Property D and pseudonormality in first countable spaces

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Abstract. In answer to a question of M. Reed, E. van Douwen and M. Wage [vDW79] constructed an example of a Moore space which had property D but was not pseudonormal. Their construction used the Martin's Axiom type principle $P(c)$. We show that there is no such space in the usual Cohen model of the failure of CH.

Keywords: property D, pseudonormal, first countable, Cohen model

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1. Introduction

A space is *pseudonormal* if any pair of disjoint closed sets, one of which is countable, can be separated by disjoint open sets. A family of subsets of a space X is said to be *discrete*, if the sets have pairwise disjoint closures and the family is locally finite. A space has *property D* if every countable closed discrete set can be separated by a discrete family of open sets. It is easy to see that every Hausdorff pseudonormal (hence regular) space will have property D. As mentioned above, van Douwen and Wage [vDW79] showed that it is consistent that there is a Moore space with property D which is not pseudonormal. John Porter and P. Nyikos have shown that there are ZFC examples of spaces which have property D and which are not pseudonormal. They have asked if there can be a first countable such example. We establish in this paper that there is no such example in the Cohen model. The reader is referred to Kunen's book [Kun83] for the necessary background on Cohen forcing.

2. First countable spaces with property D in the Cohen model

We will need many well-known facts about reflection and forcing with Cohen reals. Most of them can be found in Kunen's book [kunen] and for other facts we refer the reader to the survey [Dow92]. The proof is a somewhat standard reflection and forcing style argument.

The Cohen forcing poset for adding ω_2 Cohen reals is denoted as $\mathrm{Fn}(\omega_2, \omega)$ and consists of all finite functions into ω with domain contained in ω_2. In general, $\mathrm{Fn}(I, \omega)$ consists of all finite functions into ω with domain contained in I. The elements are ordered by $p \leq q$ if $p \supseteq q$.

Recall that if $J \subset I$, then the poset $\mathrm{Fn}(I,\omega)$ is forcing isomorphic to the iteration (or product) $\mathrm{Fn}(J,\omega) * \mathrm{Fn}(I \setminus J,\omega)$. Therefore if G is a generic filter for $\mathrm{Fn}(\omega_2,\omega)$ over the model V, and $J \subset \omega_2$, then the model $V[G]$ is equal to the model obtained by forcing with $\mathrm{Fn}(\omega_2 \setminus J,\omega)$ over the inner model $V[G \cap \mathrm{Fn}(J,\omega)]$.

Theorem 1. *It is consistent that every first countable regular space with property D is pseudonormal.*

Proof: Let V be a model of CH and let G be $\mathrm{Fn}(\omega_2,\omega)$-generic over V. In $V[G]$ assume that X is a first countable space and that Q is a countable closed subset of X. Let F denote a closed subset of X which is disjoint from Q and we will show that Q and F can be separated by disjoint open sets.

For each $x \in X$, let $\{U(x,n) : n \in \omega\}$ denote a countable base of open sets for x in the topology on X. For each $q \in Q$, we may assume that the intersection of $\overline{U(q,0)}$ and F is empty and that $\overline{U(q,1)} \subset U(q,0)$. Let $\{q_n : n \in \omega\}$ be an enumeration of Q, and for each $n \in \omega$, let $W_n = \bigcup_{k \leq n} U(q_k,1)$. If there is an $n \in \omega$ such that $Q \subset \overline{W_n}$, then it follows that Q and F can be separated. So we may assume that $A_n = Q \setminus \overline{W_n}$ is infinite for each $n \in \omega$. If $f : \omega \mapsto Q$ is any function such that $f(n) \in A_n$ for each n, i.e. $f \in \Pi_n A_n$, then $D_f = \{f(n) : n \in \omega\}$ is a closed discrete subset of X. Therefore there is a function h_f from ω to $\omega \setminus 2$ such that the family

$$\{U(f(n),h_f(n)) : n \in \omega\}$$

is a discrete family. In particular, $F \cap \overline{\bigcup_n U(f(n),h_f(n))}$ is empty.

There is no loss of generality if we assume that the base set for X is some set in V (e.g. an ordinal). In addition, we may assume that the indexing $\{q_n : n \in \omega\}$ for Q is an element of V.

Working in V now, we may choose $\mathrm{Fn}(\omega_2,\omega)$-names for each of F, $\{W_n : n \in \omega\}$ and the collection $\mathcal{U} = \{\{U(x,n) : n \in \omega\} : x \in X\}$ and let $p' \in G \subset \mathrm{Fn}(\omega_2,\omega)$ be any condition which forces the relations outlined in the previous paragraphs will hold. Let M be an elementary submodel of $H(\theta)$ for a suitably large θ so that p' and each of these names are elements of M. Since CH holds in V, we may choose M so that $M^\omega \subset M$ and $|M| = \omega_1$. With these assumptions it follows that $M \cap \omega_2$ will be some ordinal λ with cofinality ω_1. Let G_λ denote the set $G \cap \mathrm{Fn}(\lambda,\omega) = G \cap M$.

It is well known that for each $x \in X \cap M$ and each integer n, there is a $\mathrm{Fn}(\lambda,\omega)$-name $\dot{U}'(x,n)$ such that for each $y \in X \cap M$

$$y \in \mathrm{val}_{G_\lambda}(\dot{U}'(x,n)) \text{ iff } y \in \mathrm{val}_G(\dot{U}(x,n)).$$

Similarly, there are $\mathrm{Fn}(\lambda,\omega)$-names, $\dot{F}'$ and $\dot{W}'_n$ $(n \in \omega)$, so that for each $y \in X \cap M$,

$$y \in \mathrm{val}_{G_\lambda}(\dot{F}') \text{ iff } y \in \mathrm{val}_G(\dot{F})$$

and

$$y \in \mathrm{val}_{G_\lambda} \dot{W}'_n \ \text{ iff } \ y \in \mathrm{val}_G(\dot{W}_n).$$

We now work in the model $V[G_\lambda]$ and consider the forcing $\mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$. Note that since G is a coherent family of functions from ω_2 into ω, we will have that $\bigcup G$ is a function from ω_2 into ω. The function $g : \omega \mapsto \omega$ which is defined by $g(n) = \bigcup G\ (\lambda + n)$ is usually thought of as the "λ-th" Cohen real added by G. For each n, the set $Q \setminus W'_n = A_n$ is a member of $V[G_\lambda]$ and can be enumerated as $\{a(n,m) : m \in \omega\}$. We let $\dot{f}$ denote the canonical name of the element of $\Pi_n A_n$ which satisfies $\dot{f}(n) = a(n, g(n))$ for each n. Recall that there is also a name $\dot{h}_f$ which satisfies that, in $V[G]$,

$$F \cap \overline{\bigcup_n U(f(n), h_f(n))} = \emptyset.$$

We may assume that for each $p \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$,

$$p \Vdash \dot{F} \cap \overline{\bigcup_n \dot{U}(f(n), h_f(n))} = \emptyset.$$

For each $p \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$, let

$$U_p = \bigcup \{U'(q,m) : (\exists\, q \leq p, \exists\, n \in \omega)\ q \Vdash f(n) = q \ \text{ and } \ h_f(n) = m\}.$$

For each $p \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$, there is some $n_p = n$ such that $\mathrm{dom}(p) \cap [\lambda, \lambda + \omega) \subset [\lambda, \lambda + n]$. It follows then, that for each $p \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$, $Q \subset W_{n_p} \cup U_p$.

For each $x \in F'$ and $p \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$, there are $p_x \leq p \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$ and $n_x \in \omega$ such that $p_x \Vdash \dot{U}(x, n_x) \cap \bigcup_n \dot{U}(\dot{f}(n), \dot{h}_f(n))$ is empty since $1 \Vdash x \notin \overline{\bigcup_n \dot{U}(\dot{f}(n), \dot{h}_f(n))}$.

Since $\mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$ is ccc, there is a countable subset J of $\omega_2 \setminus \lambda$ such that for each $p \in \mathrm{Fn}(\omega_2 \setminus J, \omega)$, each $q \in Q$, and integers n, m, if

$$p \Vdash \dot{f}(n) = q \ \text{ and } \ \dot{h}_f(n) = m \quad \text{iff} \quad p \restriction J \Vdash \dot{f}(n) = q \ \text{ and } \ \dot{h}_f(n) = m.$$

Let $\{p_n : n \in \omega\}$ enumerate $\mathrm{Fn}(J, \omega)$ and for each n, let $h(n)$ be a large enough integer such that the closure of $U(q_n, h(n))$ is contained in $W_{n_{p_k}} \cup U_{p_k}$ for each $k \leq n$. Therefore the function h is in $V[G_\lambda]$ and, since $M^\omega \subset M$, there is a name, $\dot{h}$, for h such that $\dot{h}$ is in M. Furthermore, h is a member of $M[G_\lambda]$. By [Dow92, 4.5], $M[G_\lambda]$ is an elementary submodel of $H(\theta)[G]$. Observe that $H(\theta)[G] \models F \cap M[G_\lambda] = F'$ and that $F \in M[G_\lambda]$.

The proof will finish, in $V[G]$, by showing that

$$M[G_\lambda] \models F \cap \overline{\bigcup_n U(q_n, h(n))} = \emptyset$$

and concluding, by elementarity, that

$$H(\theta)[G] \models F \cap \overline{\bigcup_n U(q_n, h(n))} = \emptyset.$$

To show this, consider any $x \in F'$ and work in $V[G_\lambda]$. By our assumptions we know there is some $p_x \in \mathrm{Fn}(\omega_2 \setminus \lambda, \omega)$ such that x is not in the closure of U_{p_x}. Since the definition of U_{p_x} only depends on $\dot{h}_f$, it follows that we may assume that $p_x \in \mathrm{Fn}(J, \omega)$. Therefore there is some k such that $p_x = p_k$. Since $U(q_n, h(n)) \subset U_{p_x}$ for all $n > k$, it follows that x is not in the closure of $\bigcup\{U'(q_n, h(n)) : n > k\}$. In addition, x is not in the closure of $U'(q_m, h(m))$ for $m \leq k$ since $h(m) > 0$. Fix any m such that $U'(x, m) \cap \bigcup\{U'(q_n, h(n)) : n \in \omega\}$ is empty and recall that it follows then that $M[G_\lambda] \models U(x, m) \cap \bigcup\{U(q_n, h(n)) : n \in \omega\}$ is empty. Since this holds for each $x \in F \cap M$, we have proven that $M[G_\lambda] \models F \cap \overline{\bigcup\{U(q_n, h(n)) : n \in \omega\}}$ is empty and finished the proof. □

References

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