_T V|^2\,d\sigma = 2\int_{\partial B^+(0,R)} |D_T v|^2\,d\sigma.$$

□

Theorem 3.2. *Let $u \in H^1(B^+(0,R),\mathbb{R}^N)$ be a solution of the system* (13). *Then, for every $\rho \in\,]0,R]$ and $\forall\, t \in\,]0,1[$, we have*

$$\|Du\|^2_{L^2(B^+(0,t\rho))} \le t^{\gamma'}\|Du\|^2_{L^2(B^+(0,\rho))} \tag{23}$$

where

$$\gamma' = (n-1)\left[1-\left(\frac{\Lambda_2'-\Lambda_1'}{\Lambda_2'+\Lambda_1'}\right)^2\right]. \tag{24}$$

Proof: It is enough to proceed as in the proof of Theorem 3.1 and of Lemmas 3.3 and 3.4 from [12] substituting $B(x_o,R)$ with $B^+(0,R)$ and exploiting Proposition 3.2. □

4. Interior and boundary estimates

We now give the interior and boundary estimates for a vector-function u, which is a weak solution of some auxiliary problems.

Theorem 4.1. *Let $u \in H^1(\Omega,\mathbb{R}^N)$ be a solution of the system*

$$D_i\left[A_{ij}(x)D_j(u+u_o)\right] = 0 \ \text{ in } \ \Omega$$

where $u_o \in H^{1,(\lambda)}(\Omega,\mathbb{R}^N)$ with $0\le\lambda\le\gamma$ (4).

Then, for every open set $\Omega' \subset\subset \Omega$, we have

$$Du \in L^{2,\lambda}(\Omega',\mathbb{R}^{nN})$$

and the inequality

$$\|Du\|_{L^{2,\lambda}(\Omega')} \le c\left[\|Du\|_{L^2(\Omega)} + \|Du_o\|_{L^{2,\lambda}(\Omega)}\right] \tag{25}$$

(4) The same γ defined in (4).

holds, where the constant c depends also on $d = \mathrm{dist}(\overline{\Omega}', \partial\Omega)$.

PROOF: Fix $B(x_o, \rho) \subset \Omega$ with $x_o \in \Omega'$ and $\rho \le d$.

In $B(x_o, \rho)$ from (17) we get

$$\|D(u+u_o)\|^2_{L^2(B(x_o,t\rho))} \le t^\gamma \|D(u+u_o)\|^2_{L^2(B(x_o,\rho))}, \quad \forall\, t \in \,]0,1[\, . \tag{26}$$

Thus, from (26) we deduce, $\forall\, t \in \,]0,1\,[$,

$$\begin{aligned} \|Du\|^2_{L^2(B(x_o,t\rho))} &\le 2\,\|D(u+u_o)\|^2_{L^2(B(x_o,t\rho))} + 2\,\|Du_o\|^2_{L^2(B(x_o,t\rho))} \\ &\le 2\,t^\gamma \|D(u+u_o)\|^2_{L^2(B(x_o,\rho))} + 2\,(t\rho)^\lambda \|Du_o\|^2_{L^{2,\lambda}(\Omega)} \\ &\le 4\,t^\gamma \Big[\|Du\|^2_{L^2(\Omega)} + \|Du_o\|^2_{L^2(B(x_o,\rho))}\Big] + 2\,(t\rho)^\lambda \|Du_o\|^2_{L^{2,\lambda}(\Omega)}. \end{aligned} \tag{27}$$

The above inequality implies, $\forall\, t \in \,]0,1\,[$,

$$\|Du\|^2_{L^2(B(x_o,t\rho))} \le c(t\rho)^\lambda \Big[\rho^{-\lambda}\|Du\|^2_{L^2(\Omega)} + \|Du_o\|^2_{L^{2,\lambda}(\Omega)}\Big]$$

and hence (see [3, p. 59])

$$\|Du\|^2_{L^2(B(x_o,\sigma))} \le c\sigma^\lambda \Big[\|Du\|^2_{L^2(\Omega)} + \|Du_o\|^2_{L^{2,\lambda}(\Omega)}\Big], \quad \forall\, \sigma \in \,]0,d\,[\, .$$

The proof can be completed as at p. 59 of [3] (see also [4, p. 303]). □

Theorem 4.2. *Let* $u \in H^1(B^+(0,R_1), \mathbb{R}^N)$ *be a solution of the problem*

$$\begin{aligned} D_i[B_{ij}(x) D_j(u+u_o)] = 0, &\quad \textit{in}\ \ B^+(0,R_1) \\ u = 0 &\quad \textit{on}\ \ \Gamma(0,R_1) \end{aligned}$$

where $u_o \in H^{1,(\lambda)}(B^+(0,R_1), \mathbb{R}^N)$ *with* $0 \le \lambda \le \gamma'$ [(5)].

Then, for every $R < R_1$*, we have*

$$Du \in L^{2,\lambda}(B^+(0,R), \mathbb{R}^{nN})$$

and the inequality

$$\|Du\|_{L^{2,\lambda}(B^+(0,R))} \le c\Big[\|Du\|_{L^2(B^+(0,R_1))} + \|Du_o\|_{L^{2,\lambda}(B^+(0,R_1))}\Big]$$

holds, where the constant c depends also on $R_1 - R$.

PROOF: Fix $R \in \,]0,R_1\,[$ and $y_o \in \Gamma(0,R)$. In any hemisphere $B^+(y_o,\rho)$ with $0 < \rho < R_1 - R$, using (23) and estimates analogous to (27) we obtain, $\forall\, t \in \,]0,1\,[$ and $\forall\, \rho \in \,]0, R_1 - R\,[$,

$$\|Du\|^2_{L^2(B^+(y_o,t\rho))} \le c(t\rho)^\lambda \Big[\rho^{-\lambda}\|Du\|^2_{L^2(B^+(0,R_1))} + \|Du_o\|^2_{L^{2,\lambda}(B^+(0,R_1))}\Big]$$

whence

$$\|Du\|^2_{L^2(B^+(y_o,\sigma))} \le c\sigma^\lambda \Big[\|Du\|^2_{L^2(B^+(0,R_1))} + \|Du_o\|^2_{L^{2,\lambda}(B^+(0,R_1))}\Big], \ \forall\, \sigma \in \,]0,\rho\,[\, .$$

The proof now follows as at p. 312 of [4]. □

(5) The same γ' defined in (24).

5. Global regularity: proof of Theorem 1.1

Let Ω be of class C^2 and $u \in H^1(\Omega, \mathbb{R}^N)$ be the solution of the Dirichlet problem (1) where $u_o \in H^{1,(\lambda)}(\Omega, \mathbb{R}^N)$ with $0 \leq \lambda < \gamma$ (6).

Setting

$$w = u - u_o,$$

problem (1) becomes equivalent to the problem

$$\begin{aligned} &w \in H^1_o(\Omega, \mathbb{R}^N) \\ &D_i[A_{ij}(x) D_j(w + u_o)] = 0 \quad \text{ in } \Omega. \end{aligned} \tag{28}$$

Since Ω is of class C^2 (see [19, p. 314]), for each $y_o \in \partial\Omega$ there is a ball $\mathcal{B}(y_o, R_o)$ and a C^2-function ζ defined on a domain $D \subset \mathbb{R}^{n-1}$ such that with respect to a suitable system of coordinates $\{y_1, \ldots, y_n\}$, with the origin at y_o:

(a) the set $\partial\Omega \cap \mathcal{B}(y_o, R_o)$ can be represented by an equation of the type:

$$y_n = \zeta(y_1, \ldots, y_{n-1}),$$

(b) each $y \in \Omega \cap \mathcal{B}(y_o, R_o)$ satisfies

$$y_n < \zeta(y_1, \ldots, y_{n-1}).$$

Without loss of generality we can suppose that the system of coordinates is such that the hyperplane tangent to $\partial\Omega$ at y_o has equation $y_n = 0$ and

$$\zeta(y_o) = 0, \quad D\zeta(y_o) = 0. \tag{29}$$

For such domains the boundary can be locally straightened by means of the smooth transformation:

$$\begin{cases} \psi_i(y) = y_i - y_{o\,i} & \text{for } \; i = 1, 2, \ldots, n-1 \\ \psi_n(y) = y_n - \zeta(y_1, \ldots, y_{n-1}). \end{cases} \tag{30}$$

It turns out that $\psi(y) = (\psi_1(y), \ldots, \psi_n(y))$ is a C^2-diffeomorphism verifying the following properties (see e.g. [14, p. 305] or [1, Theorem V, p. 375]):

(i) $\psi(y_o) = 0$ (see $(29)_1$),

(ii) $\psi(\mathcal{B}(y_o, R_o) \cap \partial\Omega) = \{x \in \mathbb{R}^n : x_n = 0, \, |x_i| < R_o, \text{ for } i = 1, \ldots, n-1\}$,

(iii) there exist two positive constants α_1 and α_2, with $\alpha_1 \leq \alpha_2$, such that

$$\begin{gathered} \alpha_1 \, |y - y_o| \leq |\psi(y)| \leq \alpha_2 \, |y - y_o|, \quad \forall \, y \in \mathcal{B}(y_o, R_o) \cap \Omega, \\ B^+(0, \alpha_1 R_o) \subset \psi(\mathcal{B}(y_o, R_o) \cap \Omega) \subset B^+(0, \alpha_2 R_o), \\ \mathcal{B}(y_o, \alpha_1/\alpha_2 R_o) \cap \Omega \subset \psi^{-1}(B^+(0, \alpha_1 R_o)) \subset \mathcal{B}(y_o, R_o) \cap \Omega. \end{gathered} \tag{31}$$

(6) The same γ defined in (4).

Remark 5.1. The fact that $\zeta \in C^2$ and condition $(29)_2$ allow us to choose R_o so that $|D\zeta|$ be sufficiently small in $\mathcal{B}(y_o, R_o) \cap \bar{\Omega}$.

Put $R_1 = \alpha_1 R_o$, if $z \in B^+(0, R_1)$ we set

$$\begin{aligned} \tilde{A}_{ij}(z) &= A_{ij}(\psi^{-1}(z)), \\ B_{ij}(z) &= \tilde{A}_{hk}(z)\frac{\partial \psi_i}{\partial y_h}(\psi^{-1}(z))\frac{\partial \psi_j}{\partial y_k}(\psi^{-1}(z)), \\ W(z) &= w(\psi^{-1}(z)), \\ W_o(z) &= u_o(\psi^{-1}(z)). \end{aligned} \tag{32}$$

Let us observe that $B_{ij}(z)$ still satisfy hypotheses (14) [7].

Moreover, by the definitions (30) and (32) it follows that

$$\frac{\partial \psi_i}{\partial y_h} = \begin{cases} \delta_{ih} & \text{if } \ i = 1, \dots, n-1, \ h = 1, \dots, n \\ \delta_{ih} - \dfrac{\partial \zeta}{\partial y_h} & \text{if } \ i = n, \ h = 1, \dots, n \end{cases} \tag{33}$$

and that

$$\Lambda_1 \sum_{h=1}^{n} \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2 \le B_{ij}\eta_i\eta_j \le \Lambda_2 \sum_{h=1}^{n} \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2, \quad \forall\, \eta = (\eta_i) \in \mathbb{R}^{nN}. \tag{34}$$

On the other hand, exploiting (33), we obtain

$$\sum_{h=1}^{n} \left(\frac{\partial \psi_i}{\partial y_h} \eta_i \right)^2 = |\eta|^2 + \eta_n^2 |D\zeta|^2 - 2 \sum_{h=1}^{n-1} \frac{\partial \zeta}{\partial y_h} \eta_h \eta_n := I$$

whence, since $|D\,\zeta| < 1$,

$$(1 - |D\zeta|)^2\, |\eta|^2 \le I \le (1 + |D\,\zeta|)^2\, |\eta|^2.$$

Gluing together the last inequality and (34) we deduce

$$\Lambda_1 (1 - |D\zeta|)^2\, |\eta|^2 \le B_{ij}\eta_i\eta_j \le \Lambda_2 (1 + |D\zeta|)^2\, |\eta|^2, \quad \forall\, \eta = (\eta_i) \in \mathbb{R}^{nN}.$$

The above inequality and formula (24) yield

$$\gamma' = \gamma'(|D\,\zeta|) = (n-1) \left[1 - \left(\frac{\Lambda_2 (1 + |D\zeta|)^2 - \Lambda_1 (1 - |D\zeta|)^2}{\Lambda_2 (1 + |D\zeta|)^2 + \Lambda_1 (1 - |D\zeta|)^2} \right)^2 \right]. \tag{35}$$

[7] As $A_{ij}^{rs} = A_{ji}^{sr}$ we have $B_{ij}^{rs} = B_{ji}^{sr}$.

With the change of coordinates $z = \psi(y)$, since w is the solution of the problem (28) in $\Omega \cap \mathcal{B}(y_o, R_o)$, W becomes a solution of the problem

$$
(36)\qquad \begin{array}{ll} W \in H^1(B^+(0,R_1),\mathbb{R}^N) & \\ D_i[B_{ij}(z)D_j(W+W_o)] = 0 & \text{in } B^+(0,R_1)\ ^{(8)} \\ W = 0 & \text{on } \Gamma(0,R_1). \end{array}
$$

Remark 5.2. Since ψ is of class C^2 and $u_o \in H^{1,(\lambda)}(\Omega \cap \mathcal{B}(y_o,R_o),\mathbb{R}^N)$, by virtue of [1, Theorem V], $W_o \in H^{1,(\lambda)}(B^+(0,R_1),\mathbb{R}^N)$ and

$$
(37)\qquad \|DW_o\|_{L^{2,\lambda}(B^+(0,R_1))} \le c(\psi)\|Du_o\|_{L^{2,\lambda}(\Omega\cap\mathcal{B}(y_o,R_o))}.
$$

Fix now $\lambda \in [0,\gamma[$ and let $u_o \in H^{1,(\lambda)}(\Omega,\mathbb{R}^N)$. Since $\lim_{t\to 0^+}\gamma'(t) = \gamma$, Remark 5.1 implies that we can choose $R_o = R_o(\partial\Omega,1,\gamma,\lambda)$ so that $\gamma' > \lambda$. Thus, by virtue of (37), $W_o \in H^{1,(\lambda)}(B^+(0,R_1),\mathbb{R}^N)$ with $0 \le \lambda < \gamma'$.

If now $W \in H^1(B^+(0,R_1),\mathbb{R}^N)$ is a solution of the problem (36), by Theorem 4.2 we have, for every $\rho \in\,]0,R_1[$,

$$
DW \in L^{2,\lambda}(B^+(0,\rho),\mathbb{R}^{nN})
$$

and the inequality

$$
(38)\qquad \|DW\|_{L^{2,\lambda}(B^+(0,\rho))} \le c\left[\|DW\|_{L^2(B^+(0,R_1))} + \|DW_o\|_{L^{2,\lambda}(B^+(0,R_1))}\right]
$$

holds.

From (38) and the Poincaré inequality we achieve, $\forall\,\rho \in\,]0,R_1[$,

$$
[W]_{\mathcal{L}^{2,\lambda+2}(B^+(0,\rho))} \le c\left[\|DW\|_{L^2(B^+(0,R_1))} + \|DW_o\|_{L^{2,\lambda}(B^+(0,R_1))}\right].
$$

The above inequality, bearing in mind that $W = U - W_o$ and changing back to old coordinates (see Theorem V of [1]), gives (see (31))

$$
(39)\qquad [u]_{\mathcal{L}^{2,\lambda+2}(\Omega\cap\mathcal{B}(y_o,R_o))} \le c\left[\|Du\|_{L^2(\Omega)} + \|Du_o\|_{L^{2,\lambda}(\Omega)}\right].
$$

Inequalities (38) and (39) yield

$$
(40)\qquad [u]_{\mathcal{L}^{2,\lambda+2}(\Omega\cap\mathcal{B}(y_o,R_o))} + \|Du\|_{L^{2,\lambda}(\Omega\cap\mathcal{B}(y_o,R_o))} \le c\left[\|Du\|_{L^2(\Omega)} + \|Du_o\|_{L^{2,\lambda}(\Omega)}\right].
$$

---

(8) $D_i \equiv \frac{\partial}{\partial z_i}$.

Proof of Theorem 1.1. Since Ω is of class C^2, around every $y_o \in \partial\Omega$ there exists a ball $\mathcal{B}(y_o, R_o)$ and a corresponding diffeomorphism $\psi : \mathcal{B}(y_o, R_o) \to \mathbb{R}^n$ such that (29), (30), (i), (ii), (iii) are satisfied.

Because $\partial\Omega$ is compact, only a finite number of such balls are needed to cover it, say $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_\nu$. For each $\mathcal{B}_\iota$ we suppose that its radius is small enough (see Remark 5.2).

Then there exists an open set $\Omega' \subset\subset \Omega$ such that Ω', $\mathcal{B}_1$, $\mathcal{B}_2$, ..., $\mathcal{B}_\nu$ cover $\overline{\Omega}$.

Bearing in mind that problem (1) is equivalent to problem (28) and exploiting inequalities (25), (40) we obtain

$$[u]_{\mathcal{L}^{2,\lambda+2}(\Omega')} + \|Du\|_{L^{2,\lambda}(\Omega')} \le c\left[\|Du\|_{L^2(\Omega)} + \|Du_o\|_{L^{2,\lambda}(\Omega)}\right] \tag{41}$$

and, for $\iota = 1, 2, \dots, \nu$,

$$[u]_{\mathcal{L}^{2,\lambda+2}(\Omega\cap\mathcal{B}_\iota)} + \|Du\|_{L^{2,\lambda}(\Omega\cap\mathcal{B}_\iota)} \le c\left[\|Du\|_{L^2(\Omega)} + \|Du_o\|_{L^{2,\lambda}(\Omega)}\right]. \tag{42}$$

On the other hand, [3, Theorem 1.III, p. 42] yields

$$\|Du\|_{L^2(\Omega)} \le c\|Du_o\|_{L^2(\Omega)}. \tag{43}$$

Thus, inequality (6) is achieved putting together inequalities (41), (42) and (43).

In particular, (6), (7), (8) and [3, Theorem 2.I, p. 15] yield the required Hölder continuity of u and inequality (9).

Remark 5.3. It seems that an estimate similar to (6) may be proved without any regularity assumption on $\partial\Omega$ (see [13, p. 89]).

6. Maximum principle: proof of Theorem 1.2

We begin the paragraph with the fundamental interior estimate for a solution u of the system (12).

Theorem 6.1. *If $u \in H^1(\Omega, \mathbb{R}^N)$ is a solution of the system* (12) *and*

$$\frac{\Lambda_1}{\Lambda_2} > \frac{\sqrt{n-1}-1}{\sqrt{n-1}+1} \tag{44}$$

then, $\forall\, B(x_o, \rho) \subset \Omega$ and $\forall\, t \in\,]0, 1[$,

$$\|u\|^2_{L^2(B(x_o,t\rho))} \le ct^n\|u\|^2_{L^2(B(x_o,\rho))} \tag{45}$$

where c depends neither on ρ, t nor on x_o.

Proof: By Poincaré and Caccioppoli inequalities (see [3, pp. 21 and 46]), from (17) it follows that, $\forall\, B(y_o,2\sigma)\subset\Omega$ and $\forall\, t\in\,]0,1[$,

$$
\begin{aligned}
\|u-u_{B(y_o,t\sigma)}\|^2_{L^2(B(y_o,t\sigma))} &\le c(n)(t\sigma)^2\|Du\|^2_{L^2(B(y_o,t\sigma))}\\
&\le c(n)t^{2+\gamma}\sigma^2\|Du\|^2_{L^2(B(y_o,\sigma))}\\
&\le c(n,\Lambda_1,\Lambda_2)t^{2+\gamma}\|u-u_{B(y_o,2\sigma)}\|^2_{L^2(B(y_o,2\sigma))}.
\end{aligned}\tag{46}
$$

Inequality (46) implies that, $\forall\, B(x_o,\rho)\subset\Omega$,

$$
\rho^{2+\gamma}[u]^2_{\mathcal{L}^{2,2+\gamma}(B(x_o,\rho/5))}\le c(n,\Lambda_1,\Lambda_2)\|u-u_{B(x_o,\rho)}\|^2_{L^2(B(x_o,\rho))}.\tag{47}
$$

In fact, having fixed $B(x_o,\rho)\subset\Omega$, $y_o\in B(x_o,\rho/5)$ and $t\in\,]0,1[$, by (46) we obtain

$$
\begin{aligned}
&\|u-u_{B(x_o,\rho/5)\cap B(y_o,2/5t\rho)}\|^2_{L^2(B(x_o,\rho/5)\cap B(y_o,2/5t\rho))}\\
&\le c(n,\Lambda_1,\Lambda_2)t^{2+\gamma}\|u-u_{B(y_o,4/5\rho)}\|^2_{L^2(B(y_o,4/5\rho))}\\
&\le c(n,\Lambda_1,\Lambda_2)t^{2+\gamma}\|u-u_{B(x_o,\rho)}\|^2_{L^2(B(x_o,\rho))}.
\end{aligned}
$$

Thus, by virtue of [4, Lemma 2.1], inequalities (44) and (47) give, $\forall\, B(x_o,\rho)\subset\Omega$,

$$
u\in C^{0,1-\frac{n-\gamma}{2}}(\overline{B(x_o,\rho/5)},\mathbb{R}^N)
$$

and

$$
\begin{aligned}
\rho^{2+\gamma}[u]^2_{C^{0,1-\frac{n-\gamma}{2}}(\overline{B(x_o,\rho/5)})} &\le c(n)\rho^{2+\gamma}[u]^2_{\mathcal{L}^{2,2+\gamma}(B(x_o,\rho/5))}\\
&\le c(n,\Lambda_1,\Lambda_2)\|u-u_{B(x_o,\rho)}\|^2_{L^2(B(x_o,\rho))}\\
&\le c(n,\Lambda_1,\Lambda_2)\|u\|^2_{L^2(B(x_o,\rho))}.
\end{aligned}
$$

The proof can be now completed as at p. 301 of [4]. □

Now we come to the proof of Theorem 1.2 which can be carried out as in [5] or [4]. We reproduce it here for reader's convenience.

Proof of Theorem 1.2. Fix $x_o\in\Omega$, set $d=\operatorname{dist}(x_o,\partial\Omega)$ and suppose $y_o\in\partial\Omega$ be such that $|x_o-y_o|=d$.

By virtue of (7), from estimate (45) we obtain, for every $t\in\,]0,1[$,

$$
\|u\|^2_{L^2(B(x_o,td))}\le ct^n\|u\|^2_{L^2(B(x_o,d))}\le ct^n\|u\|^2_{L^2(\Omega(y_o,2d))}\tag{48}
$$

where the constant c is independent on t, d, x_o.

On the other hand, since $u-u_o \in H^1_o(\Omega,\mathbb{R}^N)$ and $\Omega(y_o,2d)$ is convex, Poincaré inequality yields

$$\begin{aligned} \|u\|^2_{L^2(\Omega(y_o,2d))} &\le 2\|u-u_o\|^2_{L^2(\Omega(y_o,2d))} + 2\|u_o\|^2_{L^2(\Omega(y_o,2d))} \\ &\le c(n)\left[d^2\|D(u-u_o)\|^2_{L^2(\Omega(y_o,2d))} + d^n\|u_o\|^2_{L^\infty(\Omega)}\right]. \end{aligned} \tag{49}$$

Moreover, since hypothesis (7) implies $\gamma > n-2$, the inequality (6) rewritten for $\lambda = n-2$ and $(10)_2$ give

$$\begin{aligned} \|D(u-u_o)\|^2_{L^2(\Omega(y_o,2d))} &\le c(n)\, d^{n-2}\|D(u-u_o)\|^2_{L^{2,n-2}(\Omega)} \\ &\le c(n)\, d^{n-2}[\|Du\|^2_{L^{2,n-2}(\Omega)} + \|Du_o\|^2_{L^{2,n-2}(\Omega)}] \\ &\le c(c_1,n,\Lambda_1,\Lambda_2,\partial\Omega)\, d^{n-2}\|u_o\|^2_{L^\infty(\Omega)}. \end{aligned} \tag{50}$$

From (48), (49) and (50) we get

$$\begin{aligned} \|u\|^2_{L^2(B(x_o,td))} &\le c\,t^n\|u\|^2_{L^2(\Omega(y_o,2d))} \\ &\le c(n)\,t^n[d^2\|D(u-u_o)\|^2_{L^2(\Omega(y_o,2d))} + d^n\|u_o\|^2_{L^\infty(\Omega)}] \\ &\le c(c_1,n,\Lambda_1,\Lambda_2,\partial\Omega)\,(td)^n\|u_o\|^2_{L^\infty(\Omega)} \end{aligned}$$

i.e.

$$\begin{aligned} &\frac{1}{\text{meas}(B(x_o,td))}\int_{B(x_o,td)} |u|^2\,dx \\ &\qquad\le c(c_1,n,\Lambda_1,\Lambda_2,\partial\Omega)\,\|u_o\|^2_{L^\infty(\Omega)}, \quad \forall\, t\in\,]0,1\,[\,. \end{aligned} \tag{51}$$

From (51), taking the limit for $t\to 0^+$, we deduce

$$|u(x_o)| \le c(c_1,n,\Lambda_1,\Lambda_2,\partial\Omega)\,\|u_o\|_{L^\infty(\Omega)}, \quad \text{for a.a. } \; x_o\in\Omega,$$

which proves the theorem.

Remark 6.1. The above stated Theorems 1.1 and 1.2 can be readily applied to the following quasilinear system (see [23])

$$D_i(A_{ij}(x,u)D_ju) = 0 \;\text{ in }\; \Omega$$

where $A_{ij}(x,u)$ are bounded Carathéodory functions in $\Omega\times\mathbb{R}^N$ satisfying the following structural conditions

$$A_{ij}(x,u) = A_{ji}(x,u),$$
$$\Lambda_2\,|\xi|^2 \ge A_{ij}(x,u)\xi_i\xi_j \ge \Lambda_1\,|\xi|^2$$
$$\text{for a.a. } x\in\Omega,\ \forall\,u\in\mathbb{R}^N,\ \forall\,\xi=(\xi_i)\in\mathbb{R}^{nN},$$
$$\frac{\Lambda_1}{\Lambda_2} > \frac{\sqrt{n-1}-1}{\sqrt{n-1}+1}\,.$$

Moreover, it is reasonable to conjecture that an assertion similar to Theorems 1.1 and 1.2 hold for variational inequalities and quasiminima (see also [13]).

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Dipartimento di Matematica ed Informatica, Viale A. Doria 6, 95125 Catania, Italy
E-mail: leonardi@dmi.unict.it

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