Complete hypersurfaces with constant scalar curvature in a sphere

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Abstract. In this paper, by using Cheng-Yau's self-adjoint operator $\square$, we study the complete hypersurfaces in a sphere with constant scalar curvature.

Keywords: hypersurface, sphere, scalar curvature
Classification: 53C42, 53A10

1. Introduction

Let S^{n+1} be an $(n+1)$-dimensional unit sphere with constant sectional curvature 1, let M^n be an n-dimensional hypersurface in S^{n+1}, and $e_1, \dots, e_n$ a local orthonormal frame field on M^n, $\omega_1, \dots, \omega_n$ its dual coframe field. Then the second fundamental form of M^n is

$$h = \sum_{i,j} h_{ij}\omega_i \otimes \omega_j. \tag{1}$$

Further, near any given point $p \in M^n$, we can choose a local frame field $e_1, \dots, e_n$ so that at p, $\sum_{i,j} h_{ij}\omega_i \otimes \omega_j = \sum_i k_i \omega_i \otimes \omega_j$. Then the Gauss equation says

$$R_{ijij} = 1 + k_i k_j, \quad i \neq j. \tag{2}$$

$$n(n-1)(R-1) = n^2H^2 - |h|^2, \tag{3}$$

where R is the normalized scalar curvature, $H = \frac{1}{n}\sum_i k_i$ the mean curvature and $|h|^2 = \sum_i k_i^2$ the norm square of the second fundamental form of M^n.

As it is well known, there are many rigidity results for minimal hypersurfaces or hypersurfaces with constant mean curvature H in S^{n+1} by use of J. Simons' method, for example, see [1], [3], [4], [6], [9], etc.

On the other hand, Cheng-Yau [2] introduced a new self-adjoint differential operator $\square$ to study the hypersurfaces with constant scalar curvature. Later, Li [5] obtained interesting rigidity results for hypersurfaces with constant scalar curvature in space-forms using the Cheng-Yau's self-adjoint operator $\square$.

In the present paper, we use Cheng-Yau's self-adjoint operator $\square$ to study the complete hypersurfaces in a sphere with constant scalar curvature, and prove the following theorem:

Theorem. *Let M^n be an n-dimensional $(n \geq 3)$ complete hypersurface with constant normalized scalar curvature R in S^{n+1}. If*

(1) $\bar{R} = R - 1 \geq 0$,

(2) *the mean curvature H of M^n satisfies*

$$\bar{R} \leq \sup H^2 \leq \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big],$$

then either

$$\sup H^2 = \bar{R}$$

and M^n is a totally umbilical hypersurface; or

$$\sup H^2 = \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big],$$

and $M^n = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $r = \sqrt{\frac{n-2}{n(\bar{R}+1)}}$.

2. Preliminaries

Let M^n be an n-dimensional complete hypersurface in S^{n+1}. We choose a local orthonormal frame $e_1, \dots, e_{n+1}$ in S^{n+1} such that at each point of M^n, $e_1, \dots, e_n$ span the tangent space of M^n and form an orthonormal frame there. Let $\omega_1, \dots, \omega_{n+1}$ be its dual coframe. In this paper, we use the following convention on the range of indices:

$$1 \leq A, B, C, \dots \leq n+1; \quad 1 \leq i, j, k, \dots \leq n.$$

Then the structure equations of S^{n+1} are given by

$$d\omega_A = \sum_B \omega_{AB} \wedge \omega_B, \quad \omega_{AB} + \omega_{BA} = 0, \tag{4}$$

$$d\omega_{AB} = \sum_C \omega_{AC} \wedge \omega_{CB} - \frac{1}{2}\sum_{C,D} K_{ABCD}\omega_C \wedge \omega_D, \tag{5}$$

$$K_{ABCD} = (\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC}). \tag{6}$$

Restricting these forms to M^n, we have

$$\omega_{n+1} = 0. \tag{7}$$

From Cartan's lemma we can write

$$\omega_{n+1i} = \sum_j h_{ij}\omega_j, \quad h_{ij} = h_{ji}. \tag{8}$$

From these formulas, we obtain the structure equations of M^n:

$$d\omega_i = \sum_j \omega_{ij} \wedge \omega_j, \quad \omega_{ij} + \omega_{ji} = 0, \tag{9}$$

$$d\omega_{ij} = \sum_k \omega_{ik} \wedge \omega_{kj} - \frac{1}{2} \sum_{k,l} R_{ijkl} \omega_k \wedge \omega_l, \tag{10}$$

$$R_{ijkl} = (\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) + (h_{ik}h_{jl} - h_{il}h_{jk}), \tag{11}$$

where R_{ijkl} are the components of the curvature tensor of M^n and

$$h = \sum_{i,j} h_{ij} \omega_i \otimes \omega_j \tag{12}$$

is the second fundamental form of M^n. We also have

$$R_{ij} = (n-1)\delta_{ij} + nHh_{ij} - \sum_k h_{ik}h_{kj}, \tag{13}$$

$$n(n-1)(R-1) = n^2H^2 - |h|^2, \tag{14}$$

where R is the normalized scalar curvature, and H the mean curvature.

Define the first and the second covariant derivatives of h_{ij}, say h_{ijk} and h_{ijkl} by

$$\sum_k h_{ijk}\omega_k = dh_{ij} + \sum_k h_{kj}\omega_{ki} + \sum_k h_{ik}\omega_{kj}, \tag{15}$$

$$\sum_l h_{ijkl}\omega_l = dh_{ijk} + \sum_m h_{mjk}\omega_{mi} + \sum_m h_{imk}\omega_{mj} + \sum_m h_{ijm}\omega_{mk}. \tag{16}$$

Then we have the Codazzi equation

$$h_{ijk} = h_{ikj}, \tag{17}$$

and the Ricci's identity

$$h_{ijkl} - h_{ijlk} = \sum_m h_{mj}R_{mikl} + \sum_m h_{im}R_{mjkl}. \tag{18}$$

For a C^2-function f defined on M^n, we define its gradient and Hessian (f_{ij}) by the following formulas

$$df = \sum_i f_i\omega_i, \quad \sum_j f_{ij}\omega_j = df_i + \sum_j f_j\omega_{ji}. \tag{19}$$

The Laplacian of f is defined by $\Delta f = \sum_i f_{ii}$.

Let $\phi = \sum_{ij} \phi_{ij}\omega_i \otimes \omega_j$ be a symmetric tensor defined on M^n, where

$$\phi_{ij} = nH\delta_{ij} - h_{ij}. \tag{20}$$

Following Cheng-Yau [2], we introduce the operator $\Box$ associated to ϕ acting on any C^2-function f by

$$\Box f = \sum_{i,j} \phi_{ij} f_{ij} = \sum_{i,j} (nH\delta_{ij} - h_{ij}) f_{ij}. \tag{21}$$

Since ϕ_{ij} is divergence-free, it follows [2] that the operator $\Box$ is self-adjoint relative to the L^2 inner product of M^n, i.e.

$$\int_{M^n} f\Box g = \int_{M^n} g\Box f. \tag{22}$$

We can choose a local frame field $e_1, \dots e_n$ at any point $p \in M^n$, such that $h_{ij} = k_i\delta_{ij}$ at p, and by use of (21) and (14), we have

$$\begin{aligned}\Box(nH) &= nH\Delta(nH) - \sum_i k_i(nH)_{ii} \\ &= \frac{1}{2}\Delta(nH)^2 - \sum_i (nH)_i^2 - \sum_i k_i(nH)_{ii} \\ &= \frac{1}{2}n(n-1)\Delta R + \frac{1}{2}\Delta|h|^2 - n^2|\nabla H|^2 - \sum_i k_i(nH)_{ii}.\end{aligned} \tag{23}$$

On the other hand, through a standard calculation by use of (17) and (18), we get

$$\frac{1}{2}\Delta|h|^2 = \sum_{i,j,k} h_{ijk}^2 + \sum_i k_i(nH)_{ii} + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{24}$$

Putting (24) into (23), we have

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + \frac{1}{2}\sum_{i,j} R_{ijij}(k_i - k_j)^2. \tag{25}$$

From (11), we have $R_{ijij} = 1 + k_ik_j, \ \ i \neq j$, and by putting this into (25), we obtain

$$\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 + n|h|^2 - n^2H^2 - |h|^4 + nH\sum_i k_i^3. \tag{26}$$

Let $\mu_i = k_i - H$ and $|Z|^2 = \sum_i \mu_i^2$. We have

$$\sum_i \mu_i = 0, \quad |Z|^2 = |h|^2 - nH^2, \tag{27}$$

$$\sum_i k_i^3 = \sum_i \mu_i^3 + 3H|Z|^2 + nH^3. \tag{28}$$

From (26)–(28), we get

$$\begin{aligned}\Box(nH) = \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 \\ + |Z|^2(n + nH^2 - |Z|^2) + nH\sum_i \mu_i^3.\end{aligned} \tag{29}$$

We need the following algebraic lemma due to M. Okumura [7] (see also [1]).

Lemma 2.1. *Let μ_i, $i = 1, \dots, n$, be real numbers such that $\sum_i \mu_i = 0$ and $\sum_i \mu_i^2 = \beta^2$, where $\beta =$ constant ≥ 0. Then*

$$-\frac{n-2}{\sqrt{n(n-1)}}\beta^3 \leq \sum_i \mu_i^3 \leq \frac{n-2}{\sqrt{n(n-1)}}\beta^3, \tag{30}$$

and the equality holds in (30) *if and only if at least* $(n-1)$ *of the* μ_i *are equal.*

By use of Lemma 2.1, we have

$$\begin{aligned}\Box(nH) \geq \frac{1}{2}n(n-1)\Delta R + |\nabla h|^2 - n^2|\nabla H|^2 \\ + (|h|^2 - nH^2)(n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}).\end{aligned} \tag{31}$$

3. Proof of Theorem

The following lemma is essentially due to Cheng-Yau [2] (see also [5]).

Lemma 3.1. *Let M be an n-dimensional hypersurface in S^{n+1}. Suppose that the normalized scalar curvature $R =$ constant and $R \geq 1$. Then $|\nabla h|^2 \geq n^2|\nabla H|^2$.*

From the assumption of Theorem that R is constant and $\bar{R} = R - 1 \geq 0$ and Lemma 3.1 we have

$$\Box(nH) \geq (|h|^2 - nH^2)(n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}). \tag{32}$$

By Gauss equation (14) we know that

$$|Z|^2 = |h|^2 - nH^2 = \frac{n-1}{n}(|h|^2 - n\bar{R}). \tag{33}$$

From (32) and (33) we have

$$\square(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_H(|h|), \tag{34}$$

where

$$\phi_H(|h|) = n + 2nH^2 - |h|^2 - \frac{n(n-2)}{\sqrt{n(n-1)}}H\sqrt{|h|^2 - nH^2}.$$

By (33) we can write $\phi_H(|h|)$ as

$$\phi_{\bar{R}}(|h|) = n+2(n-1)\bar{R} - \frac{n-2}{n}|h|^2 - \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + |h|^2)(|h|^2 - n\bar{R})}. \tag{35}$$

Therefore (34) becomes

$$\square(nH) \geq \frac{n-1}{n}(|h|^2 - n\bar{R})\phi_{\bar{R}}(|h|). \tag{36}$$

It is a direct check that our assumption

$$\sup H^2 \leq \frac{1}{n^2}\left[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\right]$$

is equivalent to

$$\sup |h|^2 \leq \frac{n}{(n-2)(n\bar{R}-2)}\left[n(n-1)\bar{R}^2 + 4(n-1)\bar{R} + n\right], \tag{37}$$

i.e.

$$\begin{aligned}(n + 2(n-1)\bar{R} - \frac{n-2}{n}\sup|h|^2)^2 \\ \geq \frac{(n-2)^2}{n^2}(n(n-1)\bar{R} + \sup|h|^2)(\sup|h|^2 - n\bar{R}).\end{aligned} \tag{38}$$

But it is clear from (37) that (38) is equivalent to

$$\begin{aligned}n + 2(n-1)\bar{R} - \frac{n-2}{n}\sup|h|^2 \\ \geq \frac{n-2}{n}\sqrt{(n(n-1)\bar{R} + \sup|h|^2)(\sup|h|^2 - n\bar{R})}.\end{aligned} \tag{39}$$

So under the hyperthesis that

$$\sup\, H^2 \le \frac{1}{n^2}\Big[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}\Big],$$

we have

$$\phi_{\bar{R}}(\sqrt{\sup|h|^2}) \ge 0. \tag{40}$$

On the other hand,

$$\begin{aligned}\Box(nH) &= \sum_{i,j}(nH\delta_{ij} - nh_{ij})(nH)_{ij} = \sum_i (nH - nh_{ii})(nH)_{ii}\\ &= n\sum_i H(nH)_{ii} - n\sum_i k_i(nH)_{ii} \le (|H|_{\max} - C)\Delta(nH),\end{aligned} \tag{41}$$

where $|H|_{\max}$ is the maximum of the mean curvature H and $C = \min k_i$ is the minimum of the principal curvatures of M^n.

Now we need the following maximum principle at infinity for complete manifolds due to Omori [8] and Yau [10]:

Lemma 3.2. *Let M^n be an n-dimensional complete Riemannian manifold whose sectional curvature is bounded from below and $f : M^n \to R$ a smooth function bounded from below. Then for each $\varepsilon > 0$ there exists a point $p_\varepsilon \in M^n$ such that*

(i) $|\nabla f|(p_\varepsilon) < \varepsilon$,
(ii) $\Delta f(p_\varepsilon) > -\varepsilon$,
(iii) $\inf\, f \le\ f(p_\varepsilon) \le \inf\, f + \varepsilon$.

Since the scalar curvature of M is a constant, from the hypothesis that $\bar{R} \le \sup\, H^2 \le \frac{1}{n^2}[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}]$, and Gauss equation (14), we know the squared norm $|h|^2$ of the second fundamental form is bounded from above, from (11) we know that the sectional curvature is bounded from below. So we may apply Lemma 3.2 to the smooth function f on M^n defined by

$$f = \frac{1}{\sqrt{1+(nH)^2}}\,.$$

It is immediate to check that

$$|\nabla f|^2 = \frac{1}{4}\frac{|\nabla(nH)^2|^2}{(1+(nH)^2)^3} \tag{42}$$

and that

$$\Delta f = -\frac{1}{2}\frac{\Delta(nH)^2}{(1+(nH)^2)^{3/2}} + \frac{3}{4}\frac{|\nabla(nH)^2|^2}{(1+(nH)^2)^{5/2}}\,. \tag{43}$$

By Lemma 3.2 we can find a sequence of points p_k, $k \in N$ in M^n, such that

$$\lim_{k\to\infty} f(p_k) = \inf f, \quad \Delta f(p_k) > -\frac{1}{k}, \quad |\nabla f|^2(p_k) < \frac{1}{k^2}. \tag{44}$$

Using (44) in equations (42) and (43) and the fact that

$$\lim_{k\to\infty} (nH)(p_k) = \sup_{p\in M^n} (nH)(p), \tag{45}$$

we get

$$-\frac{1}{k} \leq -\frac{1}{2}\frac{\Delta(nH)^2}{(1+(nH)^2)^{3/2}}(p_k) + \frac{3}{k^2}(1+(nH)^2(p_k))^{1/2}. \tag{46}$$

Hence we obtain

$$\frac{\Delta(nH)^2}{(1+(nH)^2)^2}(p_k) < \frac{2}{k}\Big(\frac{1}{\sqrt{1+(nH)^2(p_k)}} + \frac{3}{k}\Big). \tag{47}$$

On the other hand, by (36) and (41), we have

$$\frac{n-1}{n}(|h|^2 - n\bar{R})\phi_{\bar{R}}(|h|) \leq \Box(nH) \leq n(|H|_{\max} - C)\Delta(nH). \tag{48}$$

At points p_k of the sequence given in (44), this becomes

$$\begin{aligned}\frac{n-1}{n}(|h|^2(p_k) - n\bar{R})\phi_{\bar{R}}(|h|(p_k)) &\leq \Box(nH(p_k)) \\ &\leq n(|H|_{\max} - C)\Delta(nH)(p_k).\end{aligned} \tag{49}$$

Letting $k \to \infty$ and using (47) we have that the right hand side of (49) goes to zero, so we have either $\frac{n-1}{n}(\sup|h|^2 - n\bar{R}) = 0$, i.e. $\sup H^2 = \bar{R}$, or $\phi_{\bar{R}}(\sqrt{\sup|h|^2}) = 0$.

If $\sup|h|^2 = n\bar{R}$, by (33) $|Z|^2 = \frac{n-1}{n}(|h|^2 - n\bar{R})$ we have $\sup|Z|^2 = \frac{n-1}{n}(\sup|h|^2 - n\bar{R}) = 0$, hence $|Z|^2 = 0$ and M^n is totally umbilical.

If $\phi_{\bar{R}}(\sqrt{\sup|h|^2}) = 0$, it is easy to prove that $\sup H^2 = \frac{1}{n^2}[(n-1)^2\frac{n\bar{R}+2}{n-2} - 2(n-1) + \frac{n-2}{n\bar{R}+2}]$, hence equalities hold in (30) and Lemma 3.1, and it follows that k_i = constant for all i and $(n-1)$ of the k_i's are equal. After renumberation if necessary, we can assume that

$$k_1 = k_2 = \cdots = k_{n-1}, \quad k_1 \neq k_n.$$

Therefore, M^n is a isoparametric hypersurface in S^{n+1} with two distinct principal curvatures, hence $M^n = S^1(\sqrt{1-r^2}) \times S^{n-1}(r)$, $k_1 = \cdots = k_{n-1} = \sqrt{1-r^2}/r$, $k_n = -r/\sqrt{1-r^2}$. From (14), it is easy to see that $n(n-1)\bar{R} = (n-1)(n-2-nr^2)/r^2$, thus $r = \sqrt{\frac{n-2}{n(\bar{R}+1)}}$. This completes the proof of Theorem.

Acknowledgments. The authors would like to thank the referee for his comments on this paper.

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(*Received* October 7, 2004, *revised* January 7, 2005)