Function spaces on ordinals

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Abstract. We give a partial classification of spaces $C_p([1,\alpha])$ of continuous real valued functions on ordinals with the topology of pointwise convergence with respect to homeomorphisms and uniform homeomorphisms.

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1. Introduction

For a completely regular space X, $C_p(X)$ denotes the space of all continuous real-valued functions on X, equipped with the pointwise convergence topology.

Spaces X and Y are called t-equivalent (u-equivalent) if spaces $C_p(X)$ and $C_p(Y)$ are (uniformly) homeomorphic. We write $X \sim^t Y$ if the spaces X and Y are t-equivalent and $X \sim^u Y$ when X and Y are u-equivalent. Let us recall that the map $\varphi : E \to L$, where E and L are linear topological spaces, is uniformly continuous if for every neighborhood U of zero in L there is a neighborhood V of zero in E such that, for every $f, g \in E$ with $(f - g) \in V$ we have $\varphi(f) - \varphi(g) \in U$.

In this paper we are concerned with the function spaces $C_p([1,\alpha])$ on compact ordinal intervals $[1,\alpha]$ (equipped with the standard order topology).

The complete classification of such spaces with respect to linear homeomorphisms was given by Baars and de Groot [2] (see Section 3). Gul'ko proved in [3] and [4] some results concerning the problem we deal with. Some of these results and techniques have turned out to be useful for us and will be recalled bellow. In this paper we extend Gul'ko's results. Namely our goal is to prove the following:

Theorem 1.1. *Let α and β be ordinals.*
If $|\alpha| \neq |\beta|$, then

(a) *$C_p([1,\alpha])$ and $C_p([1,\beta])$ are not homeomorphic.*

If $|\alpha| = |\beta| = \kappa$, then

(b) *if $\kappa = \omega$ or κ is singular or $\alpha, \beta \geq \kappa^2$, then $C_p([1,\alpha])$ and $C_p([1,\beta])$ are uniformly homeomorphic.*

(c) *If κ is a regular uncountable cardinal and $\alpha, \beta \in [\kappa, \kappa^2)$ such that $\alpha \in [\kappa \cdot \gamma_a, \kappa \cdot (\gamma_a + 1))$ and $\beta \in [\kappa \cdot \gamma_b, \kappa \cdot (\gamma_b + 1))$, where $\gamma_a, \gamma_b \in [1, \kappa)$,* *then $C_p([1,\alpha])$ and $C_p([1,\beta])$ are (uniformly) homeomorphic if and only if $|\gamma_a| = |\gamma_b|$.*

Unfortunately this theorem does not cover all the possibilities. Namely, we do not know whether $C_p([1,\kappa^+\cdot\kappa])$ and $C_p([1,(\kappa^+)^2])$ are (uniformly) homeomorphic. The proof of Theorem 1.1 is based on ideas from [2] and [3].

2. Preliminaries

All spaces under consideration are completely regular.

Definition 2.1. For a point x of a space X, let $C_p(X,x) = \{f \in C_p(X) : f(x) = 0\}$ with the topology of the pointwise convergence. If X is compact we equip $C_p(X,x)$ with the standard sup norm.

We will use the following standard fact (see [1]):

Theorem 2.2. *If $C_p(X)$ and $C_p(Y)$ are homeomorphic then $|X| = |Y|$.*

In the sequel we will consider sets of the form $[\alpha,\beta]$ and $[\alpha,\beta)$ (α and β are ordinals) defined in the following way:
$[\alpha,\beta] = \{\gamma; \alpha \le \gamma \le \beta\}$ and $[\alpha,\beta) = \{\gamma; \alpha \le \gamma < \beta\}$.
By **Ord**, **Lim**, **Card** we denote the class of all ordinals, limit ordinals and cardinals, respectively. The topology on $[\alpha,\beta]$ and its subsets will be always the order topology. Let us point out some important properties of such spaces:

Fact 2.3. *If $\beta\cdot\omega \le \alpha$ then $[1,\alpha+\beta]$ and $[1,\alpha]$ are homeomorphic. In particular, if κ is an infinite cardinal number ($\kappa \in$ **Card** and $\kappa \ge \omega$) then for every $\beta < \kappa$, $[1,\kappa+\beta]$ and $[1,\kappa]$ are homeomorphic.*

Let us recall the definition of the c_0-product:

Definition 2.4. For every $t \in T$, let E_t be a linear topological space and $\|\cdot\|_t$ be a norm on E_t, not necessarily related to the topology. Let us define:

$$\Pi^*_{t\in T} E_t = \{(f_t)_{t\in T} \in \Pi_{t\in T} E_t : \forall\, \varepsilon > 0\ \|f_t\| < \varepsilon \text{ for all but finitely many } t \in T\}.$$

The topology on $\Pi^*_{t\in T} E_t$ is the standard product topology. Usually on $\Pi^*_{t\in T} E_t$ we consider the norm $\|(f_t)_{t\in T}\| = \max_{t\in T} \|f_t\|_t$. The symbol $\Pi^*_{t\in T} E$ denotes the c_0 product $\Pi^*_{t\in T} E_t$, where $E_t = E$ for every $t \in T$.

We will use the following relation:

Definition 2.5 (Gul'ko [3])**.** Let E and F be linear topological spaces and $\|\cdot\|_1$, $\|\cdot\|_2$ be norms, on E and F, respectively, not necessarily related to the topologies. We write $(E,\|\cdot\|_1) \doteqdot (F,\|\cdot\|_2)$ if, for every $\varepsilon > 0$, there exists a uniform homeomorphism $u_\varepsilon : E \to F$ satisfying the following condition:

$$(a_\varepsilon) \qquad (1+\varepsilon)^{-1}\|f\|_1 \le \|u_\varepsilon(f)\|_2 \le \|f\|_1 \quad \text{for every } f \in E.$$

If it is clear which norms are considered on E and F we write $E \doteqdot F$.

In the sequel on spaces of real continuous functions on compacta and its subspaces we will always consider the sup norm. Moreover, let us fix that for every two linear topological spaces E and F equipped with norms $\|\cdot\|_0$ and $\|\cdot\|_1$, respectively, on the space $E \times F$ we consider the norm $\|(e,f)\| = \max(\|e\|_0, \|f\|_1)$.

Let us point out some obvious, but important, properties of the relation $\doteqdot$.

Fact 2.6. *If $X \doteqdot X_1$ and $Y \doteqdot Y_1$ then $X \times Y \doteqdot X_1 \times Y_1$.*

Fact 2.7. *If, for every $t \in T$, $X_t \doteqdot Y_t$ then $\Pi^*_{t\in T} X_t \doteqdot \Pi^*_{t\in T} Y_t$.*

Theorem 2.8 (Gul'ko [3]). *For every compact space X and for every $x_0 \in X$ we have*

$$C_p(X) \doteqdot C_p(X, x_0) \times \mathbb{R}.$$

The above theorem of Gul'ko is very important for our consideration. We will use this in the proof of our main theorem and also some ideas from its proof will be very helpful, namely the following

Lemma 2.9 (Gul'ko [3, Lemma 1]). *Let $\mathbb{R}^2$ be the real plane equipped with the norm $\|(x_1,x_2)\| = \max(|x_1|,|x_2|)$ and let $\varepsilon > 0$. Then there exist functions $\varphi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ and $\psi_\varepsilon : \mathbb{R}^2 \to \mathbb{R}$ such that the following conditions are satisfied:*

(a) *the mapping $(x_1,x_2) \mapsto (x_1, \varphi_\varepsilon(x_1,x_2))$ is a uniform homeomorphism of the plane with the inverse of the form $(x_1,x_2) \mapsto (x_1, \psi_\varepsilon(x_1,x_2))$;*

(b) *$\varphi_\varepsilon(x_1,x_2) = 0$ if $x_1 = x_2$;*

(c) *$(1+\varepsilon)^{-1}\|(x_1,x_2)\| \le \|(x_1,\varphi_\varepsilon(x_1,x_2))\| \le \|(x_1,x_2)\|$ for $(x_1,x_2) \in \mathbb{R}^2$.*

Fact 2.10. *Let α be an infinite ordinal. Then*

$$C_p([1,\alpha]) \doteqdot C_p([1,\alpha],\alpha).$$

In particular, if $[1,\alpha]$ and $[1,\beta]$ are homeomorphic then

$$C_p([1,\alpha],\alpha) \doteqdot C_p([1,\beta],\beta).$$

Proof: From Theorem 2.8 it follows that

$$C_p([1,\alpha]) \doteqdot C_p([1,\alpha],\alpha) \times \mathbb{R}.$$

It is clear that $C_p([1,\alpha],\alpha) \times \mathbb{R}$ can be identified with $C_p([1,\alpha] \oplus \{\cdot\}, \alpha)$. On the other hand there exists a homeomorphism $h : [1,\alpha] \oplus \{\cdot\} \to [1,\alpha]$ such that $h(\alpha) = \alpha$ (we use the assumption $\alpha \ge \omega$). This homeomorphism induces the homeomorphism between function spaces such that:

$$C_p([1,\alpha] \oplus \{\cdot\}, \alpha) \doteqdot C_p([1,\alpha],\alpha).$$

$\square$

3. Linear homeomorphisms

In [5] Kislyakov gave a complete linear classification of function spaces $C_p([1,\alpha])$ with sup norm topology. Later, Baars and de Groot got analogical result for pointwise topology; more precisely, they proved that these classifications coincide.

Theorem 3.1 (Baars, de Groot [2]). *Let α and β be ordinals.*
If $|\alpha| \neq |\beta|$, then

(a) *$C_p([1,\alpha])$ and $C_p([1,\beta])$ are not linearly homeomorphic.*

If $|\alpha| = |\beta| = \kappa$, then

(b) *if $\kappa = \omega$ or κ is a singular cardinal or $\alpha, \beta \geq \kappa^2$ then $C_p([1,\alpha])$ and $C_p([1,\beta])$ are linearly homeomorphic if and only if $\max(\alpha,\beta) < [\min(\alpha,\beta)]^\omega$;*

(c) *if κ is a regular uncountable cardinal and $\alpha, \beta \in [\kappa, \kappa^2)$ such that $\alpha \in [\kappa \cdot \gamma_a, \kappa \cdot (\gamma_a + 1))$ and $\beta \in [\kappa \cdot \gamma_b, \kappa \cdot (\gamma_b + 1))$, where $\gamma_a, \gamma_b \in [1, \kappa)$, then $C_p([1,\alpha])$ and $C_p([1,\beta])$ are linearly homeomorphic if and only if $|\gamma_a| = |\gamma_b|$;*

(d) *if κ is a regular uncountable cardinal, $\alpha < \kappa^2$, and $\beta \geq \kappa^2$ then $C_p([1,\alpha])$ and $C_p([1,\beta])$ are not linearly homeomorphic.*

This theorem gives us a complete classification of $C_p([1,\alpha])$ up to linear homeomorphisms where α is an ordinal. In the next section we will use the idea of its proof to give a similar partial classification for uniform homeomorphisms and homeomorphisms.

4. Uniform homeomorphisms and homeomorphisms

Theorem 4.1. *If $\alpha \in$ **Ord** and $\tau, \kappa \in$ **Card** ($\tau \geq \omega$ and $\kappa \geq 1$) satisfy the following condition:*

$$\kappa \leq \tau \text{ and } \alpha \in [\tau \cdot \kappa, \tau \cdot \kappa^+)$$

then

$$C_p([1,\alpha]) \simeq \prod_{\beta \in \kappa}{}^{*} C_p([1,\tau]) \simeq \prod_{\beta \in \kappa}{}^{*} C_p([1,\tau],\tau).$$

To prove the above theorem we need the following

Lemma 4.2. *Let γ be a limit ordinal ($\gamma \in$ **Lim**), and $(\lambda_\xi)_{0 \leq \xi \leq \gamma}$ be a strictly increasing sequence of ordinals such that:*

(1) *$\lambda_\xi = \lim_{\beta < \xi} \lambda_\beta$ for $\xi \in (0,\gamma] \cap$ **Lim**;*

(2) *$\lambda_0 = 0$.*

Then:

$$C_p([1,\lambda_\gamma],\lambda_\gamma) \simeq C_p([1,\gamma],\gamma) \times \prod_{\xi \in \gamma}{}^{*} C_p([\lambda_\xi + 1, \lambda_{\xi+1}], \lambda_{\xi+1}).$$

Proof: The proof will be a modification of the proof of Lemma 2.5.6 in [2]. Let us define

$$X = \{f \in C_p([1,\lambda_\gamma],\lambda_\gamma) : \forall\, \xi \in [0,\gamma)\ \ f|(\lambda_\xi,\lambda_{\xi+1}] \equiv f(\lambda_{\xi+1})\}$$

and

$$Y = \{f \in C_p([1,\lambda_\gamma],\lambda_\gamma) : \forall\, \xi \in (0,\gamma]\ \ f(\lambda_\xi) = 0\}.$$

Fact 4.3. $C_p([1,\lambda_\gamma],\lambda_\gamma) \doteq X \times Y$.

Fix $\varepsilon > 0$. Let us define $\phi_1 : C_p([1,\lambda_\gamma],\lambda_\gamma) \to X$ as follows:

$\phi_1(f)|[\lambda_\xi + 1, \lambda_{\xi+1}] \equiv f(\lambda_{\xi+1})$ for $\xi \in [0,\gamma)$;
$\phi_1(f)(\lambda_\xi) = f(\lambda_\xi)$ for $\xi \in [1,\gamma] \cap \mathbf{Lim}$.

To prove that ϕ_1 is well defined we need to check the continuity of $\phi_1(f)$ at λ_ξ for $\xi \in [1,\gamma] \cap \mathbf{Lim}$. We will show that for every $\delta > 0$ there exists λ_α such that $\lambda_\alpha < \lambda_\xi$ and $\phi_1(f)((\lambda_\alpha,\lambda_\xi]) \subset (-\delta + f(\lambda_\xi), f(\lambda_\xi)+\delta)$. According to the continuity of f there exists $a \in [1,\lambda_\xi)$ such that $f((a,\lambda_\xi]) \subset (-\delta + f(\lambda_\xi), f(\lambda_\xi)+\delta)$. Because $\xi \in \mathbf{Lim}$ we have $\lim_{\eta<\xi} \lambda_\eta = \lambda_\xi$. Therefore there exists λ_α such that $a \le \lambda_\alpha < \lambda_\xi$. We have:

$$\phi_1(f)((\lambda_\alpha,\lambda_\xi]) \subset f((\lambda_\alpha,\lambda_\xi]) \subset f((a,\lambda_\xi]) \subset (-\delta + f(\lambda_\xi), f(\lambda_\xi) + \delta)$$

which finishes the proof of continuity of $\phi_1(f)$.

One can easily verify that ϕ_1 is continuous.
Let us define $\phi_2 : C_p([1,\lambda_\gamma],\lambda_\gamma) \to Y$ by

$$\phi_2(f)(x) = \varphi_\varepsilon(\phi_1(f)(x), f(x))$$

where φ_ε is as in Lemma 2.9. It is easy to observe that $\phi_2(f) \in Y$ by Lemma 2.9(b). Therefore, to check that ϕ_2 is well defined it is enough to prove that $\phi_2(f)$ is continuous for $f \in C_p([1,\lambda_\gamma],\lambda_\gamma)$. We have proved that $\phi_1(f)$ is continuous thus $\phi_2(f)$, as a composition of continuous functions, is also continuous. On the other hand the uniform continuity of ϕ_2 follows from the uniform continuity of ϕ_1 and φ_ε.
Let us check that $\phi(f) = (\phi_1(f), \phi_2(f))$ satisfies the following inequalities:

$$(1+\varepsilon)^{-1}\|f\| \le \|\phi(f)\| \le \|f\| \quad \text{for} \quad f \in C_p([1,\lambda_\gamma],\lambda_\gamma).$$

It is clear that $\|f\| \ge \|\phi_1(f)\|$. Thus, from Lemma 2.9(c) we get $\|\phi_2(f)\| \le \max(\|\phi_1(f)\|, \|f\|) \le \|f\|$. We proved that $\|\phi(f)\| \le \|f\|$. Applying once more Lemma 2.9(c) we obtain that for every $x \in [1,\lambda_\gamma]$ we have

$$\max(|\phi_1(f)(x)|, |\phi_2(f)(x)|) \ge (1+\varepsilon)^{-1} \max(|\phi_1(f)(x)|, |f(x)|) \ge (1+\varepsilon)^{-1}|f(x)|.$$

From this it follows immediately that:

$$(1+\varepsilon)^{-1}\|f\| \leq \|\phi(f)\|.$$

Let us define $\psi : X \times Y \to C_p([1,\lambda_\gamma],\lambda_\gamma)$ by $\psi(f,g)(x) = \psi_\varepsilon(f(x),g(x))$ where ψ_ε is as in Lemma 2.9. Obviously ϕ is uniformly continuous because ϕ_1 and ϕ_2 are uniformly continuous. The uniform continuity of ψ is also obvious because ψ_ε is uniformly continuous. Let us observe that the equality $\phi\circ\psi = \mathrm{id}_{C_p([1,\lambda_\gamma],\lambda_\gamma)}$ is an easy consequence of Lemma 2.9(a). The identity $\psi\circ\phi = \mathrm{id}_{X\times Y}$ also follows from Lemma 2.9(a) and from the fact that $\phi_1(\psi(f,g)) = f$ for $(f,g)\in X\times Y$, but this is an easy consequence of the equality $\psi_\varepsilon(x,0) = x$ which is a simple application of Lemma 2.9(a) and (b). This finishes the proof of Fact 4.3.

Fact 4.4.

$$C_p([1,\gamma],\gamma) \doteqdot X.$$

Fact 4.5.

$$\prod_{\xi\in\gamma}{}^{*} C_p([\lambda_\xi+1,\lambda_{\xi+1}],\lambda_{\xi+1}) \doteqdot Y.$$

In each case we can find a linear isometry with respect to the sup norm which is also continuous in the topology of pointwise convergence thus uniformly continuous. Namely, we have

$$\Phi : X \to C_p([1,\gamma],\gamma)$$

and

$$\Psi : Y \to \prod_{\xi\in\gamma}{}^{*} C_p([\lambda_\xi+1,\lambda_{\xi+1}],\lambda_{\xi+1})$$

given by the following formulas:

$$\Phi(f)(\xi) = f(\lambda_\xi)$$

and

$$\pi_\xi \circ \Psi(f) \equiv f \,|\, [\lambda_\xi+1,\lambda_{\xi+1}].$$

An easy verification is left to the reader.
As a consequence we get

$$C_p([1,\lambda_\gamma],\lambda_\gamma) \doteqdot C_p([1,\gamma],\gamma) \times \prod_{\xi\in\gamma}{}^{*} C_p([\lambda_\xi+1,\lambda_{\xi+1}],\lambda_{\xi+1}).$$

□

Proof of Theorem 4.1: We will prove this theorem by induction on $\alpha\in[\tau,\tau^+)$ for a given τ. If $\alpha=\tau$ then the thesis is obvious. Let us assume that our theorem holds for $\beta < \alpha$ and that $\alpha \in [\tau \cdot \kappa, \tau \cdot \kappa^+)$ where $\kappa \in [1, \tau] \cap \mathbf{Card}$. We know that, for some $\gamma \in [1, \tau^+)$, $\alpha \in [\tau \cdot \gamma, \tau \cdot (\gamma + 1))$. It is clear that $\alpha = \tau \cdot \gamma + \beta$ where $\beta < \tau$, thus $[1, \alpha]$ and $[1, \tau \cdot \gamma]$ are homeomorphic (see Fact 2.3). Therefore we can assume that $\alpha = \tau \cdot \gamma$. Because $\alpha \in [\tau \cdot \kappa, \tau \cdot \kappa^+)$, $|\gamma| = \kappa$. The case when κ is finite is obvious. Thus we can assume that $\kappa \geq \omega$. If $\gamma = \beta + 1$ then $\tau \cdot \gamma = \tau \cdot \beta + \tau$. Using the fact that $\gamma \geq \omega$ (because $\kappa \geq \omega$) we know that $[1, \tau \cdot \gamma]$ and $[1, \tau \cdot \beta]$ are homeomorphic (see Fact 2.3). Because $|\beta| = |\gamma|$, the thesis for $\alpha = \tau \cdot \gamma$ follows from the inductive assumption. Let $\gamma \in \mathbf{Lim}$. We have $\gamma \in [\kappa, \kappa^+)$. Let us consider a strictly increasing sequence $(\lambda_\xi)_{\xi \in \eta}$ where $\eta = \kappa + \mathrm{cf}(\gamma \setminus \kappa)$ $(\mathrm{cf}(0) = 0)$ such that the function $\xi \mapsto \lambda_\xi$ is continuous and $\lim_{\xi \in \eta} \lambda_\xi = \gamma$. The construction of such a sequence can be like that:

Let $(\lambda'_\xi)_{\xi \in \mathrm{cf}(\gamma \setminus \kappa)}$ be a strictly increasing sequence such that $\xi \mapsto \lambda'_\xi$ is continuous, $\lambda'_0 = \kappa$ and $\lim_{\xi \in \mathrm{cf}(\gamma \setminus \kappa)} \lambda'_\xi = \gamma$. Let us define $(\lambda_\xi)_{\xi \in \eta}$:

$\lambda_\xi = \xi$ for $\xi \leq \kappa$,

$\lambda_{\kappa + \xi} = \lambda'_\xi$ for $\xi < \mathrm{cf}(\gamma \setminus \kappa)$

From Lemma 4.2 from Facts 2.7 and 2.10 we get:

$$C_p([1, \tau \cdot \gamma]) \doteq C_p([1, \eta]) \times \prod_{\xi \in \eta}{}^{*} C_p([\tau \cdot \lambda_\xi + 1, \tau \cdot \lambda_{\xi+1}]). \tag{1}$$

It is obvious that for every $\xi \in \eta$ the set $[\tau \cdot \lambda_\xi + 1, \tau \cdot \lambda_{\xi+1}]$ is isomorphic to $[1, \tau \cdot \beta_\xi]$ (i.e. there exists order preserving bijection) for some $1 \leq \beta_\xi < \gamma$. Therefore from the inductive assumption we know that there exists such a cardinal number $1 \leq \theta_\xi \leq \kappa$ that:

$$C_p([\tau \cdot \lambda_\xi + 1, \tau \cdot \lambda_{\xi+1}]) \doteq C_p([1, \tau \cdot \beta_\xi]) \doteq \prod_{\zeta \in \theta_\xi}{}^{*} C_p([1, \tau]).$$

The above identity and (1) give us:

$$C_p([1, \tau \cdot \gamma]) \doteq C_p([1, \eta]) \times \prod_{\xi \in \eta}{}^{*} \Big(\prod_{\zeta \in \theta_\xi}{}^{*} C_p([1, \tau]) \Big).$$

Thus we have:

$$C_p([1, \tau \cdot \gamma]) \doteq C_p([1, \eta]) \times \prod_{t \in T}{}^{*} C_p([1, \tau])$$

where $T = \bigoplus_{\xi \in \eta} \theta_\xi$. Because $\eta = \kappa + \mathrm{cf}(\gamma \setminus \kappa)$ and $\mathrm{cf}(\gamma \setminus \kappa) \leq \kappa$ (because $|\gamma| = \kappa$) we have $|\eta| = \kappa$. From the fact that $1 \leq \theta_\xi \leq \kappa$ we know that $|T| = \kappa$.
Therefore:

$$C_p([1, \tau \cdot \gamma]) \doteq C_p([1, \eta]) \times \prod_{\xi \in \kappa}{}^{*} C_p([1, \tau]). \tag{2}$$

Let us focus on $C_p([1,\eta])$. Assume that $\kappa = \tau$. We know that if $\mathrm{cf}(\gamma \setminus \kappa) < \kappa$ then $[1,\kappa]$ and $[1,\eta]$ are homeomorphic and we can conclude that $C_p([1,\eta]) \doteq C_p([1,\kappa])$. If $\mathrm{cf}(\gamma \setminus \kappa) = \kappa$ then obviously $C_p([1,\eta]) \doteq C_p([1,\kappa]) \times C_p([1,\kappa])$. Using the assumption that $\kappa = \tau$ we get the thesis of Theorem 4.1 for $\alpha = \tau \cdot \gamma$. If $\kappa < \tau$ then also $\eta < \tau$ and:

$$C_p([1,\tau\cdot\gamma]) \doteq C_p([1,\eta]) \times C_p([1,\tau]) \times \prod_{\xi\in\kappa}{}^{*} C_p([1,\tau]).$$

It is easy to check that

$$C_p([1,\eta]) \times C_p([1,\tau]) \doteq C_p([1,\tau+\eta]) \doteq C_p([1,\tau]).$$

This way we have eliminated the factor $C_p([1,\eta])$ in (2) and this finishes the proof of Theorem 4.1. □

It is natural to ask if the spaces $\prod^*_{\xi\in\kappa} C_p([1,\tau])$ and $\prod^*_{\xi\in\kappa'} C_p([1,\tau])$ are (uniformly) homeomorphic for $\kappa' \neq \kappa$. The following result gives us the answer for finite κ and κ'.

Theorem 4.6 (S.P. Gul'ko [4]). *For every $n,m \in \mathbb{N}$, every regular cardinal number $\tau > \omega$, $\alpha \in [\tau\cdot m, \tau\cdot(m+1))$ and $\beta \in [\tau\cdot n, \tau\cdot(n+1))$, the following is true:*
$C_p([1,\alpha])$ and $C_p([1,\beta])$ are (linearly) homeomorphic iff $n = m$. □

Although Gul'ko proved this theorem for $\tau = \omega_1$ the same proof works if $\tau > \omega$ is regular. The remaining is just an obvious consequence of Theorem 3.1(c) even for all cardinals $n,m < \tau$. It appears that modifying Gul'ko's reasoning we can strengthen the above result:

Theorem 4.7. *For every regular cardinal number $\tau > \omega$, all cardinals $n,m \in \tau$, $\alpha \in [\tau\cdot m, \tau\cdot m^+)$ and $\beta \in [\tau\cdot n, \tau\cdot n^+)$, the following is true:*
$C_p([1,\alpha])$ and $C_p([1,\beta])$ are (linearly) homeomorphic iff $n = m$.

To prove this theorem let us quote the following lemmas from Gul'ko's paper [4] (obviously we assume that τ is regular and uncountable):

Lemma 4.8. *Let $X = \bigcup\{A_\alpha; \alpha\in\tau\}$, $Y = \bigcup\{B_\alpha;\alpha\in\tau\}$ where A_α and B_α are subspaces of X and Y, respectively such that $A_\alpha \subset A_\beta$ and $B_\alpha \subset B_\beta$ if $\alpha < \beta$ and $A_\beta = \bigcup\{A_\alpha;\alpha<\beta\}$, $B_\beta = \bigcup\{B_\alpha;\alpha<\beta\}$ if β is a limit ordinal and $|A_\alpha|, |B_\alpha| < \tau$. Moreover let E and F be dense subspaces of $\mathbb{R}^X$ and $\mathbb{R}^Y$, respectively, and let $T : E \to F$ be a homoeomorphism. Then the set*

$$L = \{\alpha\in\tau; \forall f,g\in E : f\,|\,A_\alpha = g\,|\,A_\alpha \Leftrightarrow Tf\,|\,B_\alpha = Tg\,|\,B_\alpha\}$$

is closed and unbounded in τ $([0,\tau))$.

Lemma 4.9. *Let E and F be linear topological spaces, $T : E \to F$ a homeomorphism and $E = \bigcup\{E_\alpha; \alpha \in \tau\}$, $F = \bigcup\{F_\alpha; \alpha \in \tau\}$, where E_α and F_α are closed subspaces of weight less than $\tau =$ weight $(E) =$ weight (F). Moreover $E_\alpha \subset E_\beta$, $F_\alpha \subset F_\beta$ if $\alpha < \beta$ and $E_\beta = \mathrm{cl}\bigcup\{E_\alpha; \alpha \in \beta\}$, $F_\beta = \mathrm{cl}\bigcup\{F_\alpha; \alpha \in \beta\}$ if β is a limit ordinal. Then the set*

$$M = \{\alpha \in \tau; T(E_\alpha) = F_\alpha\}$$

is closed and unbounded in τ $([0,\tau))$.

Although both lemmas were formulated for $\tau = \omega_1$ the proofs remain the same as in Gul'ko's paper.

Proof of Theorem 4.7: The proof will be based on the original idea of Gul'ko from [4]. Let us assume that both $C_p([1,\alpha])$ and $C_p([1,\beta])$ are homeomorphic. We can also assume without loss of generality that m, n are infinite (see Theorem 4.6). According to Theorem 4.1 we have:

$$E = \prod_{\theta\in m}{}^{*} C_p([1,\tau],\tau) \simeq C_p([1,\alpha]) \approx C_p([1,\beta]) \simeq \prod_{\theta\in n}{}^{*} C_p([1,\tau],\tau) = F.$$

Let $T : E \to F$ be a homeomorphism. We can assume, without loss of generality, that $T(0) = 0$. Obviously E and F can be identified in the natural way with the dense subsets of $\mathbb{R}^{[1,\tau)\times m}$, $\mathbb{R}^{[1,\tau)\times n}$, respectively. Let us define $A_\gamma = \{(\eta,\zeta) \in [1,\tau) \times m; \eta < \gamma\}$, $B_\gamma = \{(\eta,\zeta) \in [1,\tau) \times n; \eta < \gamma\}$ and $E_\gamma = \{(f_\zeta)_{\zeta\in m} \in \prod^*_{\theta\in m} C_p([1,\tau]); \forall\, \zeta \in m\ f_\zeta \,|\, [\gamma,\tau] \equiv 0\}$, $F_\gamma = \{(f_\zeta)_{\zeta\in n} \in \prod^*_{\theta\in n} C_p([1,\tau]); \forall\, \zeta \in n\ f_\zeta \,|\, [\gamma,\tau] \equiv 0\}$ for $\gamma < \tau$. It is easy to check that for the above defined sets we can apply Lemma 4.8 and Lemma 4.9 where L and M are defined as in these lemmas. Take $\xi \in M \cap L \cap \mathbf{Lim}$. Let us put $G_\xi = \{(f_\zeta)_{\zeta\in m} \in E; \forall\, \zeta \in m\ f_\zeta(\xi) = 0\}$ and $H_\xi = \{(f_\zeta)_{\zeta\in n} \in F; \forall\, \zeta \in n\ f_\zeta(\xi) = 0\}$. Observe that $T(G_\xi) = H_\xi$. Indeed, let us take $(f_\zeta)_{\zeta\in m} \in G_\xi$. Define $f'_\zeta \,|\, [1,\xi) \equiv f_\zeta \,|\, [1,\xi)$ and $f'_\zeta \,|\, [\xi,\tau] \equiv 0$. Since $\xi \in M$ we have that $T((f'_\zeta)_{\zeta\in m}) = (g'_\zeta)_{\zeta\in n} \in F_\xi$. But $\xi \in L$; therefore $g_\zeta(x) = g'_\zeta(x)$ for $x < \xi$ and because $\xi \in \mathbf{Lim}$ we have $g_\zeta(\xi) = g'_\zeta(\xi) = 0$ where $(g_\zeta)_{\zeta\in n} = T((f_\zeta)_{\zeta\in m})$, hence $T((f_\zeta)_{\zeta\in m}) \in H_\xi$. Analogically we prove that $T^{-1}(H_\xi) \subset G_\xi$. To finish the proof we need

Lemma 4.10. *Let E be a dense subset of $\mathbb{R}^X$ and $T : E \to \mathbb{R}^Y$ be a continuous function. Then for every infinite subset $B \subset Y$ there is a set A of cardinality not greater than cardinality of B such that for every $f, g \in E$, $f \,|\, A = g \,|\, A \Rightarrow Tf \,|\, B = Tg \,|\, B$.*

This lemma is just an immediate consequence of the Factorization lemma (see [1, 0.2.3]). Let us consider $C^m_\xi = \{(\xi,\theta); \theta \in m\}$ and $C^n_\xi = \{(\xi,\theta); \theta \in n\}$.

Applying the above lemma for C_ξ^n we get the set $A \subset [1,\tau)\times m$ of cardinality not greater than n such that for every $f,g\in E$, $f\,|\,A = g\,|\,A \Rightarrow Tf\,|\,C_\xi^n = Tg\,|\,C_\xi^n$. If $m>n$ then there exists x such that $[1,\tau)\times\{x\}\cap A=\emptyset$. Let us take $f=(f_\zeta)_{\zeta\in m}\in E$ such that $f_\zeta\equiv 0$ for $\zeta\neq x$ and $f_\zeta\equiv 1$ for $\zeta=x$. Obviously $f\notin G_\xi$ but from the choice of x we know that $Tf\in H_\xi$ which contradicts the equality $T(G_\xi)=H_\xi$. Therefore we proved that $m\le n$. Analogically we prove that $m\ge n$. □

5. Summary

Summarizing the results of the previous chapter we can prove the following:

Theorem 5.1. *Let α,β,γ are ordinals satisfying $\alpha\le\gamma\le\beta$. If $C_p([1,\alpha])$ and $C_p([1,\beta])$ are (uniformly) homeomorphic then also $C_p([1,\gamma])$ is (uniformly) homeomorphic to $C_p([1,\alpha])$ and $C_p([1,\beta])$.*

Proof: It is clear that α,β,γ have the same cardinality, say κ (see Theorem 2.2). We can assume that $\kappa\ge\omega$. From Theorem 3.1(b) and Theorem 4.6, without loss of generality, we can assume that $\kappa\cdot\omega\le\beta$. From Theorem 4.1 we get

$$C_p([1,\alpha]) \doteq \prod_{\eta\in\kappa_\alpha}{}^{*} C_p([1,\kappa])$$

$$C_p([1,\beta]) \doteq \prod_{\eta\in\kappa_\beta}{}^{*} C_p([1,\kappa])$$

$$C_p([1,\gamma]) \doteq \prod_{\eta\in\kappa_\gamma}{}^{*} C_p([1,\kappa])$$

where κ_β, κ_α are κ_γ cardinals, $\kappa_\beta\in[\omega,\kappa^+)$ and $\kappa_\alpha,\kappa_\gamma\in[1,\kappa^+)$. Moreover we know that $\kappa_\alpha\le\kappa_\gamma\le\kappa_\beta$ but, without loss of generality, we can exclude all the equalities. We have

$$\begin{aligned}
C_p([1,\gamma]) &\doteq \prod_{\eta\in\kappa_\gamma}{}^{*} C_p([1,\kappa]) \doteq \prod_{\eta\in\kappa_\gamma\setminus\kappa_\alpha}{}^{*} C_p([1,\kappa]) \times \prod_{\eta\in\kappa_\alpha}{}^{*} C_p([1,\kappa])\\
&\doteq \Big(\prod_{\eta\in\kappa_\gamma\setminus\kappa_\alpha}{}^{*} C_p([1,\kappa])\Big)\times C_p([1,\alpha])\\
(\doteq)\approx{}& \Big(\prod_{\eta\in\kappa_\gamma\setminus\kappa_\alpha}{}^{*} C_p([1,\kappa])\Big)\times C_p([1,\beta])\\
&\doteq \prod_{\eta\in\kappa_\gamma\setminus\kappa_\alpha}{}^{*} C_p([1,\kappa]) \times \prod_{\eta\in\kappa_\beta}{}^{*} C_p([1,\kappa]).
\end{aligned}$$

Because $|\kappa_\gamma\setminus\kappa_\alpha|\le\kappa_\beta$ and $\kappa_\beta\ge\omega$, we have

$$\prod_{\eta\in\kappa_\gamma\setminus\kappa_\alpha}{}^{*} C_p([1,\kappa]) \times \prod_{\eta\in\kappa_\beta}{}^{*} C_p([1,\kappa]) \doteq \prod_{\eta\in\kappa_\beta}{}^{*} C_p([1,\kappa]) \doteq C_p([1,\beta]).$$

Thus $C_p([1,\beta])$ and $C_p([1,\gamma])$ are (uniformly) homeomorphic. $\square$

Now we can prove Theorem 1.1.

Proof of Theorem 1.1: Let us observe that (a) follows from Theorem 2.2. Assume that $\alpha, \beta \geq \kappa^2$. As a consequence of Theorem 4.1 we get that $C_p([1,\alpha])$ and $C_p([1,\beta])$ are uniformly homeomorphic. If $\kappa = \omega$ or κ is singular then, from Theorem 3.1(b), we have that, for $\alpha, \beta \in [\kappa, \kappa^2]$, $C_p([1,\alpha])$ and $C_p([1,\alpha])$ are even linearly homeomorphic. Therefore the proof of (b) is completed. The remaining case (c) is just Theorem 4.7. $\square$

All the above results do not give us a complete classification of spaces $C_p([1,\alpha])$ for $\alpha \in \mathbf{Ord}$ up to t and u-equivalence what was mentioned before. To complete the classification we have to consider the case $\alpha = (\tau^+)^2$ and $\beta = \tau^+ \cdot \tau$ where τ is a cardinal. All the rest is covered by Theorem 5.1 and Theorem 1.1. Moreover one can see that in all settled cases t and u equivalences coincide. But it is still unknown whether it is true for the whole class of spaces $[1,\alpha]$ where $\alpha \in \mathbf{Ord}$ or even for the class of all compacta.

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