# On the existence of true uniform ultrafilters

Petr Simon

*Abstract.* We shall show that there is an ultrafilter on singular $\kappa$ with countable cofinality, which cannot be reached from the set of all subuniform ultrafilters by iterating the closure of sets of size $< \kappa$.

*Keywords:* uniform ultrafilter, linked family

*Classification:* Primary 54D80; Secondary 54G05, 54A20

The notation used in the present note is a standard one and follows e.g. [CN].

**Definition.** Let $\kappa$ be an infinite cardinal; consider $\kappa$ equipped with a discrete topology. In the space $\beta(\kappa)$, define subspaces $Z_\alpha$ by transfinite recursion as follows. $Z_0 = \kappa$; for a limit ordinal $\alpha$, let $Z_\alpha = \bigcup_{\beta<\alpha} Z_\beta$; and $Z_{\alpha+1} = \bigcup\{\mathrm{cl}_{\beta\kappa}(T) : T \in [Z_\alpha]^{<\kappa}\}$. Call an ultrafilter $q$ on $\kappa$ to be *true uniform*, if $q \in \beta\kappa \setminus \bigcup_{\alpha\in\mathrm{On}} Z_\alpha$.

It is self-evident that for regular $\kappa$, an ultrafilter on $\kappa$ is true uniform if and only if it is uniform. Also, one can easily produce a uniform ultrafilter on a singular $\kappa$, which is not true uniform. During the 2003 Summer Conference on Topology and its Applications, A. Dow asked a question, attributed to Wistar W. Comfort, if there are true uniform ultrafilters also on singular cardinals. The aim of this short note is to answer Comfort's question in the affirmative for the case of a singular cardinal with countable cofinality.

Fix a cardinal $\kappa > \mathrm{cf}(\kappa) = \omega$ and choose a strictly increasing sequence of regular cardinals $\langle \kappa_n : \alpha \in \omega\rangle$ converging to $\kappa$. Decompose $\kappa$ as $\kappa = \bigcup_{n<\omega} R_n$, where $R_0 = \kappa_0$, $R_{n+1} = \kappa_{n+1} \setminus \kappa_n$.

Our method of the proof resembles the standard technique for constructing remote points, see e.g., [vD].

**Definition.** Let $M$ be an infinite set, $0 < n < \omega$. A family $\mathcal{A} \subseteq \mathcal{P}(M)$ is called *precisely uniformly $n$-linked family on $M$*, if for every $\mathcal{A}' \in [\mathcal{A}]^n$, $|\bigcap \mathcal{A}'| = |M|$, and for every $\mathcal{A}'' \in [\mathcal{A}]^{n+1}$, $\bigcap \mathcal{A}'' = \emptyset$.

The author gratefully acknowledges support provided by grants GAČR 201/03/0933 and GAUK 270/2001.

**Observation.** *Let $\mu$ be an infinite cardinal, $0 < n < \omega$. Then there is a precisely uniformly $n$-linked family on $\mu$ of size $\mu$.*

Indeed, identify $\mu$ with $X = [\mu]^n \times \mu$. For each $\alpha \in \mu$, let $A_\alpha = \{K \times \mu : K \in [\mu]^n, \alpha \in K\}$. Put $\mathcal{A} = \{A_\alpha : \alpha \in \mu\}$. The family $\mathcal{A}$ on $X$ is as required.

On every set $R_n$, choose an $n$-linked family $\mathcal{A}_n = \{A_{n,\eta} : \eta \in \kappa_n\}$ with the properties from Observation. For a mapping $g \in \prod_{n\in\omega} \kappa_n$, denote by $L(g)$ the set $\{\xi \in \kappa : \text{If } \xi \in R_n, \text{ then } \xi \leq g(n)\}$.

Given two mappings $f, g \in \prod_{n\in\omega} \kappa_n$, define $A(f,g)$ to be the set $\bigcup_{n\in\omega} A_{n,f(n)} \setminus L(g)$. Observe that the family $\{A(f,g) : f, g \in \prod_{n\in\omega} \kappa_n\}$ generates a uniform filter on $\kappa$. Given $k \in \omega$ and $f_i, g_i \in \prod_{n\in\omega} \kappa_n$ for $i < k$, choose $n \in \omega$, $n > k$. Since the family $\mathcal{A}_n$ is $n$-linked and $k < n$, the intersection $\bigcap_{i<k} A_{n,f_i(n)}$ has cardinality $\kappa_n$. Since each set $\{\xi \in R_n : \xi < g_i(n)\}$ is of size $< \kappa_n$, we have that $\bigcap_{i<k} A_{n,f_i(n)} \setminus \bigcup_{i<k}\{\xi \in R_n : \xi < g_i(n)\}$ is of size $\kappa_n$ and so the intersection $\bigcap_{i<k} A(f_i, g_i)$ has full cardinality $\kappa$. — Let $\mathcal{F}$ denote the filter generated by $\{A(f,g) : f, g \in \prod_{n\in\omega} \kappa_n\}$.

We claim that every uniform ultrafilter on $\kappa$ extending $\mathcal{F}$ is true uniform.

For $h, g \in \prod_{n\in\omega} \kappa_n$, denote by $\mathcal{A}(h,g)$ the collection $\{A(f,g') : h \leq f, g \leq g'\}$.

We shall reach this goal by assigning to each ultrafilter $q \in \bigcup_{\alpha\in\mathrm{On}} Z_\alpha$ two functions $h_q, g_q \in \prod_{n\in\omega} \kappa_n$ such that for each $A \in \mathcal{A}(h_q, g_q)$, $A \notin q$.

Let $q \in Z_1$. Then $q$ is a free subuniform ultrafilter on $\kappa$ and two cases are possible:

Case 1: There is some $m \in \omega$ such that $R_m \in q$. The family $\mathcal{A}_m$ is precisely $m$-linked, so the set $J = \{\eta < \kappa_m : A_{m,\eta} \in q\}$ contains at most $m$ elements. Put $h_q(m) = \max J + 1$, $h_q(n) = 0$ for all $n \in \omega \setminus \{m\}$ and let $g_q$ be the constant function with value 0.

Apparently, if $A \in \mathcal{A}(h_q, g_q)$, then $A \notin q$.

Case 2: Not Case 1. Then there is some $m \in \omega$ and a set $U \subseteq \kappa$ such that $|U| < \kappa_m$ and for every $n \in \omega$, $U \setminus R_n \in q$. Once $n \geq m$, then $U \cap \kappa_n$ cannot be cofinal in $\kappa_n$, so there is a function $g_q \in \prod_{n\in\omega} \kappa_n$ such that $U \setminus \bigcup_{i<m} R_i \subseteq L(g_q)$. Put $h_q(n) = 0$ for all $n \in \omega$. Again, if $A \in \mathcal{A}(h_q, g_q)$, then $A \notin q$, because $U \setminus \bigcup_{i<m} R_i$ is disjoint from $A$.

Let $\alpha > 1$ be an ordinal and assume that for every $\beta < \alpha$ and every ultrafilter $q \in Z_\beta$ there is a pair of functions $h_q, g_q \in \prod_{n\in\omega} \kappa_n$ such that for every $A(f,g) \in \mathcal{A}(h_q, g_q)$, $A(f,g) \notin q$.

There is nothing to prove if $\alpha$ is a limit ordinal. So assume $\alpha = \beta + 1$ and let an ultrafilter $q$ belong to $Z_\alpha \setminus Z_\beta$. Then there is a cardinal $\kappa_m < \kappa$ and a set $P \subseteq Z_\beta$, $|P| < \kappa_m$, such that $q \in \mathrm{cl}_{\beta\kappa}(P)$.

Observe that for every $n \in \omega$, $R_n \notin q$. This follows immediately from the fact that $\beta \geq 1$ and $q \notin Z_\beta$. Define $h_q(n) = 0$ for $n < m$, $h_q(n) = \sup\{h_p(n) : p \in P\}$

for $n \geq m$, $n < \omega$. Similarly, let $g_q(n) = 0$ for $n < m$, $g_q(n) = \sup\{g_p(n) : p \in P\}$ for $n \geq m$.

Pick an arbitrary $A \in \mathcal{A}(h_q, g_q)$. Then $A = A(f,g)$ for some $f \geq h_q$ and $g \geq g_q$.

Let $P_0 = \{p \in P : \bigcup_{n<m} R_m \in p\}$, $P_1 = P \setminus P_0$. Since $\bigcup_{n<m} R_m \notin q$, we obtain $q \notin \mathrm{cl}_{\beta\kappa}(P_0)$, and so $q \in \mathrm{cl}_{\beta\kappa}(P_1)$. Let $p \in P_1$ be arbitrary. If we define functions $f'$ and $g'$ by the rule $f'(n) = h_p(n)$ for $n < m$ and $f'(n) = f(n)$ for $n \geq m$, $g'(n) = g_p(n)$ for $n < m$ and $g'(n) = g(n)$ for $n \geq m$, then $A(f',g') \in \mathcal{A}(h_p, g_p)$ and consequently $A(f',g') \notin p$. Since $A(f,g) \subseteq A(f',g') \cup \bigcup_{n<m} R_n$, $A(f,g) \notin p$. Since this holds for all $p \in P_1$, $A(f,g) \notin q$.

Hence every ultrafilter which extends the filter $\mathcal{F}$ is true uniform and we have just proved

**Theorem.** *There is a true uniform ultrafilter on every singular cardinal with countable cofinality.*

The reader undoubtedly observed that countable cofinality of a cardinal in question was crucial for the proof just given. So, a part of Comfort's question remains open: Are there true uniform ultrafilters also on singular cardinals with uncountable cofinality?

## References

[CN] Comfort W.W., Negrepontis S., *The Theory of Ultrafilters*, Springer, New York, Heidelberg, Berlin (1974).

[vD] van Douwen E.K., *Remote points*, Dissertationes Math. **188** (1981), 1–45.

Department of Computer Science and Mathematical Logic, Charles University, Malostranské nám. 25, 110 00 Praha 1, Czech Republic

*E-mail*: psimon@ms.mff.cuni.cz

(*Received* December 15, 2003, *revised* January 14, 2004)