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Stability of positive part of unit ball in Orlicz spaces

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Abstract. The aim of this paper is to investigate the stability of the positive part of the unit ball in Orlicz spaces, endowed with the Luxemburg norm. The convex set Q in a topological vector space is stable if the midpoint map $\Phi: Q \times Q \to Q$, $\Phi(x,y) = (x+y)/2$ is open with respect to the inherited topology in Q. The main theorem is established: In the Orlicz space $L^{\varphi}(\mu)$ the stability of the positive part of the unit ball is equivalent to the stability of the unit ball.

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1. Introduction

A convex set Q in a real Hausdorff topological vector space X is called *stable* if the midpoint map $\Phi: Q \times Q \to Q$, $\Phi(x,y) = (x+y)/2$ is open with respect to the inherited topology in Q ([2], [9], [16]). Stable compact sets have been studied in [10], [14], [19]. Stability is a useful tool in investigating the extremal operators between Banach spaces ([2]). Further, the set of extreme points of a stable set is closed. Thus "stability" arguments can be employed for a description of extreme points of the unit ball of $C(K,X)$, K being a compact Hausdorff space and X a Banach space, namely, applying the Michael selection theorem [12],

$$f \in \operatorname{ext} B(C(K,X)) \iff f(k) \in \operatorname{ext} B(X) \quad \text{for every} \quad k \in K$$

provided that the unit ball $B(X)$ of X is stable.

In [16] it has been proved that if $\dim X \leq 2$, then every convex set $Q \subset X$ is stable, and also that from the stability of a convex closed set Q it follows that the set of extremal points $\operatorname{ext} Q$ is closed. The converse implication is not satisfied, although for $\dim X \leq 3$ it is true. The strictly convex sets are stable, too. Finite dimensional Banach spaces can have non-stable unit balls, for example let $X = \mathbb{R}^3$ and

$$B := \operatorname{conv}\Big(\{\,(x,y,0) : x^2 + y^2 \leq 1\,\} \cup \{\,(\pm 1, 0, \pm 1)\,\}\Big), \qquad \text{(see [16]).}$$

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By Theorem from [7] the above Banach space is not Orlicz with the Luxemburg norm. Moreover,

$$B^+(X) = \operatorname{conv}\Big(\{ (x,y,0) : x \geq 0, y \geq 0, x^2 + y^2 \leq 1 \} \cup \{ (1,0,1) \}\Big)$$

is stable, which is easy to verify. Thus, the stability of $B^+(X)$ does not indicate that $B(X)$ is stable. However, it is known that in normed vector lattices, the stability of $B(X)$ implies the stability of $B^+(X)$, see [6].

In this work we give an answer to the question: does the stability of $B(X)$ in Orlicz spaces with the Luxemburg norm follow from the stability of $B^+(X)$? The main ideas of this result are contained in [22], hence some parts of the proof we omit are available in the above-mentioned work.

2. Basic definition and auxiliary results

Let (Ω, Σ, μ) be a measure space with a nonnegative, σ-finite and complete measure μ ($\mu(\Omega) > 0$), and let $\varphi\colon \mathbb{R} \to [0, +\infty]$ be a convex, even, non-identically equal to 0, vanishing at 0 and left-continuous for $t > 0$ function such that $c(\varphi) := \sup\{t > 0 : \varphi(t) < \infty\} > 0$. Such functions will be called *Young functions*. This definition is somewhat stronger than for example that in [17], but it does not really bound the class of spaces considered. We will often use the notation $a(\varphi) := \sup\{t : \varphi(t) = 0\}$. By an *Orlicz space* $L^\varphi(\mu)$ ([13], [15], [17]), we mean the set of all measurable functions $x\colon \Omega \to \mathbb{R}$ such that $I_\varphi(\lambda x) < \infty$ for some $\lambda > 0$, where *the modular* I_φ is defined by

$$I_\varphi(x) := \int_\Omega \varphi\left(x(\omega)\right)\, d\mu.$$

$L^\varphi(\mu)$ is equipped with the equality "almost everywhere" (a.e.) and *the Luxemburg norm* [11]

$$\|x\|_\varphi := \inf\left\{ \lambda > 0 : I_\varphi\left(x/\lambda\right) \leq 1 \right\}.$$

(Note that $\|x\|_\varphi \leq 1$ iff $I_\varphi(x) \leq 1$; $I_\varphi(x) = 1$ implies $\|x\|_\varphi = 1$; $I_\varphi(x) < 1 \Rightarrow$ ($\|x\|_\varphi = 1$ iff $I_\varphi(\lambda x) = +\infty$ for every $\lambda > 1$); $\|x_n - x\|_\varphi \to 0$ iff $I_\varphi(\lambda(x_n - x)) \to 0$ for every $\lambda > 0$.) The subspace

$$E^\varphi(\mu) := \left\{x \in \mathcal{M} : \forall\, \lambda > 0 \quad I_\varphi(\lambda x) < +\infty\right\}$$

is called *the space of finite elements*.

Let $r > 1$. The function φ is said to satisfy *condition* $\Delta_r(\mu)$ [20], [22] ($\varphi \in \Delta_r(\mu)$ in short) if:

(a) there exists a constant $c > 1$ such that $\varphi(rt) \leq c\varphi(t)$ for every t (respectively, every $t \geq a_0$, $\varphi(a_0) < +\infty$) provided that μ is atomless and infinite (respectively, finite);

(b) there exist $b > 0$, $c > 1$ and a nonnegative sequence (d_n) such that $\sum_n d_n < +\infty$, and $\varphi(rt)\mu(e_n) \le c\varphi(t)\mu(e_n) + d_n$ for every t with $\varphi(t)\mu(e_n) \le b$ and every $n \in \mathbb{N}$ provided that μ is purely atomic and $\{e_n : n \in N\}$, $N \subset \mathbb{N}$, is the set of all atoms of Ω;

(c) a combination of (a) and (b) if Ω has both an atomless and purely atomic part.

If $c(\varphi) = \infty$, then

$$\varphi \in \Delta_r(\mu) \text{ for some } r > 1 \Longleftrightarrow \varphi \in \Delta_r(\mu) \text{ for every } r > 1 \Longleftrightarrow \varphi \in \Delta_2(\mu).$$

The above equivalences remain true if μ is atomless (then $\varphi \in \Delta_r(\mu)$ for some $r > 1$ implies that $c(\varphi) = \infty$). If μ is purely atomic with $\sum_n \mu(e_n) = \infty$ and $\varphi \in \Delta_r(\mu)$ for some $r > 1$, then φ vanishes only at 0 (indeed, $d_n \ge \varphi(ra(\varphi))\mu(e_n)$ for every $n \in \mathbb{N}$). Thus the above equivalences are true also in the case of a purely atomic measure μ with an infinite number of atoms provided that $0 < \inf_n \mu(e_n) \le \sup \mu(e_n) < \infty$ — no matter whether φ takes only finite values or not (if $\varphi \in \Delta_{r_0}(\mu)$, then evidently $\varphi \in \Delta_r(\mu)$ for every $1 < r \le r_0$; for $r > r_0$, consider $b_r = \varphi(a' r_0/r) \cdot \inf_n \mu(e_n) > 0$, where $a' = \sup\{\, a > 0 : \varphi(a) \le b_{r_0}/\sup_n \mu(e_n) \,\} > 0$). If $\dim L^\varphi(\mu) < \infty$ (i.e., Ω consists of a finite number of atoms), then $\varphi \in \Delta_r(\mu)$ for some $r > 1$ if and only if $L^\varphi(\mu)$ is not isometric to $L^\infty(\mu)$ (take any $a_0 \in (a(\varphi), c(\varphi))$, $1 < r < c(\varphi)/a_0$ and put $b = \varphi(a_0) \cdot \inf_n \mu(e_n) > 0$, $d_n = \varphi(ra_0) \cdot \sup_n \mu(e_n) < \infty$). However, if $0 < a(\varphi) \le c(\varphi) < \infty$, then φ does not satisfy the condition $\Delta_r(\mu)$ for any $r > c(\varphi)/a(\varphi)$.

Note that if $c(\varphi) = \infty$ and $L^\varphi(\mu)$ is finite dimensional, then $L^\varphi(\mu) = E^\varphi(\mu)$. If $c(\varphi) = \infty$ and $\dim L^\varphi(\mu) = \infty$, the equality $L^\varphi(\mu) = E^\varphi(\mu)$ holds if and only if $\varphi \in \Delta_2(\mu)$ (cf. [13, Theorem 8.13, p. 52]), thus, applying the Lebesgue dominated convergence theorem, we obtain

$$(I_\varphi(x) = 1 \Longleftrightarrow \|x\|_\varphi = 1) \quad \text{if and only if} \quad \varphi \in \Delta_2(\mu).$$

In fact, we can replace condition $\Delta_2(\mu)$ by $\Delta_r(\mu)$ for some $r > 1$ in the last equivalence. Then the assumption $c(\varphi) = \infty$ is used in the "if" part of the proof only, so, in any case, we have that if $\varphi \notin \Delta_r(\mu)$ for any $r > 1$, then there exists $x \in L^\varphi(\mu)$ such that $\|x\| = 1$ but $I_\varphi(x) < 1$, and that is what we need to have.

Now we introduce another related notion.

Let $\{e_n : n \in N\}$, $N \subset \mathbb{N}$, be a set of all atoms of Ω and let $r > 1$. We shall say that a function φ satisfies the *condition* $\Delta_r^0(\mu)$ (on Ω) — $\varphi \in \Delta_r^0(\mu)$ in short — if

– there exist $a_0 > 0$ and $c > 1$ such that $0 < \varphi(a_0) < \infty$ and $\varphi(rt) \le c\varphi(t)$ for every $|t| \le a_0$, provided that the atomless part of Ω is of positive measure;

– there exist $a_0 > 0$, $b > 0$, $c > 1$ and a nonnegative sequence (d_n) such that $\sum_n d_n < +\infty$, $0 < \varphi(a_0) < \infty$ and $\varphi(rt)\mu(e_n) \le c\varphi(t)\mu(e_n) + d_n$ for every $|t| \le a_0$ with $\varphi(t)\mu(e_n) \le b$ and every $n \in N$ provided that μ is purely atomic.

If $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ on the atomless part of Ω, which is of positive measure, then evidently, $\varphi \in \Delta_r^0(\mu)$ on the whole set Ω. Further, if the measure of the atomless part of Ω is either infinite or equal to zero and $\varphi \in \Delta_r(\mu)$ for some $r > 1$, then $\varphi \in \Delta_r^0(\mu)$. Thus $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ provided that $\dim L^\varphi(\mu) < \infty$ and $L^\varphi(\mu)$ is not isometric to $L^\infty(\mu)$.

If $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$, then, (see [22, p. 509]) if $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ and $\|x\|_\infty < c(\varphi)$, then

$$I_\varphi(x) = 1 \iff \|x\|_\varphi = 1.$$

Note that $\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ iff $\varphi \in \Delta_2^0(\mu)$ provided that φ takes only finite values.

The point $z \in Q$ is called *stable* (or Q is said to be *stable at* z, (cf. [16, p. 197])) if for every $x, y \in Q$, $x \neq y$ with $\frac{x+y}{2} = z$ and every open neighborhoods U, V of x and y respectively there exists an open set W such that $W \cap Q \subset \frac{1}{2}((U \cap Q) + (V \cap Q))$.

If X is normed, then the last condition can be represented as

$$\forall \varepsilon > 0 \quad \forall x, y \in Q, z = \frac{x+y}{2} \quad \exists \delta > 0 \forall w \in Q \Big(\|w - z\| < \delta \Rightarrow$$
$$\Rightarrow \exists u, v \in Q \quad \|u - x\| < \varepsilon, \|v - y\| < \varepsilon, w = \frac{1}{2}(u + v) \Big).$$

Of course if $z \in \operatorname{int} Q$ then Q is stable at z. Moreover, Q is stable iff it is stable at each its point.

Proposition 1. *In a normed vector lattice X the positive cone X^+ is stable.*

Proof: Let the sets U, V be open. It is necessary to prove that $\frac{1}{2}\big((U \cap X^+) + (V \cap X^+)\big)$ is open in X^+. Suppose not. Then there exist $z \in \frac{1}{2}\big((U \cap X^+) + (V \cap X^+)\big)$ and a net $(z_\alpha)_{\alpha \in \Gamma}$, $\lim_{\alpha \in \Gamma} z_\alpha = z$ such that for every $\alpha \in \Gamma$ it holds $z_\alpha \notin \frac{1}{2}\big((U \cap X^+) + (V \cap X^+)\big)$, $z_\alpha \geq 0$. From the assumption it follows that there exist $x \geq 0$, $y \geq 0$, $x \in U$, $y \in V$ such that $z = \frac{x+y}{2}$. Let $x_\alpha := (2z_\alpha) \wedge x$, $y_\alpha := 2z_\alpha - x_\alpha$. Of course $x_\alpha \geq 0$, and by $x_\alpha \leq 2z_\alpha$ we have $y_\alpha \geq 0$. From the continuity of "$\wedge$" it follows that $\lim_{\alpha \in \Gamma} x_\alpha = x$ and $\lim_{\alpha \in \Gamma} y_\alpha = 2z - x = y$, too. Thus for eventually α it holds $x_\alpha \in U$, $y_\alpha \in V$. Hence for eventually α it holds $z_\alpha = \frac{1}{2}(x_\alpha + y_\alpha) \in \frac{1}{2}\big((U \cap X^+) + (V \cap X^+)\big)$ against of (z_α). □

We say that the normed vector lattice X has *property PPP* if for every $x, y \in X^+$ there exists $\sup\{x \wedge ny : n \in \mathbb{N}\}$, cf. [18, Corollary 2, p. 64].

Of course, Orlicz spaces have property PPP.

Proposition 2. *Let X be a normed vector lattice with property* PPP. *Then if $z \in B(X)$ is a point such that $B(X)$ is stable at $|z|$, then $B(X)$ is stable at z, too.*

Proof: Fix $z \in B(X)$ such that $B(X)$ is stable at $|z|$ and define a transformation $\varphi\colon X \to X$ by the formula

$$\varphi(x) := \sup_{n\in\mathbb{N}}(nz^- \wedge x^+) - \sup_{n\in\mathbb{N}}(nz^- \wedge x^-).$$

It is known that φ is the lattice projection (i.e. the vector mapping preserving the lattice operations and satisfying $\varphi \circ \varphi = \varphi$). For $z^- > 0$ it follows by Proposition 2.11 from [18, p. 63], where it is necessary to take $A = \{z^-\}$, and for $z^- = 0$ it is obvious.

At present we define a vector mapping $\widehat{\ }\colon X \to X$ in the following way:

$$\widehat{x} := x - 2\varphi(x).$$

We claim:

$$\widehat{\widehat{x}} = x, \qquad |\widehat{x}| = |x|.$$

The first equality is a consequence of simple algebraic operations. Since for $x \geq 0$

$$0 \leq \varphi(x) = \sup_{n\in\mathbb{N}}(nz^- \wedge x) \leq x \quad \text{holds,}$$

so $-x = x - 2x \leq x - 2\varphi(x) = \widehat{x} \leq x$, thus $|\widehat{x}| \leq x$ for $x \geq 0$. Hence for any $x \in X$ the inequality

$$\begin{aligned}|\widehat{x}| &= |x - 2\varphi(x)| = |(x^+ - 2\varphi(x^+)) - (x^- - 2\varphi(x^-))| \\ &\leq |x^+ - 2\varphi(x^+)| + |x^- - 2\varphi(x^-)| = |\widehat{x^+}| + |\widehat{x^-}| \leq x^+ + x^- = |x|\end{aligned}$$

holds, so $|\widehat{x}| \leq |x|$. Thus $|x| = |\widehat{\widehat{x}}| \leq |\widehat{x}| \leq |x|$.

The claim is proved, so also $\|\widehat{x}\| = \|x\|$.

Let $x, y \in B(X)$ be such that $z = (x+y)/2$ and fix $\varepsilon > 0$. Because

$$\varphi(z) = \sup_{n\in\mathbb{N}}(nz^- \wedge z^+) - \sup_{n\in\mathbb{N}}(nz^- \wedge z^-) = -z^-,$$

so $\widehat{z} = z - 2\varphi(z) = z^+ - z^- + 2z^- = z^+ + z^- = |z|$, thus $|z| = \widehat{z} = (\widehat{x} + \widehat{y})/2$. By definition of stability at a point the following statement

$$\begin{aligned}&\exists\, \delta > 0 \forall\, \widetilde{w} \in B(x)\Big(\|\widetilde{w} - |z|\| < \delta \Rightarrow \exists\, \widetilde{u}, \widetilde{v} \in B(x) \\ &\|\widetilde{u} - \widehat{x}\| < \varepsilon, \|\widetilde{v} - \widehat{y}\| < \varepsilon, \widetilde{w} = \frac{1}{2}(\widetilde{u} + \widetilde{v})\Big)\end{aligned} \tag{1}$$

is satisfied. Let $w \in B(X)$ satisfy $\|w - z\| < \delta$. Then $\|\widehat{w} - |z|\,\| = \|\widehat{w - z}\| = \|w - z\| < \delta$, so there exist $\widetilde{u}$, $\widetilde{v}$ satisfying (1) for $\widetilde{w} := \widehat{w}$.

Let $u := \widehat{\widetilde{u}}$, $v := \widehat{\widetilde{v}}$. Then $\widehat{u} = \widetilde{u}$, so $\|u - x\| = \|\widehat{u - x}\| = \|\widehat{u} - \widehat{x}\| = \|\widetilde{u} - \widehat{x}\| < \varepsilon$ and analogously $\|v - y\| < \varepsilon$. Moreover $u, v \in B(X)$ and $w = \widehat{\widehat{w}} = \widehat{(\widetilde{u} + \widetilde{v})}/2 = (\widehat{\widetilde{u}} + \widehat{\widetilde{v}})/2 = (u+v)/2$. Because $\varepsilon > 0$ has been arbitrary, $B(X)$ is stable at z. □

Now we present an elementary lemma (cf. [6]).

Lemma 1. *If X is a normed vector lattice and $x, y \in X$, the following inequalities are satisfied:*

1. $\|x^+ - y^+\| \le \|x - y\|$ *and* $\|x^- - y^-\| \le \|x - y\|$;
2. *if* $x + y \ge 0$, *then* $y^+ - x^- \ge 0$ *and* $x^+ - y^- \ge 0$.

Proof: Note that if $u, v, w \ge 0$, $u \wedge v = 0$ and $w + u \ge v$ then $w \ge v$. Indeed, from $w + u \ge v$ we get $v = (w + u) \wedge v \le (w \wedge v) + (u \wedge v) = w \wedge v \le v$. Hence $w \wedge v = v$, i.e. $w \ge v$. Put $u = x^+$, $v = x^-$, $w = y^+$. Hence $y^+ \ge x^-$. Similarly we get $x^+ - y^- \ge 0$.

Recall that if $x, x', y, y' \in X$ then $\|(x \wedge x') - (y \wedge y')\| \le \|x - y\| + \|x' - y'\|$ and $\|(x \vee x') - (y \vee y')\| \le \|x - y\| + \|x' - y'\|$. In particular, $\|x^+ - y^+\| \le \|x - y\|$ and $\|x^- - y^-\| \le \|x - y\|$. □

The following proposition is a local variant of Theorem from [6].

Proposition 3. *Let X be a normed vector lattice and $z \in B^+(X)$. If $B(X)$ is stable at z, then $B^+(X)$ is stable at z.*

Proof: Assume that $B(X)$ is stable at $z \in B^+(X)$. Let $\varepsilon > 0$ and let $x, y \in B^+(X)$ satisfy $z = (x+y)/2$. By definition of stability at a point there exists $\delta > 0$ such that for every $w \in B^+(X)$ (and even $B(X)$) satisfying $\|z - w\| < \delta$ there exist $\tilde{u}$, $\tilde{v} \in B(X)$ such that $w = (\tilde{u} + \tilde{v})/2$, and $\|x - \tilde{u}\| < \varepsilon/5$, $\|y - \tilde{v}\| < \varepsilon/5$. Then by point 1. of Lemma 1 the following inequalities $\|\tilde{u}^+ - x\| < \frac{1}{5}\varepsilon$, $\|\tilde{v}^+ - y\| < \frac{1}{5}\varepsilon$ hold, and

$$\|\tilde{u}^-\| = \|\tilde{u}^+ - x + x - \tilde{u}\| \le \|\tilde{u}^+ - x\| + \|x - \tilde{u}\| < \frac{2}{5}\varepsilon,$$

and analogously $\|\tilde{v}^-\| < \frac{2}{5}\varepsilon$. Put $u := \tilde{u}^+ - \tilde{v}^-$, $v := \tilde{v}^+ - \tilde{u}^-$. By point 2. of Lemma 1, $0 \le u \le \tilde{u}^+$ and $0 \le v \le \tilde{v}^+$ hold, so $u, v \in B^+(X)$. Of course $w = (u + v)/2$ and

$$\|u - x\| = \|\tilde{u}^- + (-\tilde{v}^-) + \tilde{u} - x\| \le \|\tilde{u}^-\| + \|\tilde{v}^-\| + \|\tilde{u} - x\| < \frac{2}{5}\varepsilon + \frac{2}{5}\varepsilon + \frac{1}{5}\varepsilon = \varepsilon,$$

and analogously $\|v - y\| \le \|\tilde{v}^-\| + \|\tilde{u}^-\| + \|\tilde{v} - y\| < \varepsilon$. Because $\varepsilon > 0$ has been arbitrary, $B^+(X)$ is stable at the point z. □

It follows from the above proposition that Theorem proved in [6] is true. It says that in normed lattices if $B(X)$ is stable then $B^+(X)$ is stable. In the case of Orlicz spaces with Luxemburg norm the converse implication is true, too.

The proof needs a lemma which differs from Proposition 1 from [22, p. 504] only in $B(L^\varphi(\mu))$ being replaced by $B^+(L^\varphi(\mu))$.

Lemma 2. *Assume that $L^\varphi(\mu)$ is neither finite dimensional nor isometric to $L^\infty(\mu)$. Let $z \in B^+(L^\varphi(\mu))$ and define, for $n \in \mathbb{N}$, $n \geq 2$,*

$$A_n := \left\{ \omega \in \Omega : |x(\omega)| < \left(1 - \frac{1}{n}\right) c(\varphi) \right\}$$

if $c(\varphi) < +\infty$ and $\varphi(c(\varphi)) < +\infty$, and $A_n = \Omega$ otherwise. If $\left\| z\chi_{A_n} \right\|_\varphi = 1$ for some $n \geq 2$, then the following conditions are equivalent:

(i) *$I_\varphi(z) < 1$;*

(ii) *there exist a subset $E \subset A_n$ of positive measure and functions $x, y \in B^+(L^\varphi(\mu))$ such that $z = \frac{1}{2}(x+y)$, $\left\| z\chi_E \right\|_\varphi < 1$ and $2\varphi\big(z(\omega)\big) < \varphi\big(x(\omega)\big) + \varphi\big(y(\omega)\big)$ for every $\omega \in E$.*

Proof: We follow the proof of Wisła [22, p. 504]. As, clearly, (ii)⇒(i), we should only prove the implication (i)⇒(ii). Let $\Omega = \Omega_1 \cup \Omega_2$, where Ω_1, Ω_2 denote the purely atomic and atomless part of the measure space (Ω, Σ, μ), respectively. Then either $\|z\chi_{\Omega_1 \cap A_n}\|_\varphi = 1$ or $\|z\chi_{\Omega_2 \cap A_n}\|_\varphi = 1$.

(1) Suppose $\|z\chi_{\Omega_2 \cap A_n}\|_\varphi = 1$.

Claim. *There exists a number $1 < \rho < 2$ such that, if $F := \{\omega \in A_n \cap \Omega_2 : 2\varphi(z(\omega)) < \varphi(\rho z(\omega)) < \infty\}$, then $\mu(F) > 0$.*

First suppose that either $c(\varphi) = \infty$ or $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$. Then, since, $\forall\, \lambda > 1$, $I_\varphi(\lambda z\chi_{\Omega_2 \cap A_n}) = \infty$, for every $1 < \rho < \infty$ such that $(1 - 1/n)\rho \leq 1$, we obtain $\mu(F_\rho) > 0$, where $F_\rho := \{\omega \in A_n \cap \Omega_2 : 2\varphi(z(\omega)) < \varphi(\rho z(\omega))\}$, and, moreover, $\varphi(\rho z(\omega)) < \infty$ for every $\omega \in F_\rho$. So, in this case we put $F = F_\rho$ for some $1 < \rho < 2$ such that $(1 - 1/n)\rho \leq 1$.

Assume now that $c(\varphi) < \infty$ and $\varphi(c(\varphi)) = \infty$. Denote $P := \{\omega \in \Omega : |z(\omega)| \geq \frac{1}{2}c(\varphi)\}$. There are two possibilities, namely:

(a) Suppose that $\mu(P \cap A_n \cap \Omega_2) > 0$. Denote $\mathbb{Q}_0 = \mathbb{Q} \cap (1, 2)$ and:

$$\forall\, q \in \mathbb{Q}_0, \quad F_q := \{\omega \in P \cap A_n \cap \Omega_2 : 2\varphi(z(\omega)) < \varphi(qz(\omega)) < \infty\}.$$

Clearly, $P \cap A_n \cap \Omega_2 = \bigcup_{q \in \mathbb{Q}_0} F_q$ a.e. (= almost everywhere), whence we conclude that there exists some $q_0 \in \mathbb{Q}_0$ such that $\mu(F_{q_0}) > 0$. We put $F = F_{q_0}$ in this case.

(b) Suppose that $\mu(P \cap A_n \cap \Omega_2) = 0$. Then for every $1 < \rho < 2$, we have $|z(\omega)| < \frac{1}{2}c(\varphi)$ and $\varphi(z(\omega)) < \infty$ a.e. on $A_n \cap \Omega_2$. Denote

$$\forall\, 1 < \rho < 2, \quad F_\rho := \{\omega \in A_n \cap \Omega_2 : 2\varphi(z(\omega)) < \varphi(\rho z(\omega))\}.$$

We claim that $\mu(F_\rho) > 0$ for every $1 < \rho < 2$. Indeed, otherwise there exists some $1 < \rho_0 < 2$ such that $\mu(F_{\rho_0}) = 0$, that is, $\varphi(\rho_0 z(\omega)) \leq 2\varphi(z(\omega))$ a.e. on $A_n \cap \Omega_2$, whence

$$+\infty = I_\varphi(\rho_0 z\chi_{\Omega_2 \cap A_n}) \leq 2I_\varphi(z\chi_{\Omega_2 \cap A_n}) < 2,$$

a contradiction. So, in this case we put $F = F_\rho$ for some $1 < \rho < 2$.

Since μ is atomless on F, we can find a measurable set $E \subset F$ such that $I_\varphi(\rho z\chi_E) < 1$. Thus $\|z\chi_E\|_\varphi \le 1/\rho < 1$. Define

$$x = z\chi_{\Omega\setminus E} + \rho\chi_E, \qquad y = z_{\Omega\setminus E} + (2-\rho)z\chi_E.$$

Clearly, $x, y \in B^+(L^\varphi(\mu))$. Further, for every $\omega \in E$,

$$\varphi\big(x(\omega)\big) + \varphi\big(y(\omega)\big) \ge \varphi(\rho z(\omega)) > 2\varphi\big(z(\omega)\big).$$

(2) Suppose that $\|z\chi_{\Omega_1\cap A_n}\|_\varphi = 1$. Then, without loss of generality, we can identify $\Omega_1 \cap A_n$ with the set $\mathbb{N}$ of all natural numbers. Since $I_\varphi(z\chi_{\mathbb{N}}) < 1$, there exists $p \in \mathbb{N}$ such that

$$I_\varphi(z\chi_{\{p,p+1,\dots\}}) < 2\eta,$$

where $\eta = 1 - I_\varphi(z) > 0$.

Define $[p,m] = \{p, p+1, \dots, m\}$ if $m \ge p$, $[p,m] = \emptyset$ otherwise. Let

$$h(m) = I_\varphi(z\chi_{\Omega\setminus[p,m]}) + I_\varphi(\rho z\chi_{[p,m]}), \quad m \in \mathbb{N}.$$

Let $q := \max\{\, m \ge p-1 : h(m) < 1 \,\}$. (In Wisła's original paper by mistake there is "min" instead of "max".) We can find $1 < \sigma \le \rho < 2$ such that $I_\varphi(\overline{x}) = 1$, where

$$\overline{x} = z\chi_{\Omega\setminus[p,q+1]} + \rho z\chi_{[p,q]} + \sigma z\chi_{\{q+1\}}.$$

Using similar arguments, we infer the existence of numbers $r \in \mathbb{N}$, $r \ge q+1$ and $1 < \tau \le \rho < 2$ such that $I_\varphi(y) = 1$, where

$$y = z\chi_{\Omega\setminus[p,r+1]} + (2-\rho)z\chi_{[p,q]} + (2-\sigma)z\chi_{\{q+1\}} + \rho z\chi_{[q+2,r]} + \tau z\chi_{\{r+1\}}.$$

Put

$$x = z\chi_{\Omega\setminus[p,r+1]} + \rho z\chi_{[p,q]} + \sigma z\chi_{\{q+1\}} + (2-\rho)z\chi_{[q+2,r]} + (2-\tau)z\chi_{\{r+1\}}.$$

Obviously $x, y \in B^+(L^\varphi(\mu))$, $\frac{1}{2}(x+y) = z$ and $I_\varphi(x) \le I_\varphi(\overline{x}) = 1$. Further

$$I_\varphi(x) \ge I_\varphi(\overline{x}) - I_\varphi(z\chi_{[q+2,r+1]}) > 1 - 2\eta.$$

Taking $E = \{i\}$, where $i \in [p, r+1]$ is such an index for which φ is not affine on the corresponding interval, all the requirements of (ii) are satisfied and the proof is concluded. □

3. Main results

Modifying Theorem 3, p. 506 from [22] we get the following lemma.

Lemma 3. *$B^+(L^\varphi(\mu))$ is stable at a point $z \in B^+(L^\varphi(\mu))$ if and only if at least one of the following conditions is satisfied:*

(i) *$L^\varphi(\mu)$ is finite dimensional,*
(ii) *$L^\varphi(\mu)$ is isometric to $L^\infty(\mu)$,*
(iii) *$\|z\|_\varphi < 1$,*
(iv) *$I_\varphi(z) = 1$,*
(v) *$c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ and $\|z\chi_{A_n}\|_\varphi < 1$ for every $n = 2, 3, \ldots$, where*

$$A_n := \left\{ \omega \in \Omega : |z(\omega)| < \left(1 - \frac{1}{n}\right) c(\varphi) \right\}.$$

Proof: ($\Leftarrow$) Let $z \in B^+(L^\varphi(\mu))$ and let at least one of the conditions (i)–(v) be satisfied. From Theorem 3 from [22] it follows that $B(L^\varphi(\mu))$ is stable at z, and by our Proposition 3 it follows that $B^+(L^\varphi(\mu))$ is stable at z.

($\Rightarrow$) (Sketch according to [22]). Suppose that none of the conditions (i)–(v) is satisfied. By Lemma 2 with its notation we can find $\varepsilon > 0$, $x, y \in B^+(L^\varphi(\mu))$ with $(x+y)/2 = z$ and a set $E \subset A_n$ of positive measure such that $\|z\chi_E\|_\varphi < 1$ and

$$2I_\varphi(z\chi_E) < I_\varphi(u\chi_E) + I_\varphi(v\chi_E)$$

for every $u, v \in B^+(L^\varphi(\mu))$ with $\|u - x\|_\varphi < \varepsilon$ and $\|v - y\|_\varphi < \varepsilon$.

Let $0 < \delta < 2/n$ and fix $k \in \mathbb{N}$ with $k > 2/\delta > n$. We have $I_\varphi(\lambda z \chi_{A_n \setminus E}) = \infty$ for every $\lambda > 1$. Let us take, if $c(\varphi) < \infty$ and $\varphi(c(\varphi)) < \infty$, any countable covering $(E_i)_{i=1}^\infty$ of the set $A_n \setminus E$ consisting of pairwise disjoint sets $E_i \subset A_n \setminus E$ of positive and finite measure and put $a_i = \varphi^{-1}(i)$,

$$E_i = \{ \omega \in \Omega \setminus E : a_{i-1} \le |z(\omega)| < a_i \}, \qquad i = 1, 2, \ldots,$$

in the other cases. Define

$$h(m) = \sum_{i=1}^{m} I_\varphi\left(\left(1 + \frac{1}{k}\right) z\chi_{E_i}\right) + I_\varphi(z\chi_{\Omega \setminus \bigcup_{i=1}^m E_i}), \qquad m = 0, 1, 2, \ldots .$$

Thus $h(m) < \infty$ for every $m \in \mathbb{N}$, and moreover $\lim_m h(m) = \infty$.

Let $p = \max\{m \ge 0 : h(m) < 1\}$ and let $0 < s \le 1/k$ be such a number that $I_\varphi(w) = 1$, where

$$w(\omega) = \begin{cases} \left(1 + \frac{1}{k}\right) z(\omega) & \text{for } \omega \in \bigcup_{i=1}^p E_i, \\ (1+s) z(\omega) & \text{for } \omega \in E_{p+1}, \\ z(\omega) & \text{otherwise.} \end{cases}$$

Suppose that there are $u, v \in B^+(L^\varphi(\mu))$ such that $\|u-x\|_\varphi < \varepsilon$, $\|v-y\|_\varphi < \varepsilon$ and $(u+v)/2 = w$. Then, by the convexity of φ, we have

$$\varphi(\alpha+\eta) \geq \varphi'_+(\alpha)\eta + \varphi(\alpha)$$

for every $\eta \in \mathbb{R}$ and $|\alpha| < c(\varphi)$, where φ'_+ denotes the right hand side derivative of φ. Because there is a minor spelling mistake in Wisła's original paper, we at present precisely give a sequence of inequalities which leads to a contradiction and ends the proof. Namely

$$\begin{aligned}
2 &\geq I_\varphi(u) + I_\varphi(v) \\
&= I_\varphi(u\chi_E) + I_\varphi(v\chi_E) + I\varphi((w+u-w)\chi_{\Omega\setminus E}) + I_\varphi((w+v-w)\chi_{\Omega\setminus E}) \\
&> 2I_\varphi(z\chi_E) + 2I_\varphi(w\chi_{\Omega\setminus E}) + \int\limits_{\Omega\setminus E} \varphi'_+(w(\omega))(u(\omega)+v(\omega)-2w(\omega))\, d\mu \\
&= 2I_\varphi(w) = 2.
\end{aligned}$$

□

By Proposition 2 and the Wisła's Theorem we have at once:

Corollary 1. *In Orlicz spaces $L^\varphi(\mu)$, for $z \in B^+(L^\varphi(\mu))$ the following conditions are equivalent:*

(i) *$B(L^\varphi(\mu))$ is stable at z;*
(ii) *$B^+(L^\varphi(\mu))$ is stable at z.* □

We connect the main theorem with Wisła's Theorem:

Theorem 1. *The following conditions are equivalent.*

(a) *$B(L^\varphi(\mu))$ is stable.*
(b) *$B^+(L^\varphi(\mu))$ is stable.*
(c) *At least one of the following conditions is satisfied:*
  (i) $\dim L^\varphi(\mu) < +\infty$,
  (ii) $L^\varphi(\mu) \cong L^\infty(\mu)$,
  (iii) *$\varphi \in \Delta_r(\mu)$ for some $r > 1$,*
  (iv) *$\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ provided $c(\varphi) < +\infty$ and $\varphi(c(\varphi)) < \infty$,*
  (v) *$\varphi \in \Delta_r^0(\mu)$ for some $r > 1$ on the purely atomic part of Ω provided $c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ and the measure of the atomless part of Ω is finite,*
  (vi) *$c(\varphi) < +\infty$, $\varphi(c(\varphi)) < +\infty$ and $\mu(\Omega) < +\infty$.*

Proof: The equivalence (a)⇔(c) is the content of Theorem 5 from [22].
(a)⇒(b) follows from Proposition 3 (or [6]).

(b)⇒(a) Let $B^+(L^\varphi(\mu))$ be stable. Let $z \in B(L^\varphi(\mu))$. Hence $|z| \in B^+(L^\varphi(\mu))$ and, by assumption, $B^+(L^\varphi(\mu))$ is stable at z, so $B(L^\varphi(\mu))$ is stable at z by Corollary 1. By Proposition 2 it follows that $B(L^\varphi(\mu))$ is stable at z. Because z has been arbitrary, $B(L^\varphi(\mu))$ is stable. □

A. Suarez Granero in [4] has proved that $B(E^\varphi(\mu))$ is stable (in general). Therefore by Proposition 3 (or [6]) it is true:

Corollary 2. *$B^+(E^\varphi(\mu))$ is stable.* □

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