On the range of a closed operator in an L_1-space of vector-valued functions

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Abstract. Let X be a reflexive Banach space and A be a closed operator in an L_1-space of X-valued functions. Then we characterize the range $R(A)$ of A as follows. Let $0 \neq \lambda_n \in \rho(A)$ for all $1 \le n < \infty$, where $\rho(A)$ denotes the resolvent set of A, and assume that $\lim_{n\to\infty} \lambda_n = 0$ and $\sup_{n\ge 1} \|\lambda_n(\lambda_n - A)^{-1}\| < \infty$. Furthermore, assume that there exists $\lambda_\infty \in \rho(A)$ such that $\|\lambda_\infty(\lambda_\infty - A)^{-1}\| \le 1$. Then $f \in R(A)$ is equivalent to $\sup_{n\ge 1} \|(\lambda_n - A)^{-1} f\|_1 < \infty$. This generalizes Shaw's result for scalar-valued functions.

Keywords: reflexive Banach space, L_1-space of vector-valued functions, closed operator, resolvent set, range and domain, linear contraction, C_0-semigroup, strongly continuous cosine family of operators

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1. Introduction

Let A be a (bounded or unbounded) closed operator in a Banach space Y with range $R(A)$ and domain $D(A)$. By assuming that the resolvent set $\rho(A)$ of A includes a countable set $\{\lambda_n : n \ge 1\}$, with $\lambda_n \neq 0$ for all $n \ge 1$, such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sup_{n\ge 1} \|\lambda_n(\lambda_n - A)^{-1}\| < \infty$, it was shown in [11] that the obviously necessary condition

$$\sup_{n\ge 1} \|(\lambda_n - A)^{-1} x\| < \infty$$

is sufficient for an element x of Y to be in the range $R(A)$ of A when Y is reflexive. This can be regarded as a generalization of a result of Browder [2]; motivated by a result of Gottschalk and Hedlund (cf. Theorem 14.11 in [5]), he studied the problem of finding a necessary and sufficient condition for an element x of a Banach space Y to be in the range of $T - I$ when T is power-bounded on Y, and proved that the obviously necessary condition

$$(*) \qquad \sup_{n\ge 1} \left\| \sum_{k=0}^{n-1} T^k x \right\| < \infty$$

is sufficient when Y is reflexive.

It was shown in [8] that also for T a contraction of $L_1(\mu)$ condition $(*)$ implies $x \in R(T-I)$, and in [6] an analogue for semigroups of contractions in $L_1(\mu)$ was proved. A unified treatment of these two results was given in [11]. The problem whether in $L_1(\mu)$ the norm condition $\|T\| \le 1$ can be replaced by power-boundedness is still unresolved; a partial answer was given in [1].

In this paper we treat the case of operators in the space $L_1((\Omega, \mathcal{B}, \mu); X)$ of vector-valued norm-integrable functions on a σ-finite measure space $(\Omega, \mathcal{B}, \mu)$, with values in a *reflexive* Banach space X. The main result (Theorem 1) is the vector-valued version of [11]. The applications extend accordingly the results of [8], [6] and [11].

Let $(X, \|\cdot\|_X)$ be a reflexive Banach space, and $(\Omega, \mathcal{B}, \mu)$ be a σ-finite measure space. For $1 \le p \le \infty$, let $L_p(\Omega; X) = L_p((\Omega, \mathcal{B}, \mu); X)$ denote the usual Banach space of all X-valued strongly measurable functions f on Ω with the norm

$$\|f\|_p := \left(\int \|f(\omega)\|_X^p \, d\mu(\omega) \right)^{1/p} < \infty \qquad \text{if} \quad 1 \le p < \infty,$$
$$\|f\|_\infty := \operatorname{ess\,sup}\{\|f(\omega)\|_X : \omega \in \Omega\} < \infty \qquad \text{if} \quad p = \infty.$$

We consider a closed operator A in $L_1(\Omega; X)$ with range $R(A)$ and domain $D(A)$. We assume that the resolvent set $\rho(A)$ of A includes a countable set $\{\lambda_n : n \ge 1\}$, with $\lambda_n \ne 0$ for all $n \ge 1$, such that $\lim_{n\to\infty} \lambda_n = 0$ and $\sup_{n\ge 1} \|\lambda_n(\lambda_n - A)^{-1}\| < \infty$. Then we prove that $\sup_{n \ge 1} \|(\lambda_n - A)^{-1} f\|_1 < \infty$ implies $f \in R(A)$, under the additional hypothesis that there exists $\lambda_\infty \in \rho(A)$ such that $\|\lambda_\infty(\lambda_\infty - A)^{-1}\| \le 1$. It would be interesting to ask whether this implication holds without the additional hypothesis. Concerning the problem the author would like to note that Assani [1] considered a power-bounded linear operator T on L_1 of scalar-valued functions, and under the hypothesis that

$$(**) \qquad \lim_{n\to\infty} h_n = 0 \quad \text{a.e.} \qquad \text{implies} \qquad \lim_{n\to\infty} Th_n = 0 \quad \text{a.e.},$$

he proved that $\sup_{n\ge 1} \|\sum_{k=1}^n T^k f\|_1 < \infty$ is equivalent to $f \in R(T-I)$. It seems to the author that it is an open problem to prove this equivalence relation without assuming condition $(**)$. (See also [10], where similar results are proved for vector-valued functions.)

As applications of the result we characterize the range $R(A)$ of A, where A is the generator of a discrete semigroup $\{T^n : n \ge 0\}$, or a C_0-semigroup $\{T(t) : t \ge 0\}$, or a strongly continuous cosine family $\{C(t) : -\infty < t < \infty\}$ of linear contractions on $L_1(\Omega; X)$. The results obtained below generalize Shaw's results (see [11, Corollaries 4, 6, and 8]) for scalar-valued functions. See also Lin and Sine [8], Krengel and Lin [6] for related topics.

2. The range of a closed operator in $L_1(\Omega; X)$

The following theorem is our main result.

Theorem 1 (cf. Theorem 2 of Shaw [11]). *Let X be a reflexive Banach space, and A be a closed operator in $L_1(\Omega; X)$ with domain $D(A)$ and range $R(A)$. Let $\rho(A)$ denote the resolvent set of A, and assume that $0 \neq \lambda_n \in \rho(A)$ for all $n \geq 1$ and $\lim_{n\to\infty} \lambda_n = 0$. If $M := \sup_{n\geq 1} \|\lambda_n(\lambda_n - A)^{-1}\| < \infty$, and there exists $\lambda_\infty \in \rho(A)$ such that $\|\lambda_\infty(\lambda_\infty - A)^{-1}\| \leq 1$, then the following conditions are equivalent for $f \in L_1(\Omega; X)$.*

(I) $\sup_{n\geq 1} \|(\lambda_n - A)^{-1} f\|_1 < \infty$.

(II) $f \in \overline{R}(A)$.

To prove this theorem we need the following lemma, which may be regarded as a generalization of the Yosida-Hewitt theorem on vector-measures (see [3, p. 30, Theorem I.5.8]).

Lemma 1. *Let X be a reflexive Banach space, and let $\ell \in L_\infty(\Omega; X^*)^*$ $(= L_1(\Omega; X)^{**})$. Then there exist unique ℓ_c and ℓ_p in $L_\infty(\Omega; X^*)^*$ such that*

(a) *$\ell = \ell_c + \ell_p$, and $\|\ell\| = \|\ell_c\| + \|\ell_p\|$;*

(b) *there exists $g \in L_1(\Omega; X)$ with*

(1) $$\ell_c(f) = \int_\Omega \langle g(\omega), f^*(\omega)\rangle \, d\mu(\omega) \quad \text{for all } \; f^* \in L_\infty(\Omega; X^*);$$

(c) *if we define a scalar-valued function G_{x^*} on $\mathcal{B}$ for each $x^* \in X^*$ by*

(2) $$G_{x^*}(B) := \ell_p(\chi_B(\cdot)x^*) \quad (B \in \mathcal{B}),$$

then G_{x^*} is a purely finitely additive measure on $\mathcal{B}$, i.e., there is no (countably additive) measure λ on $\mathcal{B}$ satisfying $0 \leq \lambda(B) \leq |G_{x^*}|(B)$ for all $B \in \mathcal{B}$, where $|G_{x^*}|$ denotes the variation of G_{x^*} (cf. [3, p. 2]).

Proof: For $B \in \mathcal{B}$, define a linear functional $F(B)$ on X^* by

(3) $$F(B)(x^*) := \ell(\chi_B(\cdot)x^*) \qquad (x^* \in X^*).$$

Since $|F(B)(x^*)| \leq \|\ell\|\|x^*\|$, it follows that $\|F(B)\| \leq \|\ell\|$. Thus we may regard $F(B)$ as an element of $X^{**} = X$, and hence we can write

(4) $$\langle F(B), x^*\rangle = \ell(\chi_B(\cdot)x^*) \qquad (B \in \mathcal{B}, x^* \in X^*).$$

Clearly, $F : \mathcal{B} \to X$ is finitely additive. To see that F is a finitely additive vector-measure of bounded variation, let $\{B_1, \dots, B_n\}$ be a finite measurable decomposition of Ω, and $x_j^* \in X^* (j = 1, \dots, n)$ be such that $\|x_j^*\| \leq 1$. Then

$$\left|\sum_{j=1}^n \langle F(B_j), x_j^*\rangle\right| = \left|\ell\Big(\sum_{j=1}^n \chi_{B_j}(\cdot)x_j^*\Big)\right| \leq \|\ell\| \cdot \left\|\sum_{j=1}^n \chi_{B_j}(\cdot)x_j^*\right\|_{L_\infty(\Omega;X^*)} \leq \|\ell\|,$$

so that $\sum_{j=1}^{n} \|F(B_j)\| \leq \|\ell\|$. Therefore, F is of bounded variation. Let $|F|$ denote the variation of F. Then, since $|F|(\Omega) \leq \|\ell\|$, it follows from Corollary I.5.3 of [3] that F is strongly additive; and hence by the Yosida-Hewitt theorem (cf. [3, p. 30, Theorem I.5.8]), there exist unique strongly additive X-valued measures F_c and F_p on $\mathcal{B}$ (which are of bounded variation) such that

(i) F_c is countably additive;
(ii) for each $x^* \in X^*$, $x^* F_p$ is purely finitely additive on $\mathcal{B}$;
(iii) $F = F_c + F_p$;
(iv) F_c and F_p are mutually singular, i.e., for each $\epsilon > 0$ there exists $E \in \mathcal{B}$ such that $|F_c|(\Omega \setminus E) + |F_p|(E) < \epsilon$;
(v) $|F| = |F_c| + |F_p|$.

Since X has the Radon-Nikodym property (cf. [3, p. 82, Corollary III.3.4]), there exists $g \in L_1(\Omega; X)$ such that $F_c(B) = \int_B g\, d\mu$ for all $B \in \mathcal{B}$. Using this g, we define a linear functional ℓ_c on $L_\infty(\Omega; X^*)$ by

$$\ell_c(f) := \int_\Omega \langle g(\omega),\, f^*(\omega) \rangle \, d\mu(\omega) \qquad (f^* \in L_\infty(\Omega; X^*)).$$

It is clear that $\ell_c \in L_\infty(\Omega; X^*)^*$ and $\|\ell_c\| = \|g\|_1$. We then put

$$\ell_p := \ell - \ell_c,$$

so that $\ell_p \in L_\infty(\Omega; X^*)^*$ and $\ell = \ell_c + \ell_p$. Let $x^* \in X^*$ and $B \in \mathcal{B}$. Then, by (2) and (3),

$$\begin{aligned} G_{x^*}(B) &= \ell_p(\chi_B(\cdot)x^*) = (\ell - \ell_c)(\chi_B(\cdot)x^*) = \ell(\chi_B(\cdot)x^*) - \ell_c(\chi_B(\cdot)x^*) \\ &= \langle F(B),\, x^* \rangle - \langle F_c(B),\, x^* \rangle = \langle F_p(B),\, x^* \rangle, \end{aligned}$$

which implies that $x^* F_p = G_{x^*}$ on $\mathcal{B}$ for each $x^* \in X^*$. Thus, G_{x^*} is purely finitely additive on $\mathcal{B}$ by (ii).

Next, we prove that $\|\ell\| = \|\ell_c\| + \|\ell_p\|$. To do this, let $\epsilon > 0$. Then, by (iv) there exists $E \in \mathcal{B}$ such that

$$|F_c|(\Omega \setminus E) + |F_p|(E) < \epsilon. \tag{5}$$

Since the set of all countably X^*-valued functions in $L_\infty(\Omega; X^*)$ is a dense subset of $L_\infty(\Omega; X^*)$, there exists $f_1^* \in L_\infty(\Omega; X^*)$ of the form

$$f_1^* = \sum_{n=1}^{\infty} \chi_{B_n}(\cdot) x_n^*,$$

where $x_n^* \in X^*$, $\|x_n^*\| \leq 1$, and $\{B_n : n \geq 1\}$ is a countable measurable decomposition of Ω, such that

$$|\ell_p(f_1^*)| > \|\ell_p\| - \epsilon. \tag{6}$$

Write $E_n = E \cap (\bigcup_{j=1}^{n} B_j)$. It follows that $E_n \uparrow E$ as $n \to \infty$, and

$$|F_c|(\Omega \setminus E) = \int_{\Omega \setminus E} \|g(\omega)\|_X \, d\mu(\omega)$$
$$= \lim_{n\to\infty} \int_{\Omega \setminus E_n} \|g(\omega)\|_X \, d\mu(\omega) = \lim_{n\to\infty} |F_c|(\Omega \setminus E_n).$$

Hence, we can choose E_N satisfying (5) with E_N in place of E. Then we can choose $f_2^* \in L_\infty(\Omega; X^*)$ of the form

$$f_2^* = \sum_{j=1}^{\infty} \chi_{D_j}(\cdot) y_j^*,$$

where $y_j^* \in X^*$, $\|y_j^*\| \le 1$, and $\{D_j : j \ge 1\}$ is a countable measurable decomposition of the set E_N, such that

$$(7) \qquad |\ell_c(f_2^*)| = \left| \int_{E_N} \langle g(\omega),\, f_2^*(\omega) \rangle \, d\mu(\omega) \right| > \int_{E_N} \|g(\omega)\|_X \, d\mu(\omega) - \epsilon.$$

Lastly, define an X^*-valued function f^* on Ω by

$$f^*(\omega) = \begin{cases} f_1^*(\omega) & \text{if } \omega \in \Omega \setminus E_N, \\ f_2^*(\omega) & \text{if } \omega \in E_N. \end{cases}$$

It is clear that $f^* \in L_\infty(\Omega; X^*)$ and $\|f^*\|_\infty \le 1$. Furthermore, by (5) with E_N in place of E, (6) and (7),

$$|\ell(f^*)| = |\ell_c(f_2^*) + \ell_p(f_1^*) + \ell_c(\chi_{\Omega\setminus E_N} f_1^*) + \ell_p(f_2^*) - \ell_p(\chi_{E_N} f_1^*)|$$
$$> \left(\int_{E_N} \|g(\omega)\|_X \, d\mu(\omega) - \epsilon \right) + (\|\ell_p\| - \epsilon) - |\ell_c(\chi_{\Omega\setminus E_N} f_1^*)|$$
$$- |\ell_p(f_2^*)| - |\ell_p(\chi_{E_N} f^*)|$$
$$> (\|\ell_c\| - 2\epsilon) + (\|\ell_p\| - \epsilon) - \epsilon - \epsilon - \epsilon = \|\ell_c\| + \|\ell_p\| - 6\epsilon.$$

Since ϵ was arbitrary, this proves $\|\ell\| \ge \|\ell_c\| + \|\ell_p\|$. Consequently, $\|\ell\| = \|\ell_c\| + \|\ell_p\|$.

The uniqueness of the decomposition $\ell = \ell_c + \ell_p$ follows from the uniqueness of the decomposition $F = F_c + F_p$, and this completes the proof. □

Proof of Theorem 1: (I) ⇒ (II). We may assume here that $\lambda_\infty \ne 0$, because $\lambda_\infty = 0$ implies $R(A) = L_1(\Omega; X)$. Since $\{(\lambda_n - A)^{-1} f : n \ge 1\}$ is a bounded subset of the dual space of $L_\infty(\Omega; X^*)$, it is relatively compact with respect to the weak*-topology. It follows that there exists $\eta \in L_1(\Omega; X)^{**}$ which is a weak*-cluster point of the sequence $\{(\lambda_n - A)^{-1} f\}_{n=1}^{\infty}$.

Let $u \in L_\infty(\Omega; X^*)$ and $0 \neq \lambda \in \rho(A)$. Then there exists a subsequence $\{n_j\}_{j=1}^{\infty}$ of the sequence $\{n\}_{n=1}^{\infty}$ such that

$$\begin{aligned}\langle (\lambda(\lambda - A)^{-1})^{**}\eta,\, u\rangle &= \langle \eta,\, (\lambda(\lambda - A)^{-1})^{*} u\rangle \\ &= \lim_{j\to\infty} \langle (\lambda_{n_j} - A)^{-1} f,\, (\lambda(\lambda - A)^{-1})^{*} u\rangle \\ &= \lim_{j\to\infty} \langle \lambda(\lambda - A)^{-1}(\lambda_{n_j} - A)^{-1} f,\, u\rangle \\ &= \lim_{j\to\infty} \frac{\lambda}{\lambda - \lambda_{n_j}} \cdot \langle (\lambda_{n_j} - A)^{-1} f - (\lambda - A)^{-1} f,\, u\rangle \\ &= \langle \eta,\, u\rangle - \langle (\lambda - A)^{-1} f,\, u\rangle,\end{aligned}$$

where the last but one equality is due to the resolvent equation. Consequently, we obtain that

$$(\lambda(\lambda - A)^{-1})^{**}\eta = \eta - (\lambda - A)^{-1} f \qquad (\lambda \in \rho(A),\, \lambda \neq 0). \tag{8}$$

Here, we apply Lemma 1 for η as follows. By Lemma 1, there exist unique η_c and η_p in $L_1(\Omega; X)^{**}$ such that

(i) there exists $g \in L_1(\Omega; X)$ with

$$\eta_c(u) = \int_\Omega \langle g(\omega),\, u^*(\omega)\rangle\, d\mu(\omega) \qquad (u \in L_\infty(\Omega; X^*));$$

(ii) if we define a scalar-valued function G_{x^*} on $\mathcal{B}$ for each $x^* \in X^*$ by

$$G_{x^*}(B) := \eta_p(\chi_B(\cdot)x^*) \qquad (B \in \mathcal{B}),$$

then G_{x^*} is a purely finitely additive measure on $\mathcal{B}$;

(iii) $\eta = \eta_c + \eta_p$, and $\|\eta\| = \|\eta_c\| + \|\eta_p\|$.

By (i) we may identify η_c with g. Then, putting $\lambda = \lambda_\infty$, we have by (8)

$$(\lambda_\infty(\lambda_\infty - A)^{-1})^{**}\eta = g + \eta_p - (\lambda_\infty - A)^{-1} f.$$

On the other hand, we also have

$$\begin{aligned}(\lambda_\infty(\lambda_\infty - A)^{-1})^{**}\eta &= (\lambda_\infty(\lambda_\infty - A)^{-1})^{**}(g + \eta_p) \\ &= \lambda_\infty(\lambda_\infty - A)^{-1} g + (\lambda_\infty(\lambda_\infty - A)^{-1})^{**}\eta_p,\end{aligned}$$

whence

$$(9)\qquad (\lambda_\infty(\lambda_\infty - A)^{-1})^{**}\eta_p = \eta_p + g - (\lambda_\infty - A)^{-1}f - \lambda_\infty(\lambda_\infty - A)^{-1}g.$$

Since $\|(\lambda_\infty(\lambda_\infty - A)^{-1})^{**}\| = \|\lambda_\infty(\lambda_\infty - A)^{-1}\| \le 1$ by hypothesis, it follows that

$$\|\eta_p\| \ge \|(\lambda_\infty(\lambda_\infty - A)^{-1})^{**}\eta_p\| = \|\eta_p + g - (\lambda_\infty - A)^{-1}f - \lambda_\infty(\lambda_\infty - A)^{-1}g\|.$$

Here we notice that $g - (\lambda_1 - A)^{-1}f - \lambda_1(\lambda_1 - A)^{-1}g$ is a function in $L_1(\Omega; X)$ and η_p is an element of $L_1(\Omega; X)^{**}$ satisfying condition (ii). Thus by Lemma 1 we have

$$\|\eta_p\| = \|\eta_p\| + \|g - (\lambda_1 - A)^{-1}f - \lambda_1(\lambda_1 - A)^{-1}g\|_1,$$

which implies

$$g = (\lambda_1 - A)^{-1}f + \lambda_1(\lambda_1 - A)^{-1}g.$$

Consequently, $g \in D(A)$ and $(\lambda_1 - A)g = f + \lambda_1 g$, so that $f = A(-g) \in R(A)$.

(II) $\Rightarrow$ (I). If $f = Ag$ for some $g \in L_1(\Omega; X)$, then

$$(\lambda_n - A)^{-1}f = (\lambda_n - A)^{-1}Ag = A(\lambda_n - A)^{-1}g = \lambda_n(\lambda_n - A)^{-1}g - g,$$

and thus $\|(\lambda_n - A)^{-1}f\|_1 \le \|\lambda_n(\lambda_n - A)^{-1}\|\|g\|_1 + \|g\|_1 \le (M+1)\|g\|_1$ for all $n \ge 1$.

This completes the proof of Theorem 1. □

Using the argument of the above proof we can prove the following proposition, which is of independent interest in view of the results of [4] and [12].

Proposition 1. *Let X be a reflexive Banach space, and A be a closed operator in $L_1(\Omega; X)$ with domain $D(A)$ and range $R(A)$. Suppose there exists $\lambda \in \rho(A)$ such that $\|\lambda(\lambda - A)^{-1}\| \le 1$. Then $A(U \cap D(A))$ is a closed subset of $L_1(\Omega; X)$, where U is the closed unit ball of $L_1(\Omega; X)$, i.e., $U = \{f \in L_1(\Omega; X) : \|f\|_1 \le 1\}$.*

Proof: Let $f_n \in U \cap D(A)$, $n = 1, 2, \ldots$, and $f \in L_1(\Omega; X)$ be such that $\lim_{n\to\infty} \|Af_n - f\|_1 = 0$. We must prove that $f \in A(U \cap D(A))$. To do this, let $\eta \in L_1(\Omega; X)^{**}$ be a weak*-cluster point of the sequence $\{f_n\}_{n=1}^\infty (\subset L_1(\Omega; X)^{**})$. Then, for $u \in L_\infty(\Omega; X^*)$ there exists a subsequence $\{n_j\}_{j=1}^\infty$ of the sequence $\{n\}_{n=1}^\infty$ such that

$$\begin{aligned}\langle(\lambda(\lambda - A)^{-1})^{**}\eta, u\rangle &= \langle\eta, (\lambda(\lambda - A)^{-1})^* u\rangle = \lim_{j\to\infty}\langle f_{n_j}, (\lambda(\lambda - A)^{-1})^* u\rangle \\ &= \lim_{j\to\infty}\langle\lambda(\lambda - A)^{-1}f_{n_j}, u\rangle = \lim_{j\to\infty}\langle(I + A(\lambda - A)^{-1})f_{n_j}, u\rangle \\ &= \lim_{j\to\infty}\langle f_{n_j} + (\lambda - A)^{-1}Af_{n_j}, u\rangle = \langle\eta + (\lambda - A)^{-1}f, u\rangle.\end{aligned}$$

It follows that $(\lambda(\lambda - A)^{-1})^{**}\eta = \eta + (\lambda - A)^{-1}f$. Thus, as in the proof of (I) $\Rightarrow$ (II) of Theorem 1, letting $\eta = \eta_c + \eta_p$ and identifying η_c with a function g in $L_1(\Omega; X)$, we see that

$$\lambda(\lambda - A)^{-1}g + (\lambda(\lambda - A)^{-1})^{**}\eta_p = g + \eta_p + (\lambda - A)^{-1}f,$$

so that

$$(\lambda(\lambda - A)^{-1})^{**}\eta_p = \eta_p + g + (\lambda - A)^{-1}f - \lambda(\lambda - A)^{-1}g.$$

By this and the fact $\|(\lambda(\lambda - A)^{-1})^{**}\| \leq 1$, it follows from Lemma 1 that

$$g + (\lambda - A)^{-1}f - \lambda(\lambda - A)^{-1}g = 0.$$

Hence $(\lambda - A)g + f - \lambda g = 0$, and we see that $f = Ag$ with $g \in D(A)$ and $\|g\|_1 \leq \|\eta\| \leq 1$. This completes the proof. □

3. Applications

Let T be a bounded linear operator on $L_1(\Omega; X)$. For $\gamma \neq -1, -2, \ldots$ we define the Cesàro means of order γ (or γ-Cesàro means) $C_n^\gamma(T)$ of the discrete semigroup $\{T^n : n \geq 0\}$ by

$$(10) \qquad C_n^\gamma(T) := \frac{1}{\sigma_n^\gamma} \sum_{k=0}^{n} \sigma_{n-k}^{\gamma-1} T^k \qquad (n \geq 0),$$

where $\sigma_n^\beta := (\beta+1)(\beta+2)\ldots(\beta+n)/n!$ for $n \geq 1$, and $\sigma_0^\beta := 1$ (cf. [15, Chapter 3]). Among them are the following two particular means: $C_n^0(T) = T^n$ and $C_n^1(T) = (n+1)^{-1}\sum_{k=0}^n T^k$. As is well-known, only the case $\gamma > -1$ is of interest. The Abel means of $\{T^n : n \geq 0\}$ are the operators

$$(11) \qquad A_r(T) := (1-r)\sum_{n=0}^{\infty} r^n T^n$$

defined for $0 < r < 1/r(T)$, where $r(T) := \lim_{n\to\infty} \|T^n\|^{1/n}$ denotes the spectral radius of T. It is known (cf. [15, Chapter 3]) that if $r(T) \leq 1$ and $0 < \gamma < \beta < \infty$, then

$$(12) \qquad \sup_{n\geq 0} \|T^n f\|_1 \geq \sup_{n\geq 0} \|C_n^\gamma(T)f\|_1 \geq \sup_{n\geq 0} \|C_n^\beta(T)f\|_1 \geq \sup_{0<r<1} \|A_r(T)f\|_1$$

for every $f \in L_1(\Omega; X)$.

The first application of Theorem 1 is the following

Theorem 2 (cf. Theorem 7 of [8]). *Let X be a reflexive Banach space, and T be a linear contraction on $L_1(\Omega;X)$. Assume that $\alpha \geq 1$. Then the following conditions are equivalent for $f \in L_1(\Omega;X)$.*

(I) $\sup_{n\geq 0} n\|C_n^\alpha(T)f\|_1 < \infty$.

(II) $\sup_{0<r<1} \|\sum_{n=0}^\infty r^n T^n f\|_1 < \infty$.

(III) $f \in R(T-I)$.

Proof: (I) $\Rightarrow$ (II). Since

$$nC_n^\alpha(T)f = \frac{n}{\sigma_n^\alpha}\sum_{k=0}^n \sigma_{n-k}^{\alpha-1}T^k f \quad \text{and} \quad \frac{\sigma_n^\alpha}{\sigma_n^{\alpha-1}} = \frac{\alpha+n}{\alpha} \sim \frac{n}{\alpha} \qquad (n\to\infty),$$

$n\|C_n^\alpha(T)f\|_1 = O(1)$ $(n\to\infty)$ is equivalent to

$$C := \sup_{n\geq 0}\|\frac{1}{\sigma_n^{\alpha-1}}\sum_{k=0}^n \sigma_{n-k}^{\alpha-1}T^k f\|_1 < \infty.$$

Then, for $0<r<1$ we have

$$\begin{aligned}\sum_{n=0}^\infty r^n T^n f &= (1-r)^\alpha (1-r)^{-\alpha}\sum_{n=0}^\infty r^n T^n f\\ &= (1-r)^\alpha\Big(\sum_{n=0}^\infty \sigma_n^{\alpha-1}r^n\Big)\Big(\sum_{n=0}^\infty r^n T^n f\Big)\\ &= (1-r)^\alpha \sum_{n=0}^\infty \sigma_n^{\alpha-1}r^n\Big(\frac{1}{\sigma_n^{\alpha-1}}\sum_{k=0}^n \sigma_{n-k}^{\alpha-1}T^k f\Big),\end{aligned}$$

so that

$$\Big\|\sum_{n=0}^\infty r^n T^n f\Big\|_1 \leq (1-r)^\alpha \sum_{n=0}^\infty \sigma_n^{\alpha-1}r^n \cdot C = C.$$

(II) $\Rightarrow$ (III). Putting $A = T-I$, we have for $\lambda>0$

$$(\lambda - A)^{-1} = (\lambda+1-T)^{-1} = \frac{1}{\lambda+1}\sum_{n=0}^\infty (\frac{1}{\lambda+1})^n T^n,$$

whence $\|T\|\leq 1$ implies

$$\|\lambda(\lambda-A)^{-1}\| \leq \frac{\lambda}{\lambda+1}\sum_{n=0}^\infty (\frac{1}{\lambda+1})^n = 1.$$

Furthermore, we get from (II) that

$$\sup_{\lambda>0}\|(\lambda-A)^{-1}f\|_1 = \sup_{\lambda>0}\left\|\frac{1}{\lambda+1}\sum_{n=0}^{\infty}(\frac{1}{\lambda+1})^n T^n f\right\|_1$$
$$\leq \sup_{\lambda>0}\left\|\sum_{n=0}^{\infty}(\frac{1}{\lambda+1})^n T^n f\right\|_1 < \infty.$$

Hence, $f \in R(A) = R(T-I)$ by Theorem 1.

(III) $\Rightarrow$ (I). Suppose $f = (T-I)g$ for some $g \in L_1(\Omega; X)$. Using the fundamental relation $C_n^{\beta}(T)(T-I) = \frac{\beta}{n+1}(C_{n+1}^{\beta-1}(T) - I)$ for $\beta > 0$ and $n \geq 0$, which can be proved by an elementary calculation (cf. [15, Chapter 3]), we see that

$$nC_n^{\alpha}(T)f = nC_n^{\alpha}(T)(T-I)g = \frac{n\alpha}{n+1}(C_{n+1}^{\alpha-1}(T) - I)g.$$

Then, since $\|C_{n+1}^{\alpha-1}(T)\| \leq 1$ (which comes from the hypotheses that $\|T\| \leq 1$ and that $\alpha \geq 1$), it follows that

$$n\|C_n^{\alpha}(T)f\|_1 \leq \alpha(\|C_{n+1}^{\alpha-1}(T)\| + 1)\|g\|_1 \leq 2\alpha\|g\|_1 \quad (n \geq 0).$$

This completes the proof of Theorem 2. □

Remarks (on Theorem 2). **(a)** If $-1 < \alpha < 1$, then (III) $\Rightarrow$ (I) does not hold in general. To see this, first suppose that $\alpha \neq 0$ and $-1 < \alpha < 1$. Then we can use the equation $C_n^{\alpha}(T)(T-I) = \frac{\alpha}{n+1}(C_{n+1}^{\alpha-1}(T) - I)$. If $f = (T-I)g$, then

$$nC_n^{\alpha}(T)f = nC_n^{\alpha}(T)(T-I)g = \frac{n\alpha}{n+1}(C_{n+1}^{\alpha-1}(T)g - g), \tag{13}$$

so that $\lim_{n\to\infty}\|C_n^{\alpha-1}(T)g\|_1 = \infty$ implies

$$\|nC_n^{\alpha}(T)f\|_1 \geq \frac{n|\alpha|}{n+1}(\|C_{n+1}^{\alpha-1}(T)g\|_1 - \|g\|_1) \longrightarrow \infty \quad \text{as } n \to \infty. \tag{14}$$

To see the possibility of the case that $\lim_{n\to\infty}\|C_n^{\alpha-1}(T)g\|_1 = \infty$, let m be the counting measure on the set $\mathbb{Z}$ of all integers, and $L_1(\mathbb{Z}, m)$ be the L_1-space of real-valued functions on $\mathbb{Z}$ with respect to the measure m. Define a positive linear isometry T on $L_1(\mathbb{Z}, m)$ by $Tf(k) = f(k-1)$ for $k \in \mathbb{Z}$. Then, the function $g = \chi_{\{0\}}$ satisfies $T^n g = \chi_{\{n\}}$ for $n \geq 0$, and hence

$$\|C_n^{\alpha-1}(T)g\|_1 = \left\|\frac{1}{\sigma_n^{\alpha-1}}\sum_{k=0}^{n}\sigma_{n-k}^{\alpha-2}T^k g\right\|_1 \geq \frac{1}{|\sigma_n^{\alpha-1}|}\|\sigma_0^{\alpha-2}\chi_{\{n\}}\|_1 = \frac{1}{|\sigma_n^{\alpha-1}|} \longrightarrow \infty$$

as $n \to \infty$, since $\sigma_n^{\alpha-1} \sim n^{\alpha-1}/\Gamma(\alpha)$ $(n \to \infty)$ (cf. [15, p. 77]). (For related topics we refer the reader to [7].)

Next, suppose that $\alpha = 0$. In this case, for any isometry $T \neq I$ and any $f \neq 0$, with $f \in R(T-I)$, we have $n\|C_n^0(T)f\| = n\|T^n f\| = n\|f\| \to \infty$ as $n \to \infty$. This completes the proof.

(b) The implication (I) $\Rightarrow$ (II) holds for every $\alpha > -1$, with $\alpha \neq 0$. To see this, it suffices to consider only the case where $-1 < \alpha < 1$ and $\alpha \neq 0$, by Theorem 2. Now, choose $\beta > 0$ satisfying $\beta + \alpha \geq 1$. Then, since

$$C := \sup_{n\geq 0} \Big\| \frac{1}{\sigma_n^{\alpha-1}} \sum_{k=0}^{n} \sigma_{n-k}^{\alpha-1} T^k f \Big\|_1 < \infty,$$

$$\sum_{k=0}^{n} \sigma_{n-k}^{\beta+\alpha-1} T^k f = \sum_{k=0}^{n} \sigma_{n-k}^{\beta-1} \sum_{l=0}^{k} \sigma_{k-l}^{\alpha-1} T^l f, \quad \text{and} \quad \sigma_n^{\beta+\alpha-1} = \sum_{k=0}^{n} \sigma_{n-k}^{\beta-1} \sigma_k^{\alpha-1},$$

it follows that

$$\left\| \frac{1}{\sigma_n^{\beta+\alpha-1}} \sum_{k=0}^{n} \sigma_{n-k}^{\beta+\alpha-1} T^k f \right\|_1 \leq \frac{\sum_{k=0}^{n} \sigma_{n-k}^{\beta-1} \sigma_k^{\alpha-1} C}{\sum_{k=0}^{n} \sigma_{n-k}^{\beta-1} \sigma_k^{\alpha-1}} = C,$$

whence $\sup_{n\geq 0} n\|C_n^{\beta+\alpha}(T)f\|_1 < \infty$, and thus $f \in R(T-I)$ by Theorem 2.

On the other hand, if $\alpha = 0$, then the implication (I) $\Rightarrow$ (II) fails to hold in general. To see this, let μ be the measure on $\mathbb{Z}$ defined by

$$\mu(\{k\}) = \begin{cases} 1 & \text{if } k \leq 0, \\ (k+1)^{-1} & \text{if } k \geq 1. \end{cases}$$

Let T be the positive linear contraction on $L_1(\mathbb{Z}, \mu)$ defined by $Tf(k) = f(k-1)$ for $k \in \mathbb{Z}$. Then the function $g = \chi_{\{0\}}$ satisfies

$$n\|C_n^0(T)g\|_1 = n\|T^n g\|_1 = n\|\chi_{\{n\}}\|_1 = \frac{n}{n+1} < 1 \qquad (n \geq 0). \tag{15}$$

By the definitions,

$$\left\| \sum_{j=0}^{n} T^j g \right\|_1 = \|\chi_{[0,n]}\|_1 = \sum_{j=0}^{n} \frac{1}{j+1} \to \infty$$

as $n \to \infty$, so $g \notin L_1(T-I)$. Thus (II) with g in place of f does not hold, by Theorem 2.

(c) In Theorem 2 the condition $\sup_{0<r<1} \|\sum_{n=0}^{\infty} r^n T^n f\|_1 < \infty$ can be replaced with the weaker condition $\liminf_{r\uparrow 1} \|\sum_{n=0}^{\infty} r^n T^n f\|_1 < \infty$, which follows from Theorem 1.

Next, we consider a C_0-semigroup $T(\cdot) \equiv \{T(t) : t \geq 0\}$ of linear contractions on $L_1(\Omega; X)$. Thus, $T(s+t) = T(s)T(t)$ for all s, $t \geq 0$, and $\lim_{t\downarrow 0} \|T(t)f - f\|_1 = 0$ for each $f \in L_1(\Omega; X)$. The infinitesimal generator A of $T(\cdot)$ is defined by $Af := \lim_{t\downarrow 0} t^{-1}(T(t)f - f)$, with domain $D(A)$ the set of all those $f \in L_1(\Omega; X)$ for which this limit exists. It is known (cf. e.g. [9]) that A is a densely defined closed operator; and since $\|T(t)\| \leq 1$ for all $t \geq 0$, if $\lambda > 0$, then $\lambda \in \rho(A)$ and $(\lambda - A)^{-1} f = \int_0^\infty e^{-\lambda s} T(s) f \, ds$ for all $f \in L_1(\Omega; X)$. Therefore we have $\sup_{\lambda>0} \|\lambda(\lambda - A)^{-1}\| \leq 1$. The Cesàro means of order γ (or γ-Cesàro means) $C_t^\gamma(T(\cdot))$ of the semigroup $T(\cdot)$, where $\gamma \geq 0$ and $t > 0$, are the operators defined by $C_t^0(T(\cdot)) := T(t)$ for $\gamma = 0$, and

$$C_t^\gamma(T(\cdot))f := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} T(s) f \, ds \qquad (\gamma > 0, f \in L_1(\Omega; X)). \tag{16}$$

In particular, if $\gamma = 1$, then we have $C_t^1(T(\cdot))f = t^{-1} \int_0^t T(s) f \, ds$. The Abel means $A_\lambda(T(\cdot))$ of $T(\cdot)$ are the operators

$$A_\lambda(T(\cdot))f := \lambda \int_0^\infty e^{-\lambda s} T(s) f \, ds \qquad (\lambda > 0, f \in L_1(\Omega; X)). \tag{17}$$

Fubini's theorem and an induction argument on n imply easily the following facts:

(i) If $0 < \gamma, \beta < \infty$ then for every $f \in L_1(\Omega; X)$ and $t > 0$,

$$C_t^{\gamma+\beta}(T(\cdot))f = \frac{\int_0^t (t-s)^{\beta-1} \left[\int_0^s (s-r)^{\gamma-1} T(r) f \, dr\right] ds}{\int_0^t (t-s)^{\beta-1} \left[\int_0^s (s-r)^{\gamma-1} \, dr\right] ds}. \tag{18}$$

(ii) If $n \geq 1$ is an integer, then for every $f \in L_1(\Omega; X)$ and $t > 0$,

$$\begin{aligned} &C_t^n(T(\cdot))f = \\ &= n!\, t^{-n} \int_0^t \left[\int_0^{s_1} \left(\int_0^{s_2} \left(\ldots \left(\int_0^{s_{n-1}} T(s_n) f \, ds_n\right) \ldots\right) ds_3\right) ds_2\right] ds_1. \end{aligned} \tag{19}$$

Furthermore, as in the discrete case (cf. (12)), we obtain that if $0 < \gamma < \beta < \infty$, then for every $f \in L_1(\Omega; X)$,

$$\begin{aligned} \sup_{t\geq 0} \|T(t)f\|_1 &\geq \sup_{t>0} \|C_t^\gamma(T(\cdot))f\|_1 \\ &\geq \sup_{t>0} \|C_t^\beta(T(\cdot))f\|_1 \geq \sup_{\lambda>0} \|A_\lambda(T(\cdot))f\|_1. \end{aligned} \tag{20}$$

Theorem 3 (cf. Corollary 8 of [11]). *Let X be a reflexive Banach space, and A be the infinitesimal generator of a C_0-semigroup $T(\cdot)$ of linear contractions on $L_1(\Omega; X)$. Assume that $\alpha \geq 1$. Then the following conditions are equivalent for $f \in L_1(\Omega; X)$.*

(I) $\sup_{t>0} t\|C_t^\alpha(T(\cdot))f\|_1 < \infty$.

(II) $\sup_{\lambda>0} \| \int_0^\infty e^{-\lambda t} T(t)f\, dt\|_1 < \infty$.

(III) $f \in R(A)$.

Proof: (I) $\Rightarrow$ (II). We first show that there exists an integer $n \geq \alpha$ such that

$$M_n := \sup_{t>0} t\|C_t^n(T(\cdot))f\|_1 < \infty. \tag{21}$$

To prove this, we may assume that $\alpha > 1$. We then notice by (16) that the condition $\sup_{t>0} t\|C_t^\alpha(T(\cdot))f\|_1 < \infty$ is equivalent to

$$M(\alpha) := \sup_{t>0} \frac{\| \int_0^t (t-s)^{\alpha-1} T(s)f\, ds\|_1}{\int_0^t (t-s)^{\alpha-2}\, ds} < \infty. \tag{22}$$

Let $\beta > 0$. By Fubini's theorem

$$\begin{aligned}\int_0^t (t-s)^{\beta-1}\left(\int_0^s (s-r)^{\alpha-1} T(r)f\, dr\right) ds &= \int_0^t \left(\int_r^t (t-s)^{\beta-1}(s-r)^{\alpha-1}\, ds\right) T(r)f\, dr \\ &= \int_0^t (t-r)^{\beta+\alpha-1}\left(\int_0^1 (1-s)^{\beta-1}s^{\alpha-1}\, ds\right) T(r)f\, dr \\ &= B(\beta,\alpha)\int_0^t (t-r)^{\beta+\alpha-1} T(r)f\, dr\end{aligned}$$

and

$$\int_0^t (t-s)^{\beta-1}\left(\int_0^s (s-r)^{\alpha-2}\, dr\right) ds = B(\beta,\alpha-1)\int_0^t (t-r)^{\beta+\alpha-2}\, dr,$$

where $B(p,q) := \int_0^1 (1-x)^{p-1}x^{q-1}\, dx$ (p, $q > 0$) is the Beta function. It follows that

$$\begin{aligned}M(\beta+\alpha) &= \sup_{t>0} \frac{\| \int_0^t (t-s)^{\beta+\alpha-1} T(s)f\, ds\|_1}{\int_0^t (t-s)^{\beta+\alpha-2}\, ds} \\ &= \frac{B(\beta,\alpha-1)}{B(\beta,\alpha)} \cdot \sup_{t>0} \frac{\left\| \int_0^t (t-s)^{\beta-1}\left(\int_0^s (s-r)^{\alpha-1} T(r)f\, dr\right) ds\right\|_1}{\int_0^t (t-s)^{\beta-1}\left(\int_0^s (s-r)^{\alpha-2}\, dr\right) ds} \\ &\leq \frac{B(\beta,\alpha-1)}{B(\beta,\alpha)} \cdot M(\alpha) < \infty \qquad \text{(by (22))}.\end{aligned}$$

Therefore, (21) holds for any integer n, with $n > \alpha$.

Next, by Fubini's theorem

$$\int_0^\infty e^{-\lambda t} T(t)f\,dt = \lambda \int_0^\infty e^{-\lambda t}\Big[\int_0^t T(s_1)f\,ds_1\Big]\,dt = \dots$$
$$= \lambda^n \int_0^\infty e^{-\lambda t}\Big[\int_0^t \Big\{\int_0^{s_1}\Big(\int_0^{s_2}\Big(\dots\Big(\int_0^{s_{n-1}} T(s_n)f\,ds_n\Big)\dots\Big)ds_3\Big)ds_2\Big\}ds_1\Big]dt$$
$$= \lambda^n \int_0^\infty e^{-\lambda t}\frac{t^{n-1}}{n!}\cdot [tC_t^n(T(\cdot))f]\,dt \qquad \text{(by (19))}.$$

Thus we apply (21) to get that for $\lambda > 0$,

$$\Big\|\int_0^\infty e^{-\lambda t}T(t)f\,dt\Big\|_1 \le \lambda^n \int_0^\infty e^{-\lambda t}\frac{t^{n-1}}{n!}\|tC_t^n(T(\cdot))f\|_1\,dt$$
$$\le \frac{M}{n!}\lambda^n \int_0^\infty e^{-\lambda t}t^{n-1}\,dt = \frac{M}{n}.$$

(II) $\Rightarrow$ (III). Since (II) implies

$$\sup_{\lambda>0}\|(\lambda - A)^{-1}f\|_1 = \sup_{\lambda>0}\|\int_0^\infty e^{-\lambda t}T(t)f\,dt\|_1 < \infty,$$

(III) follows from Theorem 1.

(III) $\Rightarrow$ (I). Suppose $f = Ag$ for some $g \in L_1(\Omega; X)$. Then, since $\int_0^t T(s)f\,ds = T(t)g - g$ for $t > 0$, it follows that

$$M_1 = \sup_{t>0} t\|C_t^1(T(\cdot))f\|_1 = \sup_{t>0}\|T(t)g - g\|_1 \le 2\|g\|_1 < \infty. \tag{23}$$

Thus, (I) holds for $\alpha = 1$. If $\alpha > 1$, then by Fubini's theorem

$$\int_0^t (t-s)^{\alpha-1}\,T(s)f\,ds = \int_0^t \Big(\int_s^t (\alpha-1)(t-r)^{\alpha-2}\,T(s)f\,dr\Big)\,ds$$
$$= (\alpha-1)\int_0^t (t-r)^{\alpha-2}\Big(\int_0^r T(s)f\,ds\Big)\,dr,$$

and thus

$$\Big\|\int_0^t (t-s)^{\alpha-1}T(s)f\,ds\Big\|_1 \le (\alpha-1)\int_0^t (t-r)^{\alpha-2}\Big\|\int_0^r T(s)f\,ds\Big\|_1\,dr$$
$$\le (\alpha-1)\int_0^t (t-r)^{\alpha-2}\,M_1\,dr = M_1\,t^{\alpha-1},$$

so that

$$\sup_{t>0} t\|C_t^\alpha(T(\cdot))f\|_1 \le \alpha M_1.$$

This completes the proof of Theorem 3. □

Remarks (on Theorem 3). **(d)** The implication (III) $\Rightarrow$ (I) does not hold for $0 \le \alpha < 1$. Indeed, let $L_1(-\infty, \infty)$ be the usual L_1-space of scalar-valued functions on the real line $\mathbb{R} := (-\infty, \infty)$. Let $T(t)$, $t \in \mathbb{R}$, be the operators on $L_1(-\infty, \infty)$ defined by

$$T(t)f(x) := f(x+t). \tag{24}$$

Then $T(\cdot) := \{T(t) : t \ge 0\}$ is a C_0-semigroup of positive invertible linear isometries on $L_1(-\infty, \infty)$. The following are well-known:

(i) $D(A) = \{g \in L_1(-\infty, \infty) : g$ is absolutely continuous, and $g' \in L_1(-\infty, \infty)\}$;
(ii) $Ag = g'$ for $g \in D(A)$.

Hence the function $f = \chi_{[0,1)} - \chi_{[1,3)} + \chi_{[3,4)}$ belongs to $D(A)$, and $f = Ag$, where $g(x) := \int_{-\infty}^{x} f(s)\,ds$.

Now, suppose $0 < \alpha < 1$. Then, for every x with $0 < t-1 < x < t$, we have

$$\begin{aligned} tC_t^{\alpha}(T(\cdot))f(-x) &= \alpha t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} f(s-x)\,ds \\ &= \alpha t^{1-\alpha} \int_x^t (t-s)^{\alpha-1}\,ds = t^{1-\alpha}(t-x)^{\alpha}, \end{aligned}$$

whence

$$\|tC_t^{\alpha}(T(\cdot))f\|_1 \ge t^{1-\alpha} \int_{t-1}^{t} (t-x)^{\alpha}\,dx = t^{1-\alpha}\frac{1}{\alpha+1}\,.$$

This implies $\lim_{t\to\infty} \|tC_t^{\alpha}(T(\cdot))f\|_1 = \infty$, because $1-\alpha > 0$. Next, suppose $\alpha = 0$. Then, clearly, we have $\|tC_t^{0}(T(\cdot))f\|_1 = t\|T(t)f\|_1 = t\|f\|_1 = 4t \to \infty$ as $t \to \infty$.

(e) The computations in the proof of (I) $\Rightarrow$ (II) apply to the case $\alpha > 0$, so that the implication (I) $\Rightarrow$ (II) holds for all $\alpha > 0$. But, if $\alpha = 0$, then the implication (I) $\Rightarrow$ (II) fails to hold in general. This can be seen by modifying the example in Remark (b). Indeed, let w be the function on $\mathbb{R}$ defined by $w(x) = 1$ if $x \ge -1$, and $w(x) = (-x)^{-1}$ if $x < -1$, and let μ be the measure on $\mathbb{R}$ defined by $\mu = w\,dx$, where dx stands for the Lebesgue measure on $\mathbb{R}$. Then the operators $T(t)$, $t \ge 0$, of the form $T(t)f(x) = f(x+t)$ define a C_0-semigroup $T(\cdot)$ of positive linear contractions on $L_1(\mathbb{R}, \mu)$ of scalar-valued integrable functions with respect to μ, and the function $f := \chi_{[0,1)}$ satisfies

$$\sup_{t>0} t\|C_t^{0}(T(\cdot))f\|_1 = \sup_{t>0} t\|T(t)f\|_1 = \sup_{t>0} t\|\chi_{[-t,-t+1)}\|_1 < 2.$$

But, it is known that

(i) $D(A) = \{g \in L_1(\mathbb{R}, \mu) : g$ is locally absolutely continuous, and $g' \in L_1(\mathbb{R}, \mu)\}$;
(ii) $Ag = g'$ for $g \in D(A)$.

Thus, if $f = Ah$ for some $h \in D(A)$, then we must have $f = \chi_{[0,1)} = h'$, and

$$h(x+t) - h(x) = \int_0^t f(x+s)\,ds \qquad (t \ge 0,\ x \in \mathbb{R}).$$

Therefore, $h(x) = h(1)$ for $x \ge 1$, and $h(x) = h(0)$ for $x \le 0$. But, since $h(1) - h(0) = \int_0^1 h'(s)\,ds = 1$, this proves that h cannot belong to $L_1(\mathbb{R}, \mu)$, a contradiction. Thus, $f \notin R(A)$, and (II) does not hold by Theorem 3.

Lastly, we give an application to the infinitesimal generator A of a strongly continuous cosine family $C(\cdot) \equiv \{C(t) : t \in \mathbb{R}\}$ of linear contractions on $L_1(\Omega; X)$. By definition, the family $C(\cdot)$ satisfies

(i) $C(s+t) + C(s-t) = 2C(s)C(t)$ for all $s, t \in \mathbb{R}$;
(ii) $C(0) = I$;
(iii) $C(t)f$ is continuous in $t \in \mathbb{R}$ for each $f \in L_1(\Omega; X)$.

The infinitesimal generator A is defined by $Af := \lim_{t\to 0}(C(2t)f - f)/2t^2$, with domain $D(A)$ the set of all those $f \in L_1(\Omega; X)$ for which this limit exists. Since $\|C(t)\| \le 1$ for all $t \in \mathbb{R}$, it is known (cf. e.g. [13], [14]) that A is a densely defined closed operator such that if $\lambda > 0$, then $\lambda \in \rho(A)$ and $\lambda(\lambda^2 - A)^{-1} f = \int_0^\infty e^{-\lambda s} C(s) f\,ds$ for all $f \in L_1(\Omega; X)$. Therefore we have $\sup_{\lambda > 0} \|\lambda(\lambda - A)^{-1}\| \le 1$.

The associated sine family $S(\cdot) \equiv \{S(t) : t \in \mathbb{R}\}$ of linear operators on $L_1(\Omega; X)$ is defined by

$$S(t)f := \int_0^t C(s) f\,ds \qquad (t \in \mathbb{R},\ f \in L_1(\Omega; X)). \tag{25}$$

Elementary properties of $S(\cdot)$ and $C(\cdot)$ can be found in [14]. The Cesàro means of order γ (or γ-Cesàro means) $C_t^\gamma(S(\cdot))$ of the sine family $S(\cdot)$, where $\gamma \ge 0$ and $t > 0$, are the operators defined by $C_t^0(S(\cdot)) := S(t)$, and

$$C_t^\gamma(S(\cdot))f := \gamma t^{-\gamma} \int_0^t (t-s)^{\gamma-1} S(s) f\,ds \qquad (\gamma > 0,\ f \in L_1(\Omega; X)). \tag{26}$$

It is direct to see that (18), (19) and (20) hold with $S(\cdot)$, $S(r)$, $S(s_n)$ and $S(t)$ in place of $T(\cdot)$, $T(r)$, $T(s_n)$ and $T(t)$, respectively.

Theorem 4 (cf. Corollary 8 of [11]). *Let X be a reflexive Banach space, and A be the infinitesimal generator of a strongly continuous cosine family $C(\cdot)$ of linear contractions on $L_1(\Omega; X)$. Assume that $\alpha \ge 1$. Then the following conditions are equivalent for $f \in L_1(\Omega; X)$.*

(I) $\sup_{t>0} t\|C_t^\alpha(S(\cdot))f\|_1 < \infty$.
(II) $\sup_{\lambda>0} \|\int_0^\infty e^{-\lambda t} S(t) f\,dt\|_1 < \infty$.
(III) $f \in R(A)$.

PROOF: (I) $\Rightarrow$ (II). We first notice, as in the proof of (I) $\Rightarrow$ (II) of Theorem 3, that there exists an integer $n \geq \alpha$ such that

$$M'_n := \sup_{t>0} t\|C_t^n(S(\cdot))f\|_1 < \infty. \tag{27}$$

Then, since

$$\int_0^\infty e^{-\lambda t} S(t)f\,dt = \lambda^n \int_0^\infty e^{-\lambda t}\frac{t^{n-1}}{n!}\cdot [tC_t^n(S(\cdot))f]\,dt$$

(cf. (19) with $S(\cdot)$ and $S(s_n)$ in place of $T(\cdot)$ and $T(s_n)$, respectively), it follows that

$$\left\|\int_0^\infty e^{-\lambda t} S(t)f\,dt\right\|_1 \leq \frac{M'_n}{n!}\lambda^n \int_0^\infty e^{-\lambda t} t^{n-1}\,dt = \frac{M'_n}{n} \qquad (\lambda > 0).$$

(II) $\Rightarrow$ (III). Since $(\lambda^2 - A)^{-1}f = \lambda^{-1}\int_0^\infty e^{-\lambda s}C(s)f\,ds = \int_0^\infty e^{-\lambda t}S(t)f\,dt$ for $\lambda > 0$, (II) implies

$$\sup_{\lambda>0}\|(\lambda^2 - A)^{-1}f\|_1 = \sup_{\lambda>0}\|\int_0^\infty e^{-\lambda t}S(t)f\,dt\|_1 < \infty.$$

Hence (III) follows from Theorem 1.

(III) $\Rightarrow$ (I). Assume that $f = Ag$ for some $g \in L_1(\Omega; X)$. By Lemma 2.15 of [13] we have $\int_0^t S(s)f\,ds = \int_0^t S(s)Ag\,ds = C(t)g - g$ for $t > 0$. Thus $M'_1 = \sup_{t>0}\|\int_0^t S(s)f\,ds\|_1 \leq 2\|g\|_1$, and hence (I) holds for $\alpha = 1$. If $\alpha > 1$, then we can obtain, as in the proof of (III) $\Rightarrow$ (I) of Theorem 3, that $\sup_{t>0} t\|C_t^\alpha(S(\cdot))f\|_1 \leq \alpha M'_1$.

This completes the proof of Theorem 4. □

Remarks (on Theorem 4)**. (f)** The implication (III) $\Rightarrow$ (I) does not hold for $0 \leq \alpha < 1$. Indeed, if $C(t)$, $t \in \mathbb{R}$, are the operators on $L_1(-\infty, \infty)$ defined by $C(t)f(x) := 2^{-1}(f(x+t) + f(x-t))$, then $C(\cdot) := \{C(t) : t \in \mathbb{R}\}$ becomes a strongly continuous cosine family of positive linear contractions on $L_1(-\infty, \infty)$. It is known (cf. e.g. [13, Theorem 4.12]) that

(i) $D(A) = \left\{g \in L_1(-\infty,\infty) : \begin{array}{l} g \text{ and } g' \text{ are absolutely continuous,} \\ \text{and } g', g'' \in L_1(-\infty,\infty)\end{array}\right\}$;

(ii) $Ag = g''$ for $g \in D(A)$.

Thus the function

$$f = \chi_{[0,1)} - \chi_{[1,3)} + \chi_{[3,4)} - \chi_{[4,5)} + \chi_{[5,7)} - \chi_{[7,8)}$$

belongs to $R(A)$. Since $S(t)f(x) = \int_0^t 2^{-1}(f(x+s) + f(x-s))\,ds$, it follows that if $t > 2$ then

$$S(t)f(x) \geq 1/4 \quad \text{for all } x \in [-t + (1/2),\ -t + (3/2)], \tag{28}$$

and thus if $s \in [t - (1/4),\ t]$, then, for all $x \in [-t + (3/4),\ -t + (3/2)]$,

$$S(s)f(x) = \int_0^s 2^{-1}(f(x+r) + f(x-r))\,dr \geq 1/4. \tag{29}$$

Now, suppose $0 < \alpha < 1$. Then by (29), for $t > 2$ and $x \in [-t + (3/4),\ -t + (3/2)]$ we have

$$\begin{aligned} tC_t^\alpha(S(\cdot))f(x) &= \alpha t^{1-\alpha} \int_0^t (t-s)^{\alpha-1} S(s)f(x)\,ds \\ &\geq \alpha t^{1-\alpha} \int_{t-\frac{1}{4}}^t (t-s)^{\alpha-1} \cdot \frac{1}{4}\,ds = \frac{t^{1-\alpha}}{4^{\alpha+1}}, \end{aligned}$$

therefore

$$t\|C_t^\alpha(S(\cdot))f\|_1 \geq \int_{-t+(3/4)}^{-t+(3/2)} \frac{t^{1-\alpha}}{4^{\alpha+1}}\,dx = \frac{3\,t^{1-\alpha}}{4^{\alpha+2}} \longrightarrow \infty \qquad (t \to \infty).$$

Next, suppose $\alpha = 0$. Then by (28) we get

$$t\|C_t^0(S(\cdot))f\|_1 = t\|S(t)f\|_1 \geq t \int_{-t+(1/2)}^{-t+(3/2)} \frac{1}{4}\,dx = \frac{t}{4} \longrightarrow \infty \qquad (t \to \infty).$$

(g) The implication (I) $\Rightarrow$ (II) of Theorem 4 holds for all $\alpha > 0$, as observed in Remark (e). Here it may be of some interest to note that if $\alpha = 0$ then the implication (I) $\Rightarrow$ (II) is trivial. Indeed, if (I) holds for $\alpha = 0$, then we have $\sup_{t>0} t\|S(t)f\|_1 < \infty$ and hence $\lim_{t\to\infty} \|S(t)f\|_1 = 0$, from which we deduce that $f = 0$ as follows. For a moment, assume that $f \neq 0$. Then there exists $s_0 > 0$ such that $g := S(s_0)f \neq 0$. Then by Proposition 2.1 of [14]

$$C(t)g = C(t)S(s_0)f = 2^{-1}(S(t+s_0)f - S(t-s_0)f) \longrightarrow 0 \quad \text{as } t \to \infty,$$

and thus $g = -C(2t)g + 2C(t)^2 g \to 0$ as $t \to \infty$. But this is a contradiction. (This proof was communicated to the author by Professor S.-Y. Shaw.)

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