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Colombeau product of currents

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Abstract. Colombeau product of de Rham's currents coincides with generalized Itano one. Sufficient conditions are found under which it is diffeomorphism invariant.

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Introduction

A diffeomorphism invariant Colombeau algebra $\mathcal{G}$ is introduced in [11] and denoted $\mathcal{G}_{\mathrm{d}}$ by Grosser et al. in [6] and [8] to distinguish it from other Colombeau algebras examined there, too. The term 'diffeomorphism invariant', introduced in [6], only indicates that the canonical embedding of distributions into $\mathcal{G}$ is diffeomorphism invariant; for de Rham's currents this no more holds. Let $\Omega, \widetilde{\Omega}$ be open sets in $\mathbb{R}^d$, $\mu : \widetilde{\Omega} \to \Omega$ a diffeomorphism, R a current on Ω, $\mu^* R$ its pullback on $\widetilde{\Omega}$ and ιR its canonical image into the space of generalized differential forms on Ω. We will see that $\overline{\mu}^*(\iota R)$ (pullback of ιR) is associated but in general not equal to $\iota(\mu^* R)$. If S is another current on Ω, then $\overline{\mu}^*(\iota R) \wedge \overline{\mu}^*(\iota S) = \overline{\mu}^*(\iota R \wedge \iota S)$ need not even be associated (see §6, Example) to $\iota(\mu^* R) \wedge \iota(\mu^* R)$. So the Colombeau product of currents (§4, Definition, §6, Example) is not diffeomorphism invariant. Using Itano's definition [9] of wedge product, we will find sufficient conditions for the Colombeau product of two currents to be diffeomorphism invariant. In general, it will be shown that the Colombeau product is equivalent to the Itano one, generalized in an appropriate way.

Notations and basic definitions

We deal with distributions, (generalized) functions, currents etc. defined on an open subset Ω (sometimes $\widetilde{\Omega}$) of $\mathbb{R}^d$. Following [14] and [8], a distribution S will be equivalently denoted e.g. by $S(x)$. Then $\mu^* S$ can be denoted by $S(\mu(x))$ that is more intuitive. However if S is a function, it will be stated explicitly (provided it is not clear) that x stands for a variable, i.e. that $S(x)$ means the same as S or $x \mapsto S(x)$ and does not mean the value of S at a fixed point x.

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We mostly refer to the book [8] by M. Grosser, M. Kunzinger, M. Oberguggenberger and R. Steinbauer. The scaling and translating operators [8, Definition 2.3.1] will be denoted by bold letters here to avoid misunderstanding if another object (e.g. a distribution) is denoted S or T. So, for a function φ on $\mathbb{R}^d$, $\varepsilon > 0$ and $x, h \in \mathbb{R}^d$, we have

$$\begin{aligned}
\mathbf{S}_\varepsilon\varphi(x) &:= \varepsilon^{-d}\varphi\Big(\frac{x}{\varepsilon}\Big),\\
\mathbf{T}_h\varphi(x) &:= \varphi(x-h),\\
\mathbf{T}(\varphi, x) &:= (\mathbf{T}_x\varphi, x).
\end{aligned}$$

As usually, define on $\mathbb{R}^d$

$$\begin{aligned}
\mathcal{A}_0 &:= \left\{\varphi \in \mathscr{D}\,;\ \textstyle\int \varphi(\xi)\,\mathrm{d}\xi = 1\right\},\\
\mathcal{A} &:= \mathcal{A}_0 - \mathcal{A}_0,\\
\mathcal{A}_q &:= \{\varphi \in \mathcal{A}_0\,; \textstyle\int \varphi(\xi)\xi^\alpha\,\mathrm{d}\xi = 0 \quad \text{whenever} \quad \alpha \in \mathbb{N}_0^d, 1 \le |\alpha| \le q\} \quad (q \in \mathbb{N}),\\
U(\Omega) &:= \{(\varphi, x) \in \mathcal{A}_0 \times \Omega\,;\ \mathbf{T}_x\varphi \in \mathcal{A}_0(\Omega)\}\,.
\end{aligned}$$

A representative of a generalized function on Ω is a smooth complex valued function on $U(\Omega)$ (element of $\mathscr{C}^\infty(U(\Omega))$) that is moderate in the following meaning:

$\forall K \Subset \Omega,\ \alpha \in \mathbb{N}_0^d,\ k \in \mathbb{N}_0 \quad \exists N \in \mathbb{N}$:

$$\partial^\alpha\,\mathrm{d}^k R_\varepsilon(\varphi, x)[\psi_1, \ldots, \psi_k] = O(\varepsilon^{-N}) \quad (\varepsilon \searrow 0)$$

uniformly if $x \in K$, φ runs over any bounded subset of $\mathcal{A}_0(\mathbb{R}^d)$ and $\psi_1, \ldots, \psi_k$ run over any bounded subset of $\mathcal{A}(\mathbb{R}^d)$. Here d^k denotes the k-th partial differential of the representative with respect to the first variable, while the derivatives with respect to the second variable are denoted ∂^α. The set of representatives is denoted by $\mathcal{E}_\mathrm{M}$.

A path is a mapping of the interval $]0,1]$ into a locally convex space, mostly into $\mathscr{C}^\infty(\Omega{\to}\mathcal{A}_0)$ and is often denoted by $\varepsilon \mapsto (x \mapsto \varphi_x^\varepsilon)$ or $\varepsilon \mapsto \left(\varphi_x^\varepsilon\right)_{x\in\Omega}$. Grosser et al. in [6], [7] and [8] use the notation $\varphi(\varepsilon, x)$ instead of φ_x^ε. A path

$$\varepsilon \mapsto \left(\varphi_x^\varepsilon\right)_{x\in\Omega} \in \mathscr{C}^\infty(\Omega{\to}\mathcal{A}_0)$$

is said to have asymptotically vanishing moments of order $q \in \mathbb{N}$ iff for every $K \Subset \Omega$ and $\beta \in \mathbb{N}_0^d$ with $1 \le |\beta| \le q$ it is

$$\sup_{x\in K}\left|\int_{\mathbb{R}^d}\varphi_x^\varepsilon(\xi)\xi^\beta\,\mathrm{d}\xi\right| = O(\varepsilon^q) \qquad (\varepsilon \searrow 0).$$

Several equivalent definitions of the ideal $\mathcal{N} \subset \mathcal{E}_{\mathrm{M}}$ of negligible representatives can be found in References. Let us recall the following one [8, Theorem 2.5.4, (0°)].
$\forall K \Subset \Omega,\ n \in \mathbb{N} \quad \exists q \in \mathbb{N} \ \ \forall \mathscr{B}$ (bounded) $\subset \mathscr{D}(\mathbb{R}^d)$:

$$R(\mathbf{S}_\varepsilon\varphi, x) = O(\varepsilon^n) \qquad (\varepsilon \searrow 0)$$

uniformly for $x \in K$, $\varphi \in \mathscr{B} \cap \mathcal{A}_q$.
Then $\mathcal{G}$ is defined to be the quotient algebra $\mathcal{E}_{\mathrm{M}}/\mathcal{N}$. For $R \in \mathcal{E}_{\mathrm{M}}$, denote $[R] := R + \mathcal{N} \in \mathcal{G}/\mathcal{N}$, i.e. the generalized function with a representative R. The canonical embedding ι of the space of Schwartz distributions $\mathscr{D}'(\Omega)$ into the algebra of representatives $\mathcal{E}_{\mathrm{M}}(\Omega)$ is defined: for $S \in \mathscr{D}'$, $\iota S(\varphi, x) := \langle S, \mathbf{T}_x\varphi\rangle$. Consequently the embedding ι of $\mathscr{C}^\infty \subset \mathscr{D}'$ into $\mathcal{E}_{\mathrm{M}}$ is defined, but we do not use the notation σ for another embedding $\mathscr{C}^\infty \to \mathcal{E}_{\mathrm{M}}$ [8, Definition 1.4.3] to avoid a confusion with the same notation of a generalized differential form.

Association and canonical embedding

§**1.** The main difference of the diffeomorphism invariant Colombeau algebra from the original one is the smoothness of representatives and the moderateness of all partial differentials with respect to the first variable. Also the notion of association is different. In order to obtain a diffeomorphism invariant notion, M. Grosser et al. [7, Definition 6.1] or [8, Definition 3.3.22] have introduced on a manifold an intrinsic definition of association that, thanks to Localisation properties [7, Lemma 4.2] or [8, Lemma 3.3.14], can be formulated on Ω as follows. We will call it G-association to distinguish it from the original Colombeau's association [4], called C-association here.

Definitions.
(1°) A generalized function $[R] \in \mathcal{G}(\Omega)$ is called C-associated to 0, denoted $[R] \overset{\mathrm{C}}{\approx} 0$, if for some (hence every) representative R of $[R]$ the following holds:
$\forall \omega \in \mathscr{D}(\Omega)\ \exists q \in \mathbb{N}_0\ \forall \varphi \in \mathcal{A}_q(\mathbb{R}^d)$ we have:

$$\lim_{\varepsilon \searrow 0} \int R(\mathbf{S}_\varepsilon\varphi, x)\,\omega(x)\,\mathrm{d}x = 0.$$

(2°) A generalized function $[R] \in \mathcal{G}(\Omega)$ is called G-associated to 0, denoted $[R] \overset{\mathrm{G}}{\approx} 0$, if for some (hence every) representative R of $[R]$ the following holds:
$\forall \omega \in \mathscr{D}(\Omega)\ \exists q \in \mathbb{N}_0$ such that for every bounded path

$$\varepsilon \mapsto \left(\varphi_x^\varepsilon\right)_{x\in\Omega} \in \mathscr{C}^\infty\big(\Omega \to \mathcal{A}_0(\mathbb{R}^d)\big)$$

that has asymptotically vanishing moments of order q, we have

$$\lim_{\varepsilon\searrow 0}\int R(\mathbf{S}_\varepsilon\varphi_x^\varepsilon,x)\,\omega(x)\,\mathrm{d}x=0.$$

In both cases two generalized functions are defined to be associated if its difference is associated to 0.

The product with $\omega(x)$ in the integrals above is understood to vanish for $x\notin\operatorname{supp}\omega$ even if the factor by $\omega(x)$ is not defined at this point x. Then one can check that, if the path is bounded, these integrals are always defined for $\varepsilon>0$ sufficiently small.

Following [8, 3.3.22] (if this is well understood and well localized) we say that a generalized function $[R]\in\mathcal{G}(\Omega)$ admits $F\in\mathscr{D}'(\Omega)$ as an associated distribution if $[R]\overset{\mathrm{G}}{\approx}\iota(F)$, i.e. if $\forall\,\omega\in\mathscr{D}(\Omega)\quad\exists\,q\in\mathbb{N}_0$ such that for every bounded path

$$\varepsilon\mapsto(\varphi_x^\varepsilon)_{x\in\Omega}\in\mathscr{C}^\infty(\Omega\to\mathcal{A}_0(\mathbb{R}^d))$$

that has asymptotically vanishing moments of order q, we have

$$\lim_{\varepsilon\searrow 0}\int R(\mathbf{S}_\varepsilon\varphi_x^\varepsilon,x)\,\omega(x)\,\mathrm{d}x=\langle F,\omega\rangle.$$

It is not proved in [8] that both formulations are equivalent, and properties of association are only briefly described with vague reference to local theory (the formulation in [8] is more general, concerning generalized functions on a manifold). The former formulation only says that $\forall\,\omega\in\mathscr{D}(\Omega)\quad\exists\,q\in\mathbb{N}_0$ such that for every bounded path

$$\varepsilon\mapsto(\varphi_x^\varepsilon)_{x\in\Omega}\in\mathscr{C}^\infty(\Omega\to\mathcal{A}_0(\mathbb{R}^d))$$

that has asymptotically vanishing moments of order q, we have

$$\lim_{\varepsilon\searrow 0}\int\big(R(\mathbf{S}_\varepsilon\varphi_x^\varepsilon,x)-\iota(F)(\mathbf{S}_\varepsilon\varphi_x^\varepsilon,x)\big)\omega(x)\,\mathrm{d}x=0.$$

Apparently the authors of [8] have known or supposed the very expected fact (quite easy for C-association) that

$$\lim_{\varepsilon\searrow 0}\int\iota(F)(\mathbf{S}_\varepsilon\varphi_x^\varepsilon,x)\,\omega(x)\,\mathrm{d}x=\langle F,\omega\rangle,\tag{1}$$

i.e. that $\iota(F)$ has F as its associated distribution by the latter formulation. We do not prove it here; the reader can prove it by himself similarly as §9, Lemma (3°) is proved.

Example. We are going to show that C-association, defined on a diffeomorphism invariant Colombeau algebra, is not diffeomorphism invariant. Consequently, it is strictly weaker than G-association. On the interval $]0,1[$ define

$$R(\varphi,x) := \cos\int|\varphi|^2 \qquad (\text{for } \ \varphi\in\mathcal{A}_0(\mathbb{R}) \quad \text{and independent on } \ x\in\,]0,1[)$$

(of course, only values at $(\varphi,x)\in U(\,]0,1[\,)$ matter) and

$$\begin{aligned}\mu:\,]1,e[&\to\,]0,1[\\ \widetilde{x}\ &\mapsto\ x=\mu(\widetilde{x}):=\ln\widetilde{x}\,.\end{aligned}$$

Then

$$R(\mathbf{S}_\varepsilon\varphi,x)=\cos\left(\frac{1}{\varepsilon}\int|\varphi|^2\right).$$

This is independent on x, so it is straightforward that $[R]$ is not C-associated to 0. On the other hand (see [8, Definition 2.8.1]), $\overline{\mu}(\widetilde{\varphi},\widetilde{x})=(\varphi,x)$ with $x=\mu(\widetilde{x})$ and $\varphi(\xi)=\widetilde{\varphi}(\mu^{-1}(x+\xi)-\widetilde{x})\cdot|\mu'(\mu^{-1}(x+\xi))|^{-1}$, so the pullback

$$\widetilde{R}(\widetilde{\varphi},\widetilde{x}):=\overline{\mu}^{*}R\,(\widetilde{\varphi},\widetilde{x})=R(\varphi,x)=\cos\int|\widetilde{\varphi}(\mu^{-1}(x+\xi)-\widetilde{x})|^2\cdot|\mu'(\mu^{-1}(x+\xi))|^{-2}\,\mathrm{d}\xi$$

(substitution $\xi=\mu(\widetilde{x}+\widetilde{\xi})-x$)

$$=\cos\int|\widetilde{\varphi}(\widetilde{\xi})|^2\cdot|\mu'(\widetilde{x}+\widetilde{\xi})|^{-1}\,\mathrm{d}\widetilde{\xi}=\cos\int|\widetilde{\varphi}(\widetilde{\xi})|^2\cdot(\widetilde{x}+\widetilde{\xi})\,\mathrm{d}\widetilde{\xi}\,.$$

Hence

$$\widetilde{R}(\mathbf{S}_\varepsilon\widetilde{\varphi},\widetilde{x})=\cos\int\frac{1}{\varepsilon^2}\Big|\widetilde{\varphi}\Big(\frac{\widetilde{\xi}}{\varepsilon}\Big)\Big|^2(\widetilde{x}+\widetilde{\xi})\,\mathrm{d}\widetilde{\xi}=\cos\int|\widetilde{\varphi}(\widetilde{\xi})|^2\Big(\frac{\widetilde{x}}{\varepsilon}+\widetilde{\xi}\Big)\,\mathrm{d}\widetilde{\xi}\,.$$

Integrating per partes in Definition (1°) above, we obtain

$$\begin{aligned}\int\widetilde{R}(\mathbf{S}_\varepsilon\widetilde{\varphi},\widetilde{x})\,\omega(\widetilde{x})\,\mathrm{d}\widetilde{x}&=\int\omega(\widetilde{x})\cdot\cos\int|\widetilde{\varphi}(\widetilde{\xi})|^2\Big(\frac{\widetilde{x}}{\varepsilon}+\widetilde{\xi}\Big)\,\mathrm{d}\widetilde{\xi}\,\mathrm{d}\widetilde{x}\\ &=\int\omega'(\widetilde{x})\cdot\frac{\varepsilon}{\int|\widetilde{\varphi}|^2}\sin\int|\widetilde{\varphi}(\widetilde{\xi})|^2\Big(\frac{\widetilde{x}}{\varepsilon}+\widetilde{\xi}\Big)\,\mathrm{d}\widetilde{\xi}\,\mathrm{d}\widetilde{x}\,,\end{aligned}$$

so $[\widetilde{R}]$ is C-associated to 0.

§**2. Notations.** Following [8], we denote by $\mathcal{T}^r_s(\Omega)$ the $\mathscr{C}^\infty$-module of smooth (r,s)-tensor fields on Ω. We often work with differential forms, so we also use (besides $\mathcal{T}^0_s$) Itano's notation (different from [8]) $\overset{s}{\mathcal{E}}$ for the $\mathscr{C}^\infty$-module of differential forms (this always means smooth, even differential forms) of order s.

Like for the simplified algebra in [8, 3.2.27(iii)], the tensor product of $\mathscr{C}^\infty$-modules $\mathcal{G}(\Omega) \otimes_{\mathscr{C}^\infty(\Omega)} \mathcal{T}^r_s(\Omega)$, briefly denoted by $\mathcal{G} \otimes \mathcal{T}^r_s(\Omega)$, is the $\mathscr{C}^\infty$-module (and $\mathcal{G}$-module) of generalized (r,s)-tensor fields on Ω. Similarly $\mathcal{G} \otimes \overset{s}{\mathcal{E}}(\Omega)$ resp. $\mathscr{D}' \otimes \overset{s}{\mathcal{E}}(\Omega)$ is the $\mathscr{C}^\infty$-module of generalized differential forms resp. of currents on Ω. Here $\mathcal{E}_{\mathrm{M}}$ resp. $\mathcal{G}$ is considered to be a $\mathscr{C}^\infty$-module with multiplication

$$fR := (\varphi, x) \mapsto f(x)R(\varphi, x) \qquad (f \in \mathscr{C}^\infty,\ R \in \mathcal{E}_{\mathrm{M}})$$

resp.

$$f[R] := [fR] = [\iota f] \cdot [R]. \tag{2}$$

The last equality is an important property of smooth functions, see e.g. [8, Theorem 2.4.6(iii)]. Let $x^1, \dots, x^d$ be standard coordinates on $\mathbb{R}^d$ (namely on Ω). Following [8, p. 245], denote

$$\mathcal{I}^d_s := \{I = (i_1, \dots, i_s) \in \mathbb{N}^s;\ 1 \le i_1 < \dots < i_s \le d\}\,,$$
$$\mathrm{d}x^I := \mathrm{d}x^{i_1} \wedge \dots \wedge dx^{i_s}.$$

Then $\left\{dx^I;\ I \in \mathcal{I}^d_s\right\}$ is a basis of the $\mathscr{C}^\infty$-module $\overset{s}{\mathcal{E}}$, so a generalized differential form $\sigma \in \mathcal{G} \otimes \overset{s}{\mathcal{E}}(\Omega)$ has a unique (up to the order of terms, if there is any) expression

$$\sigma = \sum_{I \in \mathcal{I}^d_s} [S_I] \otimes \mathrm{d}x^I \qquad ([S_I] \in \mathcal{G}(\Omega)). \tag{3}$$

Every $\sum_{I \in \mathcal{I}^d_s} S'_I \otimes \mathrm{d}x^I \in \mathcal{E}_{\mathrm{M}} \otimes \overset{s}{\mathcal{E}}(\Omega)$ where $S'_I \in [S_I]$, is called a representative of σ. We denote the set of all representatives of σ by $\left[\sum_{I \in \mathcal{I}^d_s} S_I \otimes \mathrm{d}x^I\right]$. It can be identified with σ. Similarly a current $\overset{s}{\mathrm{S}} \in \mathscr{D}' \otimes \overset{s}{\mathcal{E}}(\Omega)$ (Itano's notation in [9] for homogenous currents) has a unique expression $\sum_{I \in \mathcal{I}^d_s} S_I \otimes \mathrm{d}x^I$ with $S_I \in \mathscr{D}'(\Omega)$. The notation $\sum_{I \in \mathcal{I}^d_s} S_I \,\mathrm{d}x^I$ can be accepted, too, as it hardly can cause a misunderstanding.

Definitions. We say that two generalized differential forms

$$\rho = \sum_{I\in\mathcal{I}_s^d} [R_I]\otimes \mathrm{d}x^I \quad \text{and} \quad \sigma = \sum_{I\in\mathcal{I}_s^d} [S_I]\otimes \mathrm{d}x^I \qquad ([R_I],[S_I]\in\mathcal{G}),$$

are C-associated (resp. G-associated) if their corresponding coefficients $[R_I]$ and $[S_I]$ are C-associated (resp. G-associated).

Let $\lambda\mapsto \overset{s}{\mathrm{T}}_\lambda$ be a net of currents with $\overset{s}{\mathrm{T}}_\lambda = \sum_{I\in\mathcal{I}_s^d} T_{\lambda,I}\otimes \mathrm{d}x^I$ $(T_{\lambda,I}\in\mathscr{D}'(\Omega_\lambda)$ and λ running over a neighbourhood of 0 in $\mathbb{R}$). We say that $\lim_{\lambda\to 0}\overset{s}{\mathrm{T}}_\lambda = \overset{s}{\mathrm{T}}_0$ on the domain Ω_0 of $\overset{s}{\mathrm{T}}_0$ if for every test function $\varphi\in\mathscr{D}(\Omega_0)$ and $I\in\mathcal{I}_s^d$ we have $\lim_{\lambda\to 0}\langle T_{\lambda,I},\varphi\rangle = \langle T_{0,I},\varphi\rangle$.

§**3. Remark.** Note that we consider smooth functions to be directly elements of $\mathscr{D}'$, but not of $\mathcal{G}$. We only have a canonical embedding (let us denote it by $[\iota]$) $\mathscr{D}'\ni S\mapsto[\iota S]$ of $\mathscr{D}'$ into $\mathcal{G}$ and consequently of $\mathscr{C}^\infty$ into $\mathcal{G}$. Similarly (smooth) differential forms are currents but they are not generalized differential forms. Now we are going to extend, with the same notation, the embedding ι on currents. We will see that this extension of the canonical embedding is no more diffeomorphism invariant.

Definition. For a current $\overset{s}{\mathrm{S}} = \sum_{I\in\mathcal{I}_s^d} S_I\otimes \mathrm{d}x^I$ with $S_I\in\mathscr{D}'(\Omega)$, define

$$\iota\,\overset{s}{\mathrm{S}} := \sum_{I\in\mathcal{I}_s^d} \iota S_I\otimes \mathrm{d}x^I \in \mathcal{E}_{\mathrm{M}}\otimes\overset{s}{\mathcal{E}}(\Omega),$$

so the canonical image of $\overset{s}{\mathrm{S}}$ is

$$[\iota]\,\overset{s}{\mathrm{S}} := [\iota\,\overset{s}{\mathrm{S}}] = \sum [\iota S_I]\otimes \mathrm{d}x^I \in \mathcal{G}\otimes\overset{s}{\mathcal{E}}(\Omega).$$

This also defines the canonical embedding of differential forms.

Proposition. *Let A be a finite set of indices and $(\omega^\alpha)\in\overset{s}{\mathcal{E}}(\Omega)$ for every $\alpha\in A$. Then:*

(1°) *If for smooth functions $f_I, g_\alpha\in\mathscr{C}^\infty(\Omega)$*

$$\sum_{\alpha\in A} g_\alpha\omega^\alpha = \sum_{I\in\mathcal{I}_s^d} f_I\,\mathrm{d}x^I\,,$$

then

$$\sum_\alpha [\iota g_\alpha]\otimes\omega^\alpha = \sum_I [\iota f_I]\otimes \mathrm{d}x^I\,.$$

(2°) *If for distributions* $F_I, G_\alpha \in \mathscr{D}'(\Omega)$

$$\sum_{\alpha\in A} G_\alpha \otimes \omega^\alpha = \sum_{I\in\mathcal{I}_s^d} F_I \otimes \mathrm{d}x^I \,,$$

then

$$\sum_\alpha [\iota G_\alpha] \otimes \omega^\alpha \overset{\mathrm{G}}{\approx} \sum_I [\iota F_I] \otimes \mathrm{d}x^I \,.$$

Proof of (1°): We have $\sum_I f_I \,\mathrm{d}x^I = \sum_\alpha g_\alpha \,\omega^\alpha$. Writing

(4) $$\omega^\alpha = \sum_{I\in\mathcal{I}_s^d} a_I^\alpha \,\mathrm{d}x^I \qquad (\, a_I^\alpha \in \mathscr{C}^\infty(\Omega)),$$

we get

$$\sum_I f_I \,\mathrm{d}x^I = \sum_\alpha g_\alpha \sum_I a_I^\alpha \,\mathrm{d}x^I \,,$$

so $f_I = \sum_\alpha g_\alpha \, a_I^\alpha$ and the right hand side of the equality in (1°) is

$$\sum_I [\iota f_I] \otimes \mathrm{d}x^I = \sum_I \sum_\alpha [\iota(g_\alpha \, a_I^\alpha)] \otimes \mathrm{d}x^I \,.$$

Now we transform the left hand side of the equality in (1°) by (4) in $\mathcal{G} \otimes \overset{s}{\mathcal{E}}$:

$$\sum_\alpha [\iota g_\alpha] \otimes \omega^\alpha = \sum_\alpha [\iota g_\alpha] \otimes \sum_I a_I^\alpha \,\mathrm{d}x^I = \sum_I \sum_\alpha a_I^\alpha [\iota g_\alpha] \otimes \mathrm{d}x^I \,.$$

The coefficient at $\mathrm{d}x^I$ in this expression is $a_I^\alpha[\iota g_\alpha] = [\iota a_I^\alpha][\iota g_\alpha]$ (see (2)). This is equal to the corresponding coefficient on the right hand side above $[\iota(g_\alpha \, a_I^\alpha)]$ because the canonical embedding of $\mathscr{C}^\infty$ into $\mathcal{G}$ preserves multiplication (consequence of [8, Theorem 2.4.6(iii)]). □

Proof of (2°): is the same as the proof of (1°) above, only at the end of the proof the equality $[\iota a_I^\alpha][\iota G_\alpha] = [\iota(a_I^\alpha \, G_\alpha)]$ does not hold in general, but the association holds, see [8, formula (3.106)]. □

§**4. Notation.** Let $\mu : \widetilde{\Omega} \to \Omega = \mu(\widetilde{\Omega})$ be a diffeomorphism and

(3) $$\sigma = \sum_{I\in\mathcal{I}_s^d} [S_I] \otimes \mathrm{d}x^I \qquad ([S_I] \in \mathcal{G}(\Omega))$$

a generalized differential form. The mapping $\overline{\mu}$ [8, Definition 2.3.7] is defined

$$\begin{array}{rrcl} \overline{\mu}: & U(\widetilde{\Omega}) & \to & U(\Omega) = \overline{\mu}(U(\widetilde{\Omega})) \\ & (\widetilde{\varphi}, \widetilde{x}) & \mapsto & (\varphi, x) \end{array}$$

where

$$\begin{aligned} x &= \mu(\widetilde{x}) \\ \varphi(\xi) &= \widetilde{\varphi}\big(\mu^{-1}(x+\xi) - \widetilde{x}\big) \cdot |\operatorname{Jac} \mu^{-1}(x+\xi)| \end{aligned}$$

(Jac denotes the jacobian). If Vol stands for the standard volume on $\mathbb{R}^d$ (as an odd differential form), this notation means that the test density $\mathbf{T}_x\varphi \cdot \mathrm{Vol}$ is the pullback via μ^{-1} of the test density $\mathbf{T}_{\widetilde{x}}\widetilde{\varphi} \cdot \mathrm{Vol}$. Equivalently, φ is the unique function for which

$$\int\limits_{\operatorname{supp} \mathbf{T}_x\varphi} \mathbf{T}_x\varphi \cdot f = \int\limits_{\operatorname{supp} \mathbf{T}_{\widetilde{x}}\widetilde{\varphi}} \mathbf{T}_{\widetilde{x}}\widetilde{\varphi} \cdot (f \circ \mu)$$

for every function $f \in \mathscr{C}^\infty(\Omega)$.

Now, if $R \in \mathcal{E}_M(\Omega)$, then its pullback is $\overline{\mu}^* R \in \mathcal{E}_M(\widetilde{\Omega})$ and we consider the generalized function $[\overline{\mu}^* R] \in \mathcal{G}(\widetilde{\Omega})$ to be the pullback (via μ) of the generalized function $[R] \in \mathcal{G}(\Omega)$. In [8, §2.8] the authors have shown that the pullback of a generalized function (denoted $\widehat{\mu}$ there) is well defined. For the generalized differential form defined by (3), we define

$$\mu^*\sigma = \sum_{I \in \mathcal{I}_s^d} [\overline{\mu}^* S_I] \otimes \mu^* \, \mathrm{d}x^I \,. \tag{5}$$

We do not introduce a special notation for the pullback of a generalized differential form, different from the one for the pullback of a smooth differential form, similarly as e.g. Bishop and Goldberg in [1] also use the same notation μ^* for pullbacks via μ of different objects (functions, differential forms, connections, ...) although the definition of the pullback depends on the type of this object.

For the following proposition, if σ is given by (3) and $v_1, \dots, v_s$ are smooth vector fields on Ω, then, like for smooth differential forms, $\sigma(v_1, \dots, v_s)$ is the generalized function

$$\sum_{I \in \mathcal{I}_s^d} \mathrm{d}x^I(v_1, \dots, v_s) \cdot [S_I] \tag{6}$$

($\mathscr{C}^\infty$-module multiplication, see (2)).

Propositions.

(1°) *The application* $\overline{\mu}^* : \mathcal{E}_{\mathrm{M}}(\Omega) \to \mathcal{E}_{\mathrm{M}}(\widetilde{\Omega})$ *preserves the* $\mathscr{C}^\infty$*-module structure. This means that the multiplication with a function defined before* (2) *fulfills*

$$\overline{\mu}^*(fR) = \mu^* f \cdot \overline{\mu}^* R \qquad (f \in \mathscr{C}^\infty(\Omega),\ R \in \mathcal{E}_{\mathrm{M}}(\Omega))$$

where, as usually, $\mu^* f$ *means* $f \circ \mu$.

(2°) *Let* A *be a finite set of indices and* $\omega^\alpha \in \overset{s}{\mathcal{E}}(\Omega)$ *for every* $\alpha \in A$. *If* (*besides* (3))

$$\sigma = \sum_{\alpha \in A} [R_\alpha] \otimes \omega^\alpha \qquad ([R_\alpha] \in \mathcal{G}(\Omega)),$$

then we have (*besides* (6) *also*)

$$\sigma(v_1, \dots, v_s) = \sum_{\alpha \in A} \omega^\alpha(v_1, \dots, v_s) \cdot [R_\alpha].$$

(3°) *The generalized differential form* $\mu^*\sigma \in \mathcal{G} \otimes \overset{s}{\mathcal{E}}(\widetilde{\Omega})$ *by* (5) *is the pullback of the generalized differential form* $\sigma \in \mathcal{G} \otimes \overset{s}{\mathcal{E}}(\Omega)$. *This means that, for arbitrary smooth vector fields* $\widetilde{v}_1, \dots, \widetilde{v}_s$ *on* $\widetilde{\Omega}$ *and their direct images* $\mu_*\widetilde{v}_1, \dots, \mu_*\widetilde{v}_s$ *on* Ω, *the generalized function* $(\mu^*\sigma)(\widetilde{v}_1, \dots, \widetilde{v}_s) \in \mathcal{G}(\widetilde{\Omega})$ *is the pullback of the generalized function* $\sigma(\mu_*\widetilde{v}_1, \dots, \mu_*\widetilde{v}_s) \in \mathcal{G}(\Omega)$.

Proof: (1°) can be easily verified. (2°) follows simply, if we express the forms ω^α in the basis $\left(\mathrm{d}x^I\right)_I$ as in the proof in §3. We are going to prove (3°). We have by (5) and the already proved part (2°)

$$(\mu^*\sigma)(\widetilde{v}_1, \dots, \widetilde{v}_s) = \sum_{I \in \mathcal{I}_s^d} \left(\mu^* \,\mathrm{d}x^I\right)(\widetilde{v}_1, \dots, \widetilde{v}_s) \cdot [\overline{\mu}^* S_I], \tag{7}$$

and by (6)

$$\sigma(\mu_*\widetilde{v}_1, \dots, \mu_*\widetilde{v}_s) = \sum_{I \in \mathcal{I}_s^d} \mathrm{d}x^I(\mu_*\widetilde{v}_1, \dots, \mu_*\widetilde{v}_s) \cdot [S_I]. \tag{8}$$

The function $\left(\mu^* \,\mathrm{d}x^I\right)(\widetilde{v}_1, \dots, \widetilde{v}_s)$ is the pullback of the function $\mathrm{d}x^I(\mu_*\widetilde{v}_1, \dots, \mu_*\widetilde{v}_s)$, because this is the definition of the pullback of a differential form. Thus, by (7), (8) and the part (1°), $\sigma(\mu_*\widetilde{v}_1, \dots, \mu_*\widetilde{v}_s)$ is the pullback of $(\mu^*\sigma)(\widetilde{v}_1, \dots, \widetilde{v}_s)$, which completes the proof. □

§**5. Remark** (non commutativity of μ^* and $[\iota]$)**.** The canonical image of a current

$$\overset{s}{\mathrm{G}} = \sum_{I\in\mathcal{I}_s^d} G_I \otimes \mathrm{d}x^I \qquad (G_I \in \mathscr{D}'(\Omega))$$

into $\mathcal{G} \otimes \overset{s}{\mathcal{E}}(\Omega)$ is

$$[\iota\,\overset{s}{\mathrm{G}}] = \sum_{I\in\mathcal{I}_s^d} [\iota G_I] \otimes \mathrm{d}x^I$$

and the pullback of the latter is (see (5))

$$\mu^*[\iota\,\overset{s}{\mathrm{G}}] = \sum_{I\in\mathcal{I}_s^d} [\overline{\mu}^*(\iota G_I)] \otimes \mu^*\,\mathrm{d}x^I = \sum_{I\in\mathcal{I}_s^d} [\iota(\mu^* G_I)] \otimes \mu^*\,\mathrm{d}x^I\,, \tag{9}$$

because our algebra is diffeomorphism invariant. On the other hand, the pullback of the current $\overset{s}{\mathrm{G}}$ is a current $\mu^*\,\overset{s}{\mathrm{G}} = \sum_{I\in\mathcal{I}_s^d} \mu^* G_I \otimes \mu^*\,\mathrm{d}x^I$ on $\widetilde{\Omega}$. For calculating the canonical image of $\mu^*\,\overset{s}{\mathrm{G}}$, we have to express $\mu^*\,\overset{s}{\mathrm{G}}$ in the basis $(\,\mathrm{d}x_I)_{I\in\mathcal{I}_s^d}$:

$$\mu^*\,\overset{s}{\mathrm{G}} = \sum_{I\in\mathcal{I}_s^d} \mu^* G_I \otimes \mu^*\,\mathrm{d}x^I = \sum_{I\in\mathcal{I}_s^d} F_I \otimes \mathrm{d}x^I \qquad (F_I \in \mathscr{D}'(\widetilde{\Omega}))$$

and we obtain

$$[\iota\mu^*\,\overset{s}{\mathrm{G}}] = \sum_{I\in\mathcal{I}_s^d} [\iota F_I] \otimes \mathrm{d}x^I\,.$$

Comparing it with (9), we see due to §3, Proposition (2°) that $[\iota\mu^*\,\overset{s}{\mathrm{G}}]$ is only G-associated with $\mu^*[\iota\,\overset{s}{\mathrm{G}}]$ and we can check that in many cases the equality does not hold. In other words, the canonical embedding $[\iota]$ for currents is not diffeomorphism invariant and on a manifold it can be only defined up to association.

Wedge product

§**6.** The following definition is similar to the definition of the Colombeau product of distributions inside distributions introduced in [10]. Of course, like the product of distributions, the wedge product is not defined for arbitrary two currents. Note that in [10] no embedding ι is used and distributions are considered to be elements of $\mathcal{G}$. Instead, the product in $\mathcal{G}$ is denoted differently from the classical product. Moreover, only the original Colombeau algebra that is not diffeomorphism invariant is considered there.

Notation. In the sequel, the currents considered will always be denoted

$$\overset{r}{\mathrm{R}} = \sum_{I\in\mathcal{I}_r^d} R_I \otimes \mathrm{d}x^I \in \mathscr{D}'(\Omega)\otimes \overset{r}{\mathcal{E}} \quad \text{and} \quad \overset{s}{\mathrm{S}} = \sum_{J\in\mathcal{I}_s^d} S_J \otimes \mathrm{d}x^J \in \mathscr{D}'(\Omega)\otimes \overset{s}{\mathcal{E}}$$

$$(R_I, S_J \in \mathscr{D}'(\Omega)).$$

With the notation $I = (i_1, \dots, i_r)$, $J = (j_1, \dots, j_s)$, we denote by IJ the multi-index $(i_1, \dots, i_r, j_1, \dots, j_s)$. In relations or expressions like $I \cap J$, containing set operators, multi-indices are considered only as sets; this is used only for increasing multiindices. For an arbitrary multi-index $L = (\ell_1, \dots, \ell_t) \in \mathbb{N}^t$, we define $\operatorname{sgn} L = 0$ provided there are two equal indices $\ell_i = \ell_j \quad (i \neq j)$ in L; else $\operatorname{sgn} L$ is the sign of the permutation of L to the increasing order.

Definition. With this notation we say that a current $\overset{r+s}{\mathrm{T}} \in \mathscr{D}'(\Omega)\otimes \overset{r+s}{\mathcal{E}}$ is the Colombeau (wedge) product of $\overset{r}{\mathrm{R}}$ and $\overset{s}{\mathrm{S}}$ and denote $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ iff

$$[\iota \overset{r+s}{\mathrm{T}}] \overset{\mathrm{C}}{\approx} [\iota \overset{r}{\mathrm{R}}] \wedge [\iota \overset{s}{\mathrm{S}}] = \sum_{I\in\mathcal{I}_r^d, J\in\mathcal{I}_s^d} [\iota R_I][\iota S_J] \otimes \mathrm{d}x^I \wedge \mathrm{d}x^J .$$

It is known that $\overset{r+s}{\mathrm{T}}$, if it exists, is well defined. We will see at the end of this paper that the G-associativity gives the same result. The following example shows that the Colombeau product is not diffeomorphism invariant.

Example. On $\Omega := (-1,1)\times(-1,1) \subset \mathbb{R}^2$ define currents (do not confuse upper indices with powers)

$$\begin{aligned} \overset{1}{\mathrm{R}}(x) &= \delta(x) \otimes \big((x^2+1)\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2\big) \\ \overset{1}{\mathrm{S}}(x) &= S(x) \otimes \big((x^2+1)\,\mathrm{d}x^1 + x^1\,\mathrm{d}x^2\big) \end{aligned} \tag{10}$$

where δ is the Dirac measure on $\mathbb{R}^2$. For defining the distribution S on Ω, we first choose a function $\alpha \in \mathscr{D}([-\frac{1}{2}, \frac{1}{2}])$ with $\alpha(0) = 1$. Then

$$\langle S, \omega\rangle := \int_{\mathbb{R}^2} \frac{\omega(x) - \alpha(x^1)\,\omega(0,x^2)}{x^1}\,\mathrm{d}x.$$

We deduce

$$\begin{aligned} \iota \overset{1}{\mathrm{R}}(\varphi, y) &= \iota\big((x^2+1)\delta(x)\big)(\varphi,y)\,\mathrm{d}x^1 + \iota\big(x^1\delta(x)\big)(\varphi,y)\,\mathrm{d}x^2 \\ &= \langle (x^2+1)\delta(x)\,,\, \mathbf{T}_y\varphi(x)\rangle_x\,\mathrm{d}x^1 = \varphi(-y)\,\mathrm{d}x^1 \end{aligned}$$

and

$$\begin{aligned}\iota\overset{1}{\mathrm{S}}(\varphi,y)&=\iota\big((x^2+1)S(x)\big)(\varphi,y)\,\mathrm{d}x^1\;+\;\iota\big(x^1S(x)\big)(\varphi,y)\,\mathrm{d}x^2\\&=(\dots)\,\mathrm{d}x^1\;+\;\langle S(x),\,x^1\varphi(x-y)\rangle_x\,\mathrm{d}x^2\\&=(\dots)\,\mathrm{d}x^1+\left(\int_{\mathbb{R}^2}\frac{x^1\varphi(x-y)}{x^1}\,\mathrm{d}x\right)\mathrm{d}x^2=(\dots)\,\mathrm{d}x^1+\,\mathrm{d}x^2\,.\end{aligned}$$

So $(\iota\overset{1}{\mathrm{R}}\wedge\iota\overset{1}{\mathrm{S}})(\varphi,y)=\varphi(-y)\,\mathrm{d}x^1\wedge\mathrm{d}x^2$. This means $R\overset{\mathrm{C}}{\wedge}S=\delta\otimes\;\mathrm{d}x^1\wedge\,\mathrm{d}x^2$.

On the other hand, consider a diffeomorphism

$$\mu:\quad(\widetilde{x}^1,\widetilde{x}^2)\mapsto(x^1,x^2),\qquad \begin{aligned}x^1&=\frac{\widetilde{x}^1}{\widetilde{x}^2+1}\\x^2&=\widetilde{x}^2\end{aligned}$$

and calculate the pullbacks of the currents (10):

$$\mu^*\overset{1}{\mathrm{R}}\,(\widetilde{x})=\mu^*\delta\,(\widetilde{x})\otimes\left((\widetilde{x}^2+1)\,\mathrm{d}\frac{\widetilde{x}^1}{\widetilde{x}^2+1}+\frac{\widetilde{x}^1}{\widetilde{x}^2+1}\,\mathrm{d}\widetilde{x}^2\right)$$

and similarly for $\overset{1}{\mathrm{S}}$. We can calculate that the coefficient by $\mathrm{d}\widetilde{x}^2$ vanishes for both currents. The currents contain only the term with $\mathrm{d}\widetilde{x}^1$, so their Colombeau wedge product is zero.

§**7.** Also Itano in [9] has defined the wedge product of currents $\overset{r}{\mathrm{R}}$, $\overset{s}{\mathrm{S}}$, roughly speaking, to be the section $\overset{r+s}{\mathrm{T}}(x)$ (if it exists) of the direct product

$$\overset{r}{\mathrm{R}}(x)\times\overset{s}{\mathrm{S}}(y):=\sum_{I\in\mathcal{I}^d_r,J\in\mathcal{I}^d_s}(R_I(x)\times S_J(y))\otimes(\,\mathrm{d}x^I\wedge\mathrm{d}y^J)\in\mathscr{D}'(\Omega\times\Omega)\otimes\overset{r+s}{\mathcal{E}}$$

on the diagonal $\Delta:=\{(x,y)\in\Omega\times\Omega;\;x=y\}$. The direct product of distributions is introduced by Schwartz [15] for arbitrary two distributions. For calculating the section on a submanifold Δ, we have to choose coordinates $(\widetilde{x},\widetilde{y})$ for which $\Delta=\{(\widetilde{x},\widetilde{y});\;\widetilde{y}=0\}$. The reader need not know exactly what is the section of the current. As it is shown in [9] that the result does not depend on a particular choice of coordinates, choose e.g. $x=\widetilde{x}+\widetilde{y}\,,\;y=\widetilde{x}-\widetilde{y}$. We get, after omitting the tildes, that we can define $\overset{r+s}{\mathrm{T}}(x)$ to be the section of the current

$$\overset{r+s}{\mathrm{W}_0}(x,y):=\overset{r}{\mathrm{R}}(x+y)\,\overset{s}{\mathrm{S}}(x-y)=\sum_{I\in\mathcal{I}^d_r,J\in\mathcal{I}^d_s}(R_I(x+y)S_J(x-y))\otimes$$
$$\big((\,\mathrm{d}x^{i_1}+\,\mathrm{d}y^{i_1})\wedge\cdots\wedge(\,\mathrm{d}x^{i_r}+\,\mathrm{d}y^{i_r})\wedge(\,\mathrm{d}x^{j_1}-\,\mathrm{d}y^{j_1})\wedge\cdots\wedge(\,\mathrm{d}x^{j_s}-\,\mathrm{d}y^{j_s})\big)$$
$$\text{with the notation}\qquad I=(i_1,\dots,i_r),\;J=(j_1,\dots,j_s),$$

defined on

$$\Gamma := \Big\{(x,y) \in \mathbb{R}^{2d};\ x+y \in \Omega,\ x-y \in \Omega\Big\}, \tag{11}$$

for $y=0$. The products of distributions $R_I(x+y)S_J(x-y)$ have always sense, being pullbacks of direct products. The section is defined exactly as Lojasiewicz has introduced for distributions in [14], i.e.

$$\overset{r+s}{\mathrm{T}}(x) := \lim_{\lambda\to 0} \overset{r+s}{\mathrm{W}}(x,\lambda y)$$

provided the limit exists and does not depend on y (more precisely the left hand side could be written $\overset{r+s}{\mathrm{T}}(x) \times \mathbf{1}(y)$ where $\mathbf{1}$ stands for the constant function $=1$).

Let us do a slight generalization, replacing the current $\overset{r}{\mathrm{R}}(x) \times \overset{s}{\mathrm{S}}(y)$ above with the current $\frac{1}{2}(\overset{r}{\mathrm{R}}(x) \times \overset{s}{\mathrm{S}}(y) + \overset{r}{\mathrm{R}}(y) \times \overset{s}{\mathrm{S}}(x))$. This gives the following definition:

Definition. We say that a current $\overset{r+s}{\mathrm{T}} \in \mathscr{D}' \otimes \overset{r+s}{\mathcal{E}}(\Omega)$ is the Itano (wedge) product of currents $\overset{r}{\mathrm{R}}$, $\overset{s}{\mathrm{S}}$ by Notation 6 and denote $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{I}}{\wedge} \overset{s}{\mathrm{S}}$ iff $\overset{r+s}{\mathrm{T}}(x)$ is the section of the current

$$\overset{r+s}{\mathrm{W}}(x,y) := \frac{1}{2}\Big(\overset{r}{\mathrm{R}}(x+y)\,\overset{s}{\mathrm{S}}(x-y) + \overset{r}{\mathrm{R}}(x-y)\,\overset{s}{\mathrm{S}}(x+y)\Big)$$

for $y=0$.

The reason for this definition is (besides the generalization) that in a similar way the Colombeau product of distributions is characterized in [10]. However for currents the Itano product is not equivalent with the Colombeau one, because the Itano product, having in [9] an intrinsic definition, is diffeomorphism invariant.

§**8.** Note that by this definition the wedge product can exist although the wedge products $R_I\,\mathrm{d}x^I \wedge S_J\,\mathrm{d}x^J$ of some particular terms does not. So the Itano product cannot be calculated term by term. There is some may be unexpected displeasure that for $I \cap J \neq \emptyset$ the product $R_I\,\mathrm{d}x^I \overset{\mathrm{I}}{\wedge} S_J\,\mathrm{d}x^J$ need not always be $=0$. Although it cannot be nonzero, it need not exist. So we generalize the Itano's definition, setting the wedge products of these terms $=0$ by definition. Thus we obtain a generalized Itano product denoted by $\overset{r}{\mathrm{R}} \overset{\mathrm{gI}}{\wedge} \overset{s}{\mathrm{S}}$, but this notation becomes superfluous when we prove that the generalized Itano product is equivalent to the Colombeau one. Consequently the (non-generalized or only slightly generalized in §7) Itano product is strictly stronger and represents sufficient conditions under which the Colombeau product is diffeomorphism invariant.

Definition (see §6, Notation).
(1°) We say that a current

$$\overset{r+s}{\mathrm{T}} = \sum_{L\in\mathcal{I}^d_{r+s}} T_L \otimes \mathrm{d}x^L \in \mathscr{D}'(\Omega)\otimes \overset{r+s}{\mathcal{E}}$$

is the generalized Itano product of $\overset{r}{\mathrm{R}}$ and $\overset{s}{\mathrm{S}}$ and denote $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{gI}}{\wedge} \overset{s}{\mathrm{S}}$ iff for all $L \in \mathcal{I}^d_{r+s}$ the distribution $T_L(x)$ is the section for $y=0$ of the distribution

$$W_L(x,y) := \sum_{\substack{I\in\mathcal{I}^d_r, J\in\mathcal{I}^d_s\\ I\cap J=\emptyset, I\cup J=L}} \frac{\mathrm{sgn}(IJ)}{2}\big(R_I(x+y)S_J(x-y)+R_I(x-y)S_J(x+y)\big)$$

defined on Γ, see (11).

(2°) Equivalently, $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{gI}}{\wedge} \overset{s}{\mathrm{S}}$ is the section for $y=0$ of the current $\overset{r+s}{\mathrm{W}}$' obtained from the current

$$\overset{r+s}{\mathrm{W}}(x,y) := \frac{1}{2}\big(\overset{r}{\mathrm{R}}(x+y)\,\overset{s}{\mathrm{S}}(x-y) + \overset{r}{\mathrm{R}}(x-y)\,\overset{s}{\mathrm{S}}(x+y)\big)$$

expressed in the form $\displaystyle\sum_{t=0}^{r+s} \sum_{\substack{L\in\mathcal{I}^d_t\\ M\in\mathcal{I}^d_{r+s-t}}} W_{L,M}(x,y)\otimes \mathrm{d}x^L \wedge \mathrm{d}y^M \in \mathscr{D}'(\Omega)$

by eliminating all terms with $t < r+s$, i.e. keeping only $M=\emptyset$ in the last expression.

The equivalence is straightforward.

§**9.** The rest of the paper is devoted to prove the equivalence of the Colombeau product with the generalized Itano product. We often refer to [10] where similar things are done for distributions.

Lemma. *Let* $\overset{r+s}{\mathrm{T}}$ *by* §8, *Definition* (1°) *and* $\overset{r}{\mathrm{R}}$, $\overset{s}{\mathrm{S}}$ *by* §6, *Notation be currents on* Ω. *Then*
(1°) $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ *on* Ω *iff the following holds:*

$$\forall\,\omega\in\mathscr{D}(\Omega) \quad \exists\, q\in\mathbb{N}_0 \quad \forall\,\varphi\in\mathcal{A}_q \quad \forall L\in\mathcal{I}^d_{r+s}$$

we have

$$\langle T_L,\omega\rangle = \lim_{\varepsilon\searrow 0}\langle W_L,\eta_\varepsilon\rangle$$

where

$$\eta_\varepsilon(x,y) := 2^d \mathbf{S}_\varepsilon\varphi(x-y)\,\mathbf{S}_\varepsilon\varphi(x+y) *_x \omega(x)$$
$$= 2^d \int \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,\omega(x+h)\,\mathrm{d}h.$$

Moreover, the number q *in the definition of C-association for the relation* $[\iota \overset{r+s}{\mathrm{T}}] \overset{\mathrm{C}}{\approx} [\iota \overset{r}{\mathrm{R}}] \wedge [\iota \overset{s}{\mathrm{S}}]$ *in* §6, *Definition can be chosen the same as in this statement.*

(2°) $[\iota \overset{r+s}{\mathrm{T}}] \overset{\mathrm{G}}{\approx} [\iota \overset{r}{\mathrm{R}}] \wedge [\iota \overset{s}{\mathrm{S}}]$ *on* Ω *iff the following holds:* $\forall\, \omega \in \mathscr{D}(\Omega) \quad \exists\, q \in \mathbb{N}_0$ *such that for every bounded path*

$$\varepsilon \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty\big(\Omega{\to}\mathcal{A}_0(\mathbb{R}^d)\big)$$

that has asymptotically vanishing moments of order q, *and* $\forall L \in \mathcal{I}^d_{r+s}$, *we have*

$$\langle T_L, \omega\rangle = \lim_{\varepsilon\searrow 0} \langle W_L, \zeta_\varepsilon\rangle$$

where

$$\zeta_\varepsilon(x,y) := 2^d \int \mathbf{S}_\varepsilon\varphi_z^\varepsilon(x-z-y)\,\mathbf{S}_\varepsilon\varphi_z^\varepsilon(x-z+y)\,\omega(z)\,\mathrm{d}z$$
$$= 2^d \int \mathbf{S}_\varepsilon\varphi_{x+h}^\varepsilon(-h-y)\,\mathbf{S}_\varepsilon\varphi_{x+h}^\varepsilon(-h+y)\,\omega(x+h)\,\mathrm{d}h.$$

Moreover, the number q *in the definition of G-association* §1, *Definition* (2°) *for the relation* $[\iota \overset{r+s}{\mathrm{T}}] \overset{\mathrm{G}}{\approx} [\iota \overset{r}{\mathrm{R}}] \wedge [\iota \overset{s}{\mathrm{S}}]$ *can be chosen the same as in this statement.*

(3°) *For* $\varphi \in \mathcal{A}_0$ *the functions* $x, y \mapsto \varepsilon^d \eta_\varepsilon(x, \varepsilon y)$ *converge in* $\mathscr{D}(\Omega\times\mathbb{R}^d)$ *to*

$$2^d\omega(x) \int \varphi(-h-y)\varphi(-h+y)\,\mathrm{d}h = 2^d\omega(x)\cdot(\check\varphi * \varphi)(2y)$$

$(\check\varphi(x) := \varphi(-x))$. *For arbitrary bounded path*

$$\varepsilon \mapsto \big(\varphi_x^\varepsilon\big)_{x\in\Omega} \in \mathscr{C}^\infty\big(\Omega{\to}\mathcal{A}_0(\mathbb{R}^d)\big),$$

there is an $\varepsilon_0 > 0$ *such that the set of functions* $\Big\{x, y \mapsto \varepsilon^d\zeta_\varepsilon(x,\varepsilon y); 0<\varepsilon<\varepsilon_0\Big\}$ *is bounded in* $\mathscr{D}(\Omega\times\mathbb{R}^d)$, *and*

$$\lim_{\varepsilon\searrow 0}\Big(x \mapsto \int \zeta_\varepsilon(x,y)\,\mathrm{d}y\Big) = \omega$$

in $\mathscr{D}(\Omega)$.

The proof of (1°) is left to the reader, being similar to the following

PROOF OF (2°): The hypothesis means that the corresponding coefficients are associated (§2, Definition):

$$[\iota T_L] \overset{\mathrm{G}}{\approx} \sum_{I\cup J=L} \operatorname{sgn}(IJ)[\iota R_I][\iota S_J].$$

By §1, Definition (2°) this means: $\forall\omega\in\mathscr{D}(\Omega)$ $\exists q\in\mathbb{N}_0$ such that for every bounded path

$$\varepsilon\mapsto\big(\varphi_x^\varepsilon\big)_{x\in\Omega}\in\mathscr{C}^\infty\big(\Omega\to\mathcal{A}_0(\mathbb{R}^d)\big)$$

that has asymptotically vanishing moments of order q, we have

$$\lim_{\varepsilon\searrow 0}\int\Big(\sum_{I\cup J=L}\operatorname{sgn}(IJ)[\iota R_I](\mathbf{S}_\varepsilon\varphi_z^\varepsilon,z)\cdot[\iota S_J](\mathbf{S}_\varepsilon\varphi_z^\varepsilon,z)-[\iota T_L](\mathbf{S}_\varepsilon\varphi_z^\varepsilon,z)\Big)\omega(z)\,\mathrm{d}z=0.$$

By the definition of ι and (1), this means

$$\lim_{\varepsilon\searrow 0}\int\Big(\sum_{I\cup J=L}\operatorname{sgn}(IJ)\cdot\langle R_I,\mathbf{T}_z\mathbf{S}_\varepsilon\varphi_z^\varepsilon\rangle\cdot\langle S_J,\mathbf{T}_z\mathbf{S}_\varepsilon\varphi_z^\varepsilon\rangle\Big)\omega(z)\,\mathrm{d}z=\langle T_L,\omega\rangle,$$

i.e.

$$\lim_{\varepsilon\searrow 0}\int\sum_{I\cup J=L}\operatorname{sgn}(IJ)\cdot\langle R_I(u),\mathbf{S}_\varepsilon\varphi_z^\varepsilon(u-z)\rangle_u\cdot\langle S_J(v),\mathbf{S}_\varepsilon\varphi_z^\varepsilon(v-z)\rangle_v\cdot\omega(z)\,\mathrm{d}z=\langle T_L,\omega\rangle.$$

On the left hand side there is an action of the direct product $R_I\times S_J$. After changing variables $u=x-y$, $v=x+y$, $\det\frac{\partial(u,v)}{\partial(x,y)}=2^d$, the left hand side is

$$\begin{aligned}
&=\lim_{\varepsilon\searrow 0}\int\sum_{I\cup J=L}\operatorname{sgn}(IJ)\cdot\langle R_I(x-y)S_J(x+y)\,,\\
&\qquad\qquad 2^d\mathbf{S}_\varepsilon\varphi_z^\varepsilon(x-y-z)\mathbf{S}_\varepsilon\varphi_z^\varepsilon(x+y-z)\rangle\,\omega(z)\,\mathrm{d}z\\
&=\lim_{\varepsilon\searrow 0}\sum_{I\cup J=L}\operatorname{sgn}(IJ)\cdot\Big\langle R_I(x-y)S_J(x+y)\,,\\
&\qquad\qquad 2^d\int\mathbf{S}_\varepsilon\varphi_z^\varepsilon(x-y-z)\mathbf{S}_\varepsilon\varphi_z^\varepsilon(x+y-z)\,\omega(z)\,\mathrm{d}z\Big\rangle.
\end{aligned}$$

As the test function is even in y, this is equal to $\lim\langle W_L,\zeta_\varepsilon\rangle$ (see §8, Definition (1°) for the notation) which completes the proof of (2°). □

Proof of (3°): Again, we are going to prove it only for ζ_ε, the proof for η_ε being similar.

$$\begin{aligned}\varepsilon^d\zeta_\varepsilon(x,\varepsilon y) &= 2^d\varepsilon^d\int \mathbf{S}_\varepsilon\varphi^\varepsilon_{x+h}(-h-\varepsilon y)\,\mathbf{S}_\varepsilon\varphi^\varepsilon_{x+h}(-h+\varepsilon y)\,\omega(x+h)\,\mathrm{d}h\\ &= 2^d\varepsilon^d\int \varepsilon^{-d}\varphi^\varepsilon_{x+h}(-\frac{h}{\varepsilon}-y)\,\varepsilon^{-d}\varphi^\varepsilon_{x+h}(-\frac{h}{\varepsilon}+y)\,\omega(x+h)\,\mathrm{d}h.\end{aligned}$$

After substitution $h\to\varepsilon h$ we get

$$\varepsilon^d\zeta_\varepsilon(x,\varepsilon y) = 2^d\int \varphi^\varepsilon_{x+\varepsilon h}(-h-y)\,\varphi^\varepsilon_{x+\varepsilon h}(-h+y)\,\omega(x+\varepsilon h)\,\mathrm{d}h. \tag{12}$$

The set of applications $\{z\mapsto\varphi^\varepsilon_z;\ \varepsilon\in\,]0,1]\}$ is bounded in $\mathscr{C}^\infty(\Omega\to\mathscr{D}(\mathbb{R}^d))$. So for an $r\in\mathbb{R}$ we have $\operatorname{supp}\varphi^\varepsilon_z\subseteq\{y;|y|\le r\}$ $(r>0)$ (euclidian norm) whenever $z\in\operatorname{supp}\omega$. For $|y|>r$ we have $|2y|=|-(-h-y)+(-h+y)|\le|-h-y|+|-h+y|$, so either $|-h-y|>r$ or $|-h+y|>r$ and by (12) $\zeta_\varepsilon(x,\varepsilon y)=0$. By the similar reason, the domain of integration need not exceed $\{|h|\le r\}$, so evidently if $\operatorname{dist}(x,\operatorname{supp}\omega)>\varepsilon r$, then $\omega(x+\varepsilon h)=0$ and again $\zeta_\varepsilon(x,\varepsilon y)=0$. Thus we have proved that, for ε small enough, the supports of the functions $x,y\mapsto\varepsilon^d\zeta_\varepsilon(x,\varepsilon y)$ are contained in a compact set in $\Omega\times\mathbb{R}^d$. By (12) it is clear that (for ε small enough) these functions form a bounded set in $\mathscr{D}(\Omega\times\mathbb{R}^d)$. Similar reasoning can be done for η_ε if we write φ instead of φ^ε_z and in that case the proof is evidently already completed.

It remains to estimate $\int\zeta_\varepsilon(x,y)\,\mathrm{d}y$. As above, for ε small enough, also the functions

$$x,y\mapsto 2^d\int\varphi^{\varepsilon'}_{x+\varepsilon h}(-h-y)\,\varphi^{\varepsilon'}_{x+\varepsilon h}(-h+y)\,\omega(x+\varepsilon h)\,\mathrm{d}h$$

have supports contained in a compact set in $\Omega\times\mathbb{R}^d$, independent on $\varepsilon'\in\,]0,1]$. As the set of applications $\{z\mapsto\varphi^{\varepsilon'}_z;\ \varepsilon'\in\,]0,1]\}$ is bounded in $\mathscr{C}^\infty(\Omega\to\mathscr{D}(\mathbb{R}^d))$, we deduce by the mean value theorem, e.g. [11, Theorem 11] that

$$\lim_{\varepsilon\searrow0}(x\mapsto\varphi^{\varepsilon'}_{x+\varepsilon h}(z)) = x\mapsto\varphi^{\varepsilon'}_x(z)$$

in $\mathscr{C}^\infty(\Omega)$ uniformly with respect to $\varepsilon'\in\,]0,1]$ and z. Consequently we have in $\mathscr{C}^\infty(\Omega)$, so in $\mathscr{D}(\Omega)$ uniformly with respect to $\varepsilon'\in\,]0,1]$

$$\begin{aligned}&\lim_{\varepsilon\searrow0}\left(x\mapsto 2^d\int\!\!\int\varphi^{\varepsilon'}_{x+\varepsilon h}(-h-y)\,\varphi^{\varepsilon'}_{x+\varepsilon h}(-h+y)\,\omega(x+\varepsilon h)\,\mathrm{d}h\,\mathrm{d}y\right)\\ &=\left(x\mapsto 2^d\int\!\!\int\varphi^{\varepsilon'}_x(-h-y)\,\varphi^{\varepsilon'}_x(-h+y)\,\omega(x)\,\mathrm{d}h\,\mathrm{d}y\right)\\ &= x\mapsto\omega(x)\cdot2^d\int\check{\varphi}^{\varepsilon'}_x*\varphi^{\varepsilon'}_x(2y)\,\mathrm{d}y=\omega.\end{aligned}$$

By (12), putting $\varepsilon'=\varepsilon$ gives the result. □

§**10.** Now we are able to prove one implication of the expected equivalence while for the converse we will have to prove more auxiliary results.

Partial result. If $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{gI}}{\wedge} \overset{s}{\mathrm{S}}$ on Ω, then also $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ on Ω. Moreover $[\iota \overset{r+s}{\mathrm{T}}] \overset{\mathrm{G}}{\approx} [\iota \overset{r}{\mathrm{R}}] \wedge [\iota \overset{s}{\mathrm{S}}]$ with $q = 0$ in the definition of G-association, i.e. without requirement on moments.

Proof: By §8, Definition (1°), the hypothesis means

$$T_L(x) \times \mathbf{1}(y) = \lim_{\varepsilon\to 0} W_L(x, \varepsilon y) \qquad (\forall\, L \in \mathcal{I}^d_{r+s}). \tag{13}$$

We are going to verify the conditions of §9, Lemma (2°). For an arbitrary bounded path

$$\varepsilon \mapsto \big(\varphi^\varepsilon_x\big)_{x\in\Omega} \in \mathscr{C}^\infty\big(\Omega \to \mathcal{A}_0(\mathbb{R}^d)\big),$$

if the function ζ_ε is defined by §9, Lemma (2°), we have

$$\lim_{\varepsilon\searrow 0} \langle W_L, \zeta_\varepsilon\rangle = \lim_{\varepsilon\searrow 0} \langle W_L(x, \varepsilon y),\ \varepsilon^d \zeta_\varepsilon(x, \varepsilon y)\rangle.$$

The convergence of distributions means uniform convergence on bounded sets in $\mathscr{D}$. By §9, Lemma (3°), the test functions in the last expression form a bounded set, so by (13) and again by §9, Lemma (3°), the last limit is

$$= \lim_{\varepsilon\searrow 0} \Big\langle T_L(x),\ \varepsilon^d \int \zeta_\varepsilon(x, \varepsilon y)\,\mathrm{d}y\Big\rangle = \Big\langle T_L(x)\,,\ \lim_{\varepsilon\searrow 0} \varepsilon^d \int \zeta_\varepsilon(x, \varepsilon y)\,\mathrm{d}y\Big\rangle$$
$$= \Big\langle T_L(x)\,,\ \lim_{\varepsilon\searrow 0} \int \zeta_\varepsilon(x, y)\,\mathrm{d}y\Big\rangle = \langle T_L, \omega\rangle.$$

These are the required sufficient conditions in §9, Lemma (2°). □

§**11. Lemma.** *Let* $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ *on* Ω, $\omega \in D(\Omega)$ *and let a natural number* o *be greater than or equal to the orders of all distributions* W_L $(L \in \mathcal{I}^d_{r+s}$, *the notation in* §8, *Definition* (1°)) *on some neighbourhood of* $\operatorname{supp}\omega \times 0$ *in* $\Gamma \subset \mathbb{R}^d \times \mathbb{R}^d$ (*see* (11)). *Then the following holds:*

(1°) $\exists\, q \in \mathbb{N}_0$ (*the same as in* §9, *Lemma* (1°)) $\forall\, \varphi \in \mathcal{A}_q$ $\forall L \in \mathcal{I}^d_{r+s}$ *we have*

$$\langle T_L, \omega\rangle =$$
$$\lim_{\varepsilon\searrow 0} \Bigg\langle W_L(x, y),\ \sum_{\substack{j\in\mathbb{N}_0^d\\ |j|\le d+o}} \frac{2^d}{j!}\Big(\frac{\partial}{\partial x}\Big)^j \omega(x)\cdot \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,h^j\,\mathrm{d}h\Bigg\rangle.$$

(2°) *If* $i \in \mathbb{N}_0^d$, $|i| > d+o$, $\varphi \in \mathcal{A}_0(\mathbb{R}^d)$, *then*

$$\lim_{\varepsilon \searrow 0} \Big\langle W_L(x,y),\ \omega(x) \cdot \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,h^i\,\mathrm{d}h \Big\rangle = 0.$$

Proof: The Taylor development of ω is

$$\omega(x+h) = \sum_{|j| \le d+o} \Big(\frac{\partial}{\partial x}\Big)^j \omega(x) \cdot \frac{h^j}{j!} + \rho(x,h)$$

where the remainder ρ fulfils $|\rho(x,h)| \le c|h|^{d+o+1}$ with c not depending on x,h. Also we have

$$\Big| \Big(\frac{\partial}{\partial x}\Big)^j \rho(x,h) \Big| \le c'_j |h|^{d+o+1}, \tag{14}$$

because $\left(\frac{\partial}{\partial x}\right)^j \rho(x,h)$ is the remainder of the Taylor development of $\left(\frac{\partial}{\partial x}\right)^j \omega(x+h)$. By §9, Lemma (1°) we deduce

$$\langle T_L, \omega\rangle = \lim_{\varepsilon \searrow 0} \Big\langle W_L(x,y),\ 2^d \int \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,\omega(x+h)\,\mathrm{d}h \Big\rangle_{x,y}$$

$$= \lim_{\varepsilon \searrow 0} \Big\langle W_L(x,y),\ 2^d \int \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y) \Big(\sum_{|j| \le d+o} \Big(\frac{\partial}{\partial x}\Big)^j \omega(x) \cdot \frac{h^j}{j!} + \rho(x,h) \Big)\,\mathrm{d}h \Big\rangle_{x,y}.$$

For proving (1°), it remains to show that

$$\lim_{\varepsilon \searrow 0} \Big\langle W_L(x,y)\ ,\ 2^d \int \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,\rho(x,h)\,\mathrm{d}h \Big\rangle_{x,y} = 0. \tag{15}$$

As W_L has an order not exceeding o on a neighbourhood of $\operatorname{supp}\omega \times 0$, it suffices to prove that the net of test functions in (15) tends to zero in $\mathscr{D}_o(\Omega \times \mathbb{R}^d)$. We are proving it even for $\varphi \in \mathcal{A}_0(\mathbb{R}^d)$. Let $\operatorname{supp}\varphi \subseteq \{y; |y| \le r\}$ $(r > 0)$ (euclidian norm). Like in the proof of §9, Lemma (3°), the test function $= 0$ if $|y| > \varepsilon r$. The domain of integration need not exceed $\{|h| \le \varepsilon r\}$. For these h, if $\operatorname{dist}(x, \operatorname{supp}\omega) > \varepsilon r$, the remainder of the Taylor development $\rho(x,h) = 0$, so we see that, for ε sufficiently small, the test function in (15) has the support in a compact neighbourhood of $\operatorname{supp}\omega \times 0$ where W_L, by hypothesis, is of order not greater than o.

Now we estimate the derivatives of order not greater than o of this test function:

$$
\begin{aligned}
&\Big(\frac{\partial}{\partial x}\Big)^j \Big(\frac{\partial}{\partial y}\Big)^k 2^d \int \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,\rho(x,h)\,\mathrm{d}h \\
&\qquad = 2^d \int \Big(\frac{\partial}{\partial y}\Big)^k \big(\mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\big)\Big(\frac{\partial}{\partial x}\Big)^j \rho(x,h)\,\mathrm{d}h.
\end{aligned}
\tag{16}
$$

(1°) The integration domain $\{|h| \leq \varepsilon r\}$ is of volume $= c_1\varepsilon^d$ for some constant c_1 not depending on ε.

(2°) If $\Big|\Big(\frac{\partial}{\partial y}\Big)^k \varphi(y)\Big| \leq c_2$ for all y and $|k| \leq o$, then $\Big|\Big(\frac{\partial}{\partial y}\Big)^k \mathbf{S}_\varepsilon\varphi(y)\Big| \leq c_2\varepsilon^{-d-|k|}$ and by Euler's rule

$$
\Big|\Big(\frac{\partial}{\partial y}\Big)^k \big(\mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\big)\Big| \leq c_3\varepsilon^{-2d-|k|}
$$

with a constant c_3 depending on φ but not on ε.

(3°) For $\Big(\frac{\partial}{\partial x}\Big)^j \rho(x,h)$ we have estimates (14).

For $|j|, |k| \leq o$, these items give

$$
\begin{aligned}
&\Big|2^d \int \Big(\frac{\partial}{\partial y}\Big)^k \big(\mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\big)\Big(\frac{\partial}{\partial x}\Big)^j \rho(x,h)\,\mathrm{d}h\Big| \\
&\qquad \leq c\varepsilon^{d-2d-|k|}|h|^{d+o+1} \leq \varepsilon^{-d-|k|}(\varepsilon r)^{d+o+1} \leq c'\varepsilon^{-|k|+o+1}
\end{aligned}
$$

that tends to zero for $|k| \leq o$. Thus the proof of the part (1°) of Lemma is completed. The part (2°) can be proved in the same way as (15), because the function $x, h \mapsto \omega(x)h^i$ on bounded sets has the same properties required in this proof as the function ρ. □

§**12. Lemma.** *For $p, q \in \mathbb{N}$, let a function (net of polynomials of variable $t = (t_1, \dots, t_p)$)*

$$
\varepsilon, t_1, \dots, t_p \mapsto P(\varepsilon, t_1, \dots, t_p) = P(\varepsilon, t) := \sum_{\substack{I \in \mathbb{N}_0^p \\ |I| \leq q}} a_I(\varepsilon) t^I
$$

be defined on $\{\varepsilon \in (0,1], t \in \mathbb{R}^p\}$. If for all t the limit $Q(t) = \lim\limits_{\varepsilon \searrow 0} P(\varepsilon, t)$ exists, then Q is a polynomial $Q(t) = \sum b_I(t)$ and $b_I = \lim\limits_{\varepsilon \searrow 0} a_I(\varepsilon)$.

Proof: This is a well known property of polynomials. Let us give an idea of the proof. For a convenable sufficiently large finite set of points $t^{(1)}, \dots, t^{(n)} \in \mathbb{R}^p$ a polynomial is uniquely determined by its values at these points. Choose polynomials $P_1, \dots, P_n$ with $P_j(t_k) = \delta_{j,k}$ $(j, k = 1, \dots, n)$ (Kronecker's delta). Then $P(\varepsilon, t) = \sum\limits_{j=1}^{n} P(\varepsilon, t^{(j)}) P_j(t)$, that gives the result. □

§**13. Notation.** Following [15], for multiindices $i=(i_1,\dots,i_d)$, $j=(j_1,\dots,j_d)$ $\in \mathbb{N}_0^d$ we write $i \le j$ iff $i_\alpha \le j_\alpha \quad (\alpha = 1,\dots,d)$; we write $i<j$ iff $i\le j$ and $i \ne j$.

Lemma. *If* $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ *on* Ω, *then* $\forall\,\omega\in\mathscr{D}(\Omega)\quad \exists\, q'\in\mathbb{N}_0$ *such that* $\forall\, i > (0,\dots,0)\in\mathbb{N}_0^d$, $\forall\,\varphi\in\mathcal{A}_{q'}(\mathbb{R}^d)$

$$\lim_{\varepsilon\searrow 0}\Big\langle W_L(x,y),\ \omega(x)\cdot\int_{\mathbb{R}^d}\mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,h^i\,\mathrm{d}h\Big\rangle = 0. \tag{17}$$

Proof: We will prove an apparently weaker but equivalent formulation of the lemma: If $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ on Ω, then $\forall\, i>(0,\dots,0)\in\mathbb{N}_0^d$, $\forall\omega\in\mathscr{D}(\Omega)\quad \exists\, q'\in\mathbb{N}_0$ such that $\forall\,\varphi\in\mathcal{A}_{q'}(\mathbb{R}^d)$ (17) holds. This is indeed equivalent, because for a greater q' so much the more the assertion holds; by §11, Lemma (2°), (17) holds even for $q'=0$ (i.e. for every q') provided $|i|>d+o$ and there are only a finite number of multiindices i with $|i|\le d+o$. Hence, q' in this formulation can be chosen independent on i.

Fix a compact $K\subset\Omega$. If we confine ourself on $\omega\in\mathscr{D}(K)$, the number o in §11, Lemma can be independent on ω. We are going to prove our weaker formulation above by contradiction. We know that there are only a finite number of multiindices i for which the assertion of this formulation does not hold. So, if there is any, choose a maximal such i and denote $\bar{i}$. So for $i=\bar{i}$ the assertion does not hold and for all $i>\bar{i}$ even the stronger assertion with q' independent on i holds. For finitely many functions $\omega\in\mathscr{D}(K)$, q' can be the same. Thus we have:

$$\begin{aligned}&\forall\,\omega\in\mathscr{D}(K)\ \exists\, q'\ge q\ \text{ in §11, Lemma } (1^\circ)\\ &\forall\, i>\bar{i},\ \varphi\in\mathcal{A}_{q'}(\mathbb{R}^d),\ j\in\mathbb{N}_0^d\ \text{ with }\ j\le o+d:\\ &\lim_{\varepsilon\searrow 0}\Big\langle W_L(x,y),\ \Big(\frac{\partial}{\partial x}\Big)^j\omega(x)\cdot\int_{\mathbb{R}^d}\mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,h^i\,\mathrm{d}h\Big\rangle = 0.\end{aligned} \tag{18}$$

We deduce that for $i=\bar{i}$ the assertion holds, too, that will be a contradiction. Namely, we deduce: $\forall\,\omega\in\mathscr{D}(K)\ \exists\, q''\ge q'\ \forall\,\psi\in\mathcal{A}_{q''}(\mathbb{R}^d)$

$$\lim_{\varepsilon\searrow 0}\Big\langle W_L(x,y)\,,\ \omega(x)\cdot\int_{\mathbb{R}^d}\mathbf{S}_\varepsilon\psi(-h-y)\,\mathbf{S}_\varepsilon\psi(-h+y)\,h^{\bar{i}}\,\mathrm{d}h\Big\rangle = 0. \tag{19}$$

Denote by $n_1,\dots,n_{|\bar{i}|}\in\{1,2,\dots,d\}$ indices for which

$$x_{n_1}\cdot x_{n_2}\cdots x_{n_{|\bar{i}|}} = x^{\bar{i}} \qquad (x=(x_1,\dots,x_d)\,). \tag{20}$$

Denote $q'' := q' + |\bar{i}|$ and choose numbers $t_1, \dots, t_{|\bar{i}|}$. If $\psi \in \mathcal{A}_{q''}$, we easily deduce

$$\varphi(x) := \psi(x) \cdot \prod_{k=1}^{|\bar{i}|} (1 + t_k x_{n_k}) \in \mathcal{A}_{q'} \,. \tag{21}$$

Then

$$\mathbf{S}_\varepsilon \varphi(x) = \mathbf{S}_\varepsilon \psi(x) \cdot \prod_{k=1}^{|\bar{i}|} (1 + \frac{t_k}{\varepsilon} x_{n_k})$$

and

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{S}_\varepsilon \varphi(-h-y)\, \mathbf{S}_\varepsilon \varphi(-h+y)\, h^j \,\mathrm{d}h \\
&= \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon \psi(-h-y)\, \mathbf{S}_\varepsilon \psi(-h+y)\, h^j \\
&\quad \cdot \prod_{k=1}^{|\bar{i}|} \Big(1 + \frac{t_k}{\varepsilon}(-h_{n_k} - y_{n_k})\Big)\Big(1 + \frac{t_k}{\varepsilon}(-h_{n_k} + y_{n_k})\Big) \,\mathrm{d}h \\
&= \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon \psi(-h-y)\, \mathbf{S}_\varepsilon \psi(-h+y)\, h^j \\
&\quad \cdot \prod_{k=1}^{|\bar{i}|} \Big(1 + \frac{t_k}{\varepsilon}(-2h_{n_k}) + \Big(\frac{t_k}{\varepsilon}\Big)^2 (h_{n_k}^2 - y_{n_k}^2)\Big) \,\mathrm{d}h.
\end{aligned}$$

This is a polynomial of the variable $t = (t_1, \dots, t_{|\bar{i}|})$ whose coefficient at the power $t^{\mathbf{1}} = (t_1 \cdots t_{|\bar{i}|})$ is

$$\begin{aligned}
&\int_{\mathbb{R}^d} \mathbf{S}_\varepsilon \psi(-h-y)\, \mathbf{S}_\varepsilon \psi(-h+y)\, h^j \cdot \Big(\prod_{k=1}^{|\bar{i}|} h_{n_k}\Big) \Big(\frac{-2}{\varepsilon}\Big)^{|\bar{i}|} \,\mathrm{d}h \\
&\qquad = \Big(\frac{-2}{\varepsilon}\Big)^{|\bar{i}|} \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon \psi(-h-y)\, \mathbf{S}_\varepsilon \psi(-h+y)\, h^{j+\bar{i}} \,\mathrm{d}h
\end{aligned}$$

due to (20). Substituting φ defined by (21) into §11, Lemma (1°) and applying §12, Lemma give

$$\begin{aligned}
&0 = \lim_{\varepsilon \searrow 0} \\
&\Big\langle W_L(x,y), \sum_{\substack{j \in \mathbb{N}_0^d \\ |j| \le d+o}} \frac{2^d}{j!} \Big(\frac{\partial}{\partial x}\Big)^j \omega(x) \cdot \Big(\frac{-2}{\varepsilon}\Big)^{|\bar{i}|} \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon \psi(-h-y)\, \mathbf{S}_\varepsilon \psi(-h+y)\, h^{j+\bar{i}} \,\mathrm{d}h \Big\rangle.
\end{aligned}$$

First we multiply this relation with $\left(\frac{\varepsilon}{-2}\right)^{|\bar{i}|}$. Then we see by (18) that every term in the sum with $j > (0,\dots,0)$ gives the result zero, so the same holds for the term with $j = (0,\dots,0)$. Thus we have deduced (19) from (18) that is the required contradiction. □

Corollary. *If* $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ *on* Ω*, then* $\forall \omega \in \mathscr{D}(\Omega)$ $\exists\, q \in \mathbb{N}_0$ *such that* $\forall \varphi \in \mathcal{A}_q(\mathbb{R}^d)$

$$\begin{aligned}\langle T_L, \omega\rangle &= \lim_{\varepsilon\searrow 0} \big\langle W_L(x,y),\ \omega(x)\cdot \mathbf{S}_\varepsilon \check{\varphi} * \mathbf{S}_\varepsilon \varphi\,(y)\,\big\rangle \\ &= \lim_{\varepsilon\searrow 0} \big\langle W_L(x,y),\ \omega(x)\cdot \mathbf{S}_\varepsilon(\check{\varphi} * \varphi)(y)\big\rangle\end{aligned}$$

$(\check{\varphi} := y \mapsto \varphi(-y))$.

Proof: By §11, Lemma (1°) and §13, Lemma we have

$$\begin{aligned}\langle T_L, \omega\rangle &= \lim_{\varepsilon\searrow 0} \Big\langle W_L(x,y),\ 2^d\omega(x)\cdot \int_{\mathbb{R}^d} \mathbf{S}_\varepsilon\varphi(-h-y)\,\mathbf{S}_\varepsilon\varphi(-h+y)\,\mathrm{d}h\Big\rangle \\ &= \lim_{\varepsilon\searrow 0} \Big\langle W_L(x,y),\ 2^d\omega(x)\cdot \mathbf{S}_\varepsilon\check{\varphi} * \mathbf{S}_\varepsilon\varphi(2y)\Big\rangle.\end{aligned}$$

Replacing $\mathbf{S}_\varepsilon$ with $\mathbf{S}_{\varepsilon/2}$ gives the result. □

§**14.** For completing the proof of equivalence of the Colombeau product with the generalized Itano product, we refer to [10, Theorem 3]. It is proved there that, for a distribution F defined on a neighbourhood of 0 in $\mathbb{R}^d$, the following are equivalent.

(1°) F has at 0 a value in Lojasiewicz sense $= a$. This means that for every $\eta \in \mathcal{A}_0(\mathbb{R}^d)$

$$\lim_{\varepsilon\searrow 0} \langle F, \mathbf{S}_\varepsilon \eta\rangle = a. \tag{22}$$

(3°) If a $q \in \mathbb{N}_0$ is fixed, then for every $\varphi \in \mathcal{A}_q$, $\eta := \varphi * \varphi$ (22) holds.

Theorem. *Let* $\overset{r}{\mathrm{R}} \in \mathscr{D}' \otimes \overset{r}{\mathcal{E}}$, $\overset{s}{\mathrm{S}} \in \mathscr{D}' \otimes \overset{s}{\mathcal{E}}$ *and* $\overset{r+s}{\mathrm{T}} \in \mathscr{D}' \otimes \overset{r+s}{\mathcal{E}}$ *be currents on* Ω*. Then* $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$ *iff* $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{gI}}{\wedge} \overset{s}{\mathrm{S}}$*. In that case* $[\iota\,\overset{r+s}{\mathrm{T}}]$ *is even G-associated with* $\iota[\overset{r}{\mathrm{R}}] \wedge \iota[\overset{s}{\mathrm{S}}]$ *with* $q = 0$ *in the definition of G-association* §1(2°)*, i.e. without requirements on moments.*

Proof: A partial result is given already in §10. We only have to prove: If $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$, then $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{gI}}{\wedge} \overset{s}{\mathrm{S}}$. So let $\overset{r+s}{\mathrm{T}} = \overset{r}{\mathrm{R}} \overset{\mathrm{C}}{\wedge} \overset{s}{\mathrm{S}}$. By §8, Definition (1°), we have to prove that, for every $L \in \mathcal{I}^d_{r+s}$, the distribution $W_L(x,y)$ has for $y=0$ a section equal to $T_L(x)$. Equivalently, we have to prove that, for $\omega \in \mathscr{D}(\Omega)$, the distribution

$$F(y) := \langle W_L(x,y), \omega(x)\rangle_x$$

depending on ω and defined by

$$\langle F, \varphi\rangle = \langle W_L(x,y), \omega(x)\varphi(y)\rangle$$

has at 0 a value equal to $\langle T_L, \omega\rangle$. By the last Corollary, $\exists\, q \in \mathbb{N}_0 \quad \forall\, \varphi \in \mathcal{A}_q(\mathbb{R}^d)$

$$\langle T_L, \omega\rangle = \lim_{\varepsilon \searrow 0} \big\langle F, \mathbf{S}_\varepsilon(\check{\varphi} * \varphi)\big\rangle. \tag{23}$$

Thanks to the reference $(3^\circ) \Rightarrow (1^\circ)$ above, we need

$$\langle T_L, \omega\rangle = \lim_{\varepsilon \searrow 0} \big\langle F, \mathbf{S}_\varepsilon(\varphi * \varphi)\big\rangle. \tag{24}$$

$W(x,y)$ is even in y, so F is even, that means $\langle F, \varphi\rangle = \langle F, \check{\varphi}\rangle$, so $\langle F, \varphi * \varphi\rangle = \langle F, \check{\varphi} * \check{\varphi}\rangle$. (24) follows from (23), because

$$\begin{aligned}\big\langle F, (\varphi * \varphi)\big\rangle &= \Big\langle F, \frac{1}{2}(\varphi * \varphi) + \frac{1}{2}(\check{\varphi} * \check{\varphi})\Big\rangle \\ &= 2\Big\langle F\,,\ \frac{\varphi + \check{\varphi}}{2} * \frac{\check{\varphi} + \varphi}{2}\Big\rangle - \langle F, \check{\varphi} * \varphi\rangle\end{aligned}$$

which completes the proof. □

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