Minimal KC–spaces are countably compact

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Abstract. In this paper we show that a minimal space in which compact subsets are closed is countably compact. This answers a question posed in [1].

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1. Introduction

A topological space (X, τ) is said to be a KC-space if every compact set is closed. Since every KC-space is T_1 and every T_2 space is KC, the KC-property can be thought of as a separation axiom between T_1 and T_2.

In 1943 E. Hewitt [3] proved that a compact T_2 space is minimal T_2 and maximal compact, see also [5], [6], [7]. R. Larson [4] asked whether a space is maximal compact iff it is minimal KC. A related question is whether every KC-topology contains a minimal KC-topology. W. Fleissner proved that this is not always true. In [2] he constructed a KC-topology which does not contain a minimal KC-topology.

In a recent paper, [1], the authors proved that every minimal KC-topology on a countable set is compact and posed the question whether minimal KC-spaces are countably compact.

In this paper we answer affirmatively this question by proving that *every KC-space which is not countably compact has a strictly weaker KC-topology.*

2. Preliminaries and notations

A filter over a set X is a collection $\mathcal{F}$ of subsets of X such that:

(i) $\emptyset \notin \mathcal{F}$;
(ii) if $F_1 \in \mathcal{F}$ and $F_2 \in \mathcal{F}$ then $F_1 \cap F_2 \in \mathcal{F}$;
(iii) if $A, B \subset X, A \in \mathcal{F}$ and $B \supset A$ then $B \in \mathcal{F}$.

A filter $\mathcal{F}$ over a set X is an ultrafilter if

$$\forall A \subset X \quad \text{either} \quad A \in \mathcal{F} \quad \text{or} \quad X - A \in \mathcal{F}.$$

With $|A|$ we denote the cardinality of a set A, and with A^c the complement of a set A.

For κ an infinite cardinal number, an ultrafilter $\mathcal{F}$ over κ is uniform if $|F| = \kappa$ for all $F \in \mathcal{F}$.

3. Minimal KC-spaces are countably compact

Let (X,τ) be a KC-space which is not countably compact. Then there exists a set $\{x_n : n \in \omega\} \subset X$ which has no accumulation points. We define a new topology τ' on X as follows:

(NT) For every $x \in X$ with $x \neq x_0$ the open neighborhoods of x in τ' coincide with the open neighborhoods of x in τ. An open neighborhood of x_0 in τ' is every τ-open set containing x_0 and a member of $\mathcal{F}$, where $\mathcal{F}$ is a uniform ultrafilter defined over the set $\{x_n : 0 < n < \omega\}$.

Remark 3.1. It is clear that τ' is a T_1-topology and that x_0 is the unique point which can be τ'-accumulation point for a set $K \subset X$ while it is not τ-accumulation point of it.

Our aim is to show that if (X,τ) is a KC-space, which is not countably compact, then the topology τ' defined by (NT) is also a KC-topology.

Let $K \subset X$ be τ'-compact. If $x_0 \notin K$ then K is τ-compact, thus τ-closed, and since $\{x_n : n \in \omega\}$ has no accumulation points we have that $\{x_n : n \in \omega\} \cap K$ is finite. Hence x_0 is not a τ'-accumulation point of K and it follows that K is τ'-closed.

So it remains to prove that if $K \subset X$ is τ'-compact and $x_0 \in K$, then K is τ'-closed, or equivalently it is τ-closed. Therefore we assume for the rest of the paper that $x_0 \in K$.

To prove that a τ'-compact set K is τ'-closed we consider the following cases for a member of the ultrafilter $\mathcal{F}$ in relation with K:

(1) $F \subset K$;
(2) $F \cap \overline{K}^{\tau} = \emptyset$;
(3) $F \subset (\overline{K}^{\tau} - K)$.

Lemma 3.2 below refers to case (1), Lemma 3.3 to case (2), while Lemmas 3.4 and 3.5 to case (3).

Lemma 3.2. *Let (X,τ) be a KC-space which is not countably compact, $\{x_n : n \in \omega\}$ a set without accumulation points, $\mathcal{F}$ a uniform ultrafilter defined over $\{x_n : 0 < n < \omega\}$, τ' the topology defined by* (NT) *and K a τ'-compact set. Then there is an $F \in \mathcal{F}$, such that $F \cap K = \emptyset$.*

Proof: Since $\mathcal{F}$ is an ultrafilter, either there exists an $F \in \mathcal{F}$ such that $F \subset K$, or there is an $F \in \mathcal{F}$ with $F \cap K = \emptyset$.

In the first case let $F = F_1 \cup F_2$ with $F_1 \cap F_2 = \emptyset$ and $|F_1| = |F_2| = \omega$.

Then if $F_1 \in \mathcal{F}$, there exists an open set $U(F_1)$ containing F_1 with

$$U(F_1) \cap F_2 = \emptyset.$$

Thus there is a τ'-open neighborhood of x_0, $U'(x_0)$, with

$$F_2 \cap U'(x_0) = \emptyset,$$

and F_2 will be an infinite subset of K without τ'-accumulation points, which is impossible. So there must be an $F \in \mathcal{F}$ such that: $F \cap K = \emptyset$. □

Lemma 3.3. *With the assumptions of Lemma* 3.2 *if there exists an $F_0 \in \mathcal{F}$ such that $F_0 \cap \overline{K}^\tau = \emptyset$, then K is τ'-closed.*

PROOF: Since $x_0 \in K$ it suffices to show that K is τ-closed.

Let $\{U_i : i \in I\}$, be a τ-open cover of K and let V_0 be an open set containing F_0 such that $V_0 \cap K = \emptyset$.

Then the collection $\{U_i \cup V_0 : i \in I\}$, is a τ'-open cover of K and thus it has a finite subcover, say, $U_{i_1} \cup U_{i_2} \cup \ldots \cup U_{i_n} \cup V_0$.

The set $\bigcup\{U_{i_k} : k = 1, 2, \ldots, n\}$ covers K, so K is τ-compact and therefore τ-closed. □

It remains to consider the case where there is an $F \in \mathcal{F}$ such that $F \subset (\overline{K}^\tau - K)$. We will show first that in this case K is countably compact.

Lemma 3.4. *Let (X, τ) be a KC-space which is not countably compact, τ' the topology defined by* (NT), *K a τ'-compact set, $x_0 \in K$ and $F_0 \in \mathcal{F}$ with $F_0 \subset (\overline{K}^\tau - K)$. Then K is τ-countably compact.*

PROOF: Let $F_0 \in \mathcal{F}$ be such that $F_0 \subset (\overline{K}^\tau - K)$, with $F_0 = \{x_{n_k} : k \in \omega\}$ and suppose for a contradiction that K is not τ-countably compact.

Then there exists a set $\{y_n : n \in \omega\} \subset K$ without τ-accumulation points in K and since $x_0 \in K$, there is a τ-open neighborhood $U(x_0)$ of x_0 with

$$U(x_0) \cap \{y_n : n \in \omega\} = \emptyset.$$

We claim that for every infinite subset $\{y_{n_k} : k \in \omega\}$ of $\{y_n : n \in \omega\}$ and for every $z \in F_0$ there is a τ-open neighborhood of z, $U(z)$, such that

$$|U(z)^c \cap \{y_{n_k} : k \in \omega\}| = \omega.$$

Actually, for otherwise $\{y_{n_k} : k \in \omega\} \to z$ and since τ is a KC-topology, z will be the unique τ-accumulation point of $\{y_{n_k} : k \in \omega\}$.

But, there is an $F \in \mathcal{F}$ with $z \notin F$, thus there is an open set $W(F)$ containing F with $z \notin W(F)$. So $z \notin U(x_0) \cup W(F)$, and consequently x_0 is not a τ'-accumulation point of $\{y_{n_k} : k \in \omega\}$.

It follows that $\{y_{n_k} : k \in \omega\}$ is an infinite subset of K with no τ'-accumulation points in K which is impossible, since K is τ'-compact.

So, let $U(x_{n_1})$ be an open neighborhood of x_{n_1} such that

$$|U(x_{n_1})^c \cap \{y_n : n \in \omega\}| = \omega,$$

and let

$$z_1 \in U(x_{n_1})^c \cap \{y_n : n \in \omega\}.$$

Let $U(x_{n_2})$ be an open neighborhood of x_{n_2} with

$$|U(x_{n_2})^c \cap U(x_{n_1})^c \cap \{y_n : n \in \omega\}| = \omega,$$

and let

$$z_2 \in U(x_{n_2})^c \cap U(x_{n_1})^c \cap \{y_n : n \in \omega\},$$

with $z_2 \neq z_1$ and inductively, let $U(x_{n_k})$ be an open neighborhood of x_{n_k} with

$$|U(x_{n_1})^c \cap U(x_{n_2})^c \cap \ldots \cap U(x_{n_k})^c \cap \{y_n : n \in \omega\}| = \omega,$$

and let

$$z_k \in U(x_{n_1})^c \cap U(x_{n_2})^c \cap \ldots \cap U(x_{n_k})^c \cap \{y_n : n \in \omega\},$$

with

$$z_k \notin \{z_1, z_2, \ldots , z_{k-1}\}.$$

The so defined sequence $\{z_n : n \in \omega\}$ is a subset of K and since

$$\{z_n : n \in \omega\} \cap [U(x_0) \cup \bigcup\{U(x_{n_k}) : k \in \omega\}] = \emptyset,$$

it follows that it has no τ'-accumulation points in K, contrary to the hypothesis. □

Lemma 3.5. *Let (X, τ) be a KC-space which is not countably compact. Then X can be condensed onto a weaker KC-topology.*

PROOF: Let τ' be the topology defined by (NT). We will prove that (X, τ') is a KC-space.

For this we will show that there is an $F \in \mathcal{F}$ with $F \cap \overline{K}^{\tau} = \emptyset$ and the proof will be a consequence of Lemma 3.3.

Indeed, suppose for a contradiction that there is $F_0 \in \mathcal{F}$ such that $F_0 \subset \overline{K}^{\tau}$. Let F_1, F_2 be subsets of F_0 with $|F_1| = |F_2| = \omega$, $F_1 \cup F_2 = F_0$, and $F_1 \cap F_2 = \emptyset$.

Suppose that $F_1 \in \mathcal{F}$. We claim that $F_1 \cup K$ is τ-compact.

Actually let $\{U_i : i \in I\}$ be a τ-open cover of $F_1 \cup K$. Then countably many of the $U_i's$, say, $\{U_{i_n} : n \in \omega\}$, cover the countable set F_1, and if we write

$$U'(x_0) = U(x_0) \cup \bigcup\{U_{i_n} : n \in \omega\},$$

where $U(x_0)$ is a member of $\{U_i : i \in I\}$ which contains x_0 then $U'(x_0)$ is a τ'-open neighborhood of x_0, and we will have

$$\bigcup\{U_i : i \in I\} = U'(x_0) \cup \bigcup\{V_j : j \in J\},$$

where $\{V_j : j \in J\}$ is a subcollection of $\{U_i : i \in I\}$ which covers $U'(x_0)^c \cap K$. But $\{U_i : i \in I\}$ is also a τ'-open cover of K. So it contains a finite subcover.

It turns out that finitely many V_j's, say, $V_{j_1}, V_{j_2}, \dots, V_{j_k}$, cover the set

$$K \cap (U(x_0) \cup \bigcup\{U_{i_n} : n \in \omega\})^c = K \cap U'(x_0)^c.$$

Now

$$\bigcup\{V_{j_m} : m = 1, 2, \dots, k\} \cup \bigcup\{U_{i_n} : n \in \omega\} \cup U(x_0)$$

is a countable τ-open cover of K and in view of Lemma 3.4 it has a finite subcover.

So $K \cup F_1$ is τ-compact and therefore τ-closed. But this is impossible since every $x \in F_2$ is a τ-accumulation point of K.

So there must be an $F \in \mathcal{F}$ with

$$F \cap \overline{K}^{\tau} = \emptyset$$

and Lemma 3.3 implies that K is τ-closed. Now from Remark 3.1 it follows that K is τ'-closed. □

The following theorem answers a question posed in [1]. Its proof is an immediate consequence of Lemma 3.5.

Theorem 3.6. *Every minimal KC-space is countably compact.*

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