Comment.Math.Univ.Carolin. 46,4 (2005)637–643

On a weak form of uniform convergence

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Dedicated to the memory of Professor Miroslav Katětov.

Abstract. The notion of Δ-convergence of a sequence of functions is stronger than pointwise convergence and weaker than uniform convergence. It is inspired by the investigation of ill-posed problems done by A.N. Tichonov. We answer a question posed by M. Katětov around 1970 by showing that the only analytic metric spaces X for which pointwise convergence of a sequence of continuous real valued functions to a (continuous) limit function on X implies Δ-convergence are σ-compact spaces. We show that the assumption of analyticity cannot be omitted.

Keywords: continuous functions on metric spaces, pointwise convergence, Δ-convergence, analytic spaces, Hurewicz theorem, K_σ-spaces

Classification: 54H05, 40A30, 54E35

1. Introduction and main results

The investigation of ill-posed problems looks for properties that ensure some kind of stability of numerical solutions to problems which are in principle nonstable. Inspired by the work of A.N. Tichonov (see [8]), M. Katětov introduced the following notion of, in a sense, weakly uniform convergence on metric spaces.

Definition 1.1. Let f_n, f, $n \in \mathbb{N}$, be mappings of a metric space (X, ρ) to a metric space (Y, τ). We say that f_n Δ-*converges* to f if there is a sequence $\{\delta_n\}_{n=1}^{\infty}$ of positive reals such that $\lim_{n\to\infty} \tau(f_n(x_n), f(x)) = 0$, whenever $\rho(x_n, x) < \delta_n$ for $n \in \mathbb{N}$. We write $f_n \overset{\Delta}{\to} f$ in such a case. We say that a metric space (X, ρ) is a Δ-*space* if every pointwise convergent sequence of continuous functions $f_n : X \to \mathbb{R}$ to a continuous function $f : X \to \mathbb{R}$ is Δ-convergent. We use the notation $f_n \to f$ for the *pointwise convergence* of f_n's to f.

It was an idea of M. Katětov that separable Δ-spaces might be possibly characterized just by the properties of their topology. J. Fuka found an example of a topologically complete separable metric space (X, ρ) which is not a Δ-space around 1970. It is almost straightforward that σ-compact metric spaces are Δ-spaces. These results were announced in [2] published in 1999. The example of

The work is a part of the research project MSM 0021620839 financed by MSMT; the research was partly supported also by grants GAČR 201/03/0933 and GAČR 201/03/0931.

J. Fuka lead M. Katětov to the hypothesis that σ-compactness should also be a necessary topological property of Δ-spaces. We show that this conjecture is essentially true. More exactly, among analytic spaces the only Δ-spaces are the σ-compact ones. This is the main contribution of our Theorem 2.1. We also show that it is consistent with ZFC that there are nonanalytic (possibly coanalytic) separable Δ-spaces.

We begin with the proof of the above mentioned fact.

Proposition 1.2. *Let (X,ρ) be a σ-compact metric space. Then for an arbitrary metric space (Y,τ) the pointwise convergence $f_n \to f$ of continuous mappings $f_n : X \to Y$ to a mapping $f : X \to Y$ implies $f_n \overset{\Delta}{\to} f$. In particular, (X,ρ) is a Δ-space.*

Proof: We fix a sequence of compact subsets X_n of X such that $X_n \subset X_{n+1}$ for $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} X_n = X$.

We consider a fixed $n \in \mathbb{N}$ for a while. Given an $x \in X_n$ the continuity of f_n ensures the existence of a $\delta_n(x) > 0$ such that $\tau(f_n(x), f_n(x')) < \frac{1}{n}$ if $\rho(x,x') < \delta_n(x)$ and $x' \in X$. By compactness of X_n there are $k_n \in \mathbb{N}$ and $a_1,\dots,a_{k_n} \in X_n$ such that $X_n \subset \bigcup_{i=1}^{k_n} U(a_i, \delta_n(a_i)/2)$, where $U(a,\delta)$ stands for the open ball in (X,ρ) with center a and radius δ. Put $\delta_n = \min\{\delta_n(a_i)/2 : i = 1,\dots,k_n\}$. We consider now an arbitrary pair of points $x \in X_n$ and $x' \in X$ with $\rho(x,x') < \delta_n$. By the choice of $a_1,\dots,a_{k_n}$ there is an $l \in \{1,\dots,k_n\}$ such that $\rho(a_l,x) < \delta_n(a_l)/2$ which implies due to the choice of $\delta_n(x)$ that $\tau(f_n(x), f_n(a_l)) < \frac{1}{n}$. Using the triangle inequality we also get that $\rho(a_l,x') < \delta_n(a_l)/2 + \delta_n \le \delta_n(a_l)$, and the choice of $\delta_n(a_l)$ implies that $\tau(f_n(a_l), f_n(x')) < \frac{1}{n}$. Hence, using triangle inequality once more, we have

$$\tau(f_n(x), f_n(x')) < \frac{2}{n} \ \text{ if } \ x \in X_n,\ x' \in X, \ \text{ and } \ \rho(x,x') < \delta_n. \tag{1}$$

We are going to conclude the proof by showing that the previously found sequence $\{\delta_n\}_{n=1}^{\infty}$ is the required sequence of positive numbers which ensures the Δ-convergence of f_n to f. Let $x, x_n \in X$, $n \in \mathbb{N}$, be such that $\rho(x,x_n) < \delta_n$. There is an $n_0 \in \mathbb{N}$ such that $x \in X_{n_0}$, and so, by the monotonicity of X_n's, $x \in X_n$ for $n \ge n_0$. Given $\varepsilon > 0$ we find $n_1 \ge n_0$ such that $2/n_1 < \varepsilon/2$ and $\tau(f_n(x), f(x)) < \varepsilon/2$ for $n \ge n_1$. Thus, using (1), we have that $\tau(f_n(x_n), f(x)) \le \tau(f_n(x_n), f_n(x)) + \tau(f_n(x), f(x)) < 2/n + \varepsilon/2 < \varepsilon$ for $n \ge n_1$. So $f_n(x_n)$ converge to $f(x)$. □

2. Analytic spaces and Δ-convergence

In this section we show that the assertion of Proposition 1.2 may be reversed within the class of analytic metric spaces. In this sense, we give the positive answer to the above stated question of M. Katětov for analytic spaces. Let us point out that although the Δ-convergence on a metric space depends on the metric, Theorem 2.1 shows that in fact being a Δ-space is a topological property of analytic metric spaces.

A metrizable space is *analytic* if it is a continuous image of some Polish, i.e., separable and completely metrizable, space. In what follows we also use the notation $\mathbf{K}_\sigma$ for the class of σ-compact spaces.

Theorem 2.1. *Let (X, ρ) be an analytic space. Then the following are equivalent.*

(a) *(X, ρ) is a Δ-space.*
(b) *If $f_n \to f$ for some continuous mappings f_n of (X, ρ) to a metric space (Y, τ), then $f_n \xrightarrow{\Delta} f$.*
(c) *X is $\mathbf{K}_\sigma$.*

Obviously, (a) implies (b). Due to Proposition 1.2, (c) implies (a), and so it is enough to show that any analytic metric space X which is not $\mathbf{K}_\sigma$ is not a Δ-space. We show it first for a particular space. Let us endow the Baire space $\mathbb{N}^\mathbb{N}$ with the only metric ρ_B for which $\rho_B(x, y) = \max\{\frac{1}{k} : k \in \mathbb{N},\ x(k) \neq y(k)\}$ if $x, y \in \mathbb{N}^\mathbb{N}$ and $x \neq y$. This is a metric compatible with the topology τ_B of the Baire space, i.e., the product topology of the countable product of discrete copies of $\mathbb{N} = \{1, 2, \dots\}$. Our proof proceeds in several steps. The example constructed in the following lemma is the crucial part of it.

Lemma 2.2. *The metric space $(\mathbb{N}^\mathbb{N}, \rho_B)$ is not a Δ-space.*

Proof: Let $x \in \mathbb{N}^\mathbb{N}$ be arbitrary. Define $N(x) = \min\{n \in \mathbb{N} : x(n) = 1\}$ ($N(x)$ is not defined if the set $\{n \in \mathbb{N} : x(n) = 1\}$ is empty).

We will consider the elements of $\mathbb{N}^\mathbb{N}$ as composed of blocks, revealing their inner structure as indicated below. For this reason, we put $N_0(x) = 0$ and $N_k(x) = N_{k-1}(x) + x(N_{k-1}(x) + 1)$ for $k \in \mathbb{N}$.

$$\overbrace{x(1), \dots, x(N_1(x))}^{\text{1st block}}, \dots, \overbrace{x(N_{n-1}(x) + 1), \dots, x(N_n(x))}^{n\text{th block}}, \dots$$

Further, let $K(x)$ be the smallest natural number k such that $N_k(x) \geq N(x)$. Thus a typical sequence $x \in \mathbb{N}^\mathbb{N}$ with $N(x) \in \mathbb{N}$ looks as follows

$$\overbrace{x(1), \dots, x(N_1(x))}^{x(1)\text{ times}}, \dots, \overbrace{x(N_{K(x)-1}(x) + 1), \dots, x(N(x)) = 1, \dots, x(N_{K(x)}(x))}^{x(N_{K(x)-1}(x)+1)\text{ times}}, \dots$$

Finally, we define $f_k(x) = 1$ if $K(x)$ is defined and $k = K(x)$, or $f_k(x) = 0$ otherwise.

Obviously, $\lim_{k\to\infty} f_k(x) = 0$ (as $f_k(x) \neq 0$ for at most one k) for every $x \in \mathbb{N}^{\mathbb{N}}$.

Fix $n \in \mathbb{N}$ and $x \in \mathbb{N}^{\mathbb{N}}$. If $N(x) \in \mathbb{N}$ is defined and $y(1) = x(1), \dots, y(N(x)) = x(N(x)) = 1$, then $N_k(x) = N_k(y)$ for $k = 1, 2, \dots, K(x)$, so $K(y) = K(x)$ and $f_n(x) = f_n(y)$. Thus f_n is continuous at x. If $x(k) \neq 1$ for every $k \in \mathbb{N}$ and $y(1) = x(1), \dots, y(N_n(x)) = x(N_n(x))$, then $K(y) > n$ if defined and $f_n(y) = 0$. Thus f_n is continuous at x again.

Let $\delta_n > 0$ be arbitrary. We put $N_0 = 0$ and choose $N_n \in \mathbb{N}$ so that $N_n - N_{n-1} > 1$ and $1/N_n < \delta_n$ for $n \in \mathbb{N}$. We define $x \in \mathbb{N}^{\mathbb{N}}$ by $x(N_k+1) = N_{k+1} - N_k$ for $k = 0, 1, \dots$ and $x(N) = 2$ otherwise. The sequence x has the form

$$\overbrace{N_1, 2, \dots, 2}^{N_1 \text{ times}}, \dots, \overbrace{x(N_{n-1}+1) = N_n - N_{n-1}, 2, \dots, x_n(N_n) = 2}^{N_n - N_{n-1} \text{ times}}, \dots$$

Similarly we define $x_n(N_k + 1) = N_{k+1} - N_k$ for $k = 0, \dots, n-1$, $x_n(N_n) = 1$, and $x_n(N) = 2$ otherwise. Thus we have a sequence $x_n \in \mathbb{N}^{\mathbb{N}}$ of the form

$$\overbrace{N_1, 2, \dots, 2}^{N_1 \text{ times}}, \dots, \overbrace{x_n(N_{n-1}+1) = N_n - N_{n-1}, 2, \dots, x_n(N_n) = 1}^{N_n - N_{n-1} \text{ times}}, \dots$$

Now $N(x_n) = N_n$ because of $N_k - N_{k-1} > 1$ for $k \in \mathbb{N}$. The equality $N_n = N_n(x_n)$ follows from the definition of the last quantity. So $K(x_n) = n$, $f_n(x_n) = 1$, and $\rho_B(x_n, x) = 1/N_n < \delta_n$. Thus the f_n's do not Δ-converge to the zero function. □

The following easy observation shows that $\mathbb{N}^{\mathbb{N}}$ is not a Δ-space if it is endowed with any metric compatible with τ_B.

Lemma 2.3. *Let ρ be a metric on the Baire space $\mathbb{N}^{\mathbb{N}}$ compatible with the product topology τ_B. Then there is a homeomorphism h of the Baire space $\mathbb{N}^{\mathbb{N}}$ onto itself such that $\rho(h(x), h(y)) \leq \rho_B(x, y)$ if $x, y \in \mathbb{N}^{\mathbb{N}}$ are such that $\rho_B(x, y) \leq \frac{1}{2}$. Consequently, $(\mathbb{N}^{\mathbb{N}}, \rho)$ is not a Δ-space.*

Proof: It is easy to find partitions $\mathcal{P}_n = \{P^n_{k_1,\dots,k_n} : k_1, \dots, k_n \in \mathbb{N}\}$, $n \in \mathbb{N}$, of $\mathbb{N}^{\mathbb{N}}$ consisting of clopen pairwise disjoint sets $P^n_{k_1,\dots,k_n}$ so that the ρ-diameter of each $P \in \mathcal{P}_n$ is at most $\frac{1}{n+1}$ and so that $P^{n+1}_{k_1,\dots,k_n,k_{n+1}} \subset P^n_{k_1,\dots,k_n}$ (we use the existence of a basis consisting of clopen sets without isolated points and the separability of $\mathbb{N}^{\mathbb{N}}$). Given an $x \in \mathbb{N}^{\mathbb{N}}$ let $h(x)$ be the only element of $\bigcap_{n\in\mathbb{N}} P^n_{x(1),\dots,x(n)}$. Note that $h(\{x \in \mathbb{N}^{\mathbb{N}} : x(1) = x_0(1), \dots, x(n) = x_0(n)\}) = P^n_{x_0(1),\dots,x_0(n)}$ for $x_0(1), \dots, x_0(n) \in \mathbb{N}$. Let $0 < \rho_B(x, y) = \frac{1}{n+1} \leq \frac{1}{2}$. Then both $h(x)$ and $h(y)$ are elements of the same $P^n_{x(1),\dots,x(n)} \in \mathcal{P}^n$ and so $\rho(h(x), h(y)) \leq \frac{1}{n+1} = \rho_B(x, y)$.

Let f_n, f be continuous functions on $\mathbb{N}^\mathbb{N}$ witnessing the fact that $(\mathbb{N}^\mathbb{N}, \rho_B)$ is not a Δ-space. Considering $f_n - f$ and the zero function, we may suppose without loss of generality that f is the zero function. Thus we have $f_n \to 0$ and for every sequence of $\delta_n > 0$ there are $x_n, x \in \mathbb{N}^\mathbb{N}$ with $\rho_B(x_n, x) < \delta_n$ such that the statement $\lim_{n\to\infty} f_n(x_n) = 0$ is false.

Put $g_n = f_n \circ h^{-1}$. Consider an arbitrary sequence of positive real numbers δ_n. Let $0 < \delta'_n \le \delta_n$ be such that $\delta'_n \le \frac{1}{2}$. By our assumptions there are $x_n, x \in \mathbb{N}^\mathbb{N}$ such that $\rho_B(x_n, x) < \delta'_n$ and the statement $\lim_{n\to\infty} f_n(x_n) = 0$ is false. Now, for $y_n = h(x_n)$ and $y = h(x)$ we have $\rho(y_n, y) \le \rho_B(x_n, x) < \delta'_n \le \delta_n$ and the statement $\lim_{n\to\infty} g_n(y_n) = 0$ is false. □

We are going to use a classical result of Hurewicz to reduce the proof of Theorem 2.1 to the preceding lemma. We need one more simple observation.

Lemma 2.4. *If (F, ρ) is a closed subspace of a Δ-space (X, ρ), then (F, ρ) is also a Δ-space.*

Proof: As in the proof of Lemma 2.3 we may suppose that there are continuous functions $f_n : F \to \mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = 0$ for all $x \in F$ but the sequence $\{f_n\}_{n=1}^\infty$ does not Δ-converge to the zero function. Since X is normal we may find continuous functions $g_n : X \to \mathbb{R}$ such that $g_n = f_n$ on F and such that $g_n(x) = 0$ if $\operatorname{dist}_\rho(x, F) \ge \frac{1}{n}$. Obviously, the functions g_n converge to zero pointwise and the convergence is not the Δ-convergence. □

We now conclude the proof of Theorem 2.1. Let (X, ρ) be an analytic metric space which is not $\mathbf{K}_\sigma$. So it is not $\mathbf{F}_\sigma$ in its metrizable compactification and it contains a closed subspace F which is homeomorphic to the Baire space $\mathbb{N}^\mathbb{N}$ (see, e.g., [5, Theorem 21.18]). The space (F, ρ) is not a Δ-space by Lemma 2.3. Finally, (X, ρ) is not a Δ-space by Lemma 2.4.

3. Nonanalytic spaces and Δ-convergence

Remark 3.1. If $A \subset [0, 1]$ is any $\mathbf{G}_\delta$ set that is not $\mathbf{K}_\sigma$, and $B \subset [2, 3]$ is arbitrary, then the space $A \cup B$ endowed with the Euclidean metric ρ_E is not a Δ-space. It is easy to observe that a suitable choice of B shows that such a set can be coanalytic non-Borel, nonmeasurable, etc. In the following proposition we show that, at least under ZFC with additional axioms, there are nonanalytic Δ-spaces.

A subset B of $\mathbb{N}^\mathbb{N}$ is *bounded* if there is an $x \in \mathbb{N}^\mathbb{N}$ such that the set $\{n \in \mathbb{N} : x(n) < y(n)\}$ is finite for every $y \in B$. The smallest cardinal of an unbounded subset of $\mathbb{N}^\mathbb{N}$ is denoted by $\mathfrak{b}$.

Proposition 3.2. (a) *There is a model of ZFC which ensures the existence of a coanalytic subset C of the real line which is not analytic and (C, ρ_E) is a Δ-space.*

(b) *There is a model of ZFC which ensures the existence of a subset C of the real line which is not Lebesgue measurable and (C, ρ_E) is a Δ-space.*

Moreover, in both cases if $f_n \to f$ on C for continuous mappings $f_n : X \to (Y, \tau)$, then $f_n \xrightarrow{\Delta} f$.

Proof: To prove (a) we consider a model of ZFC in which Martin's axiom holds together with the negation of continuum hypothesis obtained by a ccc forcing in [7]. By [3, Corollary on page 178] ccc forcing preserves cardinals. So, taking $V = L$ as the ground model, we may suppose that moreover $\omega_1^L = \omega_1$ in the model of our consideration. Consequently, by [6, Theorem 3.2], we have that every set $C \subset \mathbb{R}$ of cardinality $\aleph_1$ is coanalytic. At the same time, as $\aleph_1 < 2^{\aleph_0}$, the set C is not analytic (it follows, e.g., from [5, Theorem 29.1]). Let a sequence of continuous mappings $f_n : X \to Y$ converge pointwise to f on C. For every $x \in C$ we may find a $\delta_n(x)$ such that $\tau(f_n(y), f_n(x)) < \frac{1}{n}$ whenever $|y - x| < \delta_n(x)$. We may find natural numbers $k_n(x)$ such that $\frac{1}{k_n(x)} < \delta_n(x)$ for every $x \in C$ and every $n \in \mathbb{N}$. The set $\{\{k_n(x)\}_{n=1}^{\infty} : x \in C\}$ is a bounded subset of $\mathbb{N}^{\mathbb{N}}$ in the sense recalled above as $\mathfrak{b} > \aleph_1$ by [4, Theorem 19.22], i.e., there is a $\{k(n)\}_{n=1}^{\infty} \in \mathbb{N}^{\mathbb{N}}$ with the sets $\{n \in \mathbb{N} : k(n) < k_n(x)\}$, $x \in C$, finite. Put $\delta_n = \frac{1}{k(n)}$. Given a sequence of x_n's in C such that $|x_n - x| < \delta_n$ for some $x \in C$, we have $|x_n - x| < \delta_n = \frac{1}{k(n)} \leq \frac{1}{k_n(x)} < \delta_n(x)$ for sufficiently large n's. As $\lim_{n\to\infty} f_n(x) = f(x)$ and $\tau(f_n(x_n), f_n(x)) < \frac{1}{n}$ we have that $\lim_{n\to\infty} f_n(x_n) = f(x)$. Thus (C, ρ_E) is a Δ-space.

According to [1, table on page 5] there is a model of ZFC in which $\mathfrak{b} = 2^{\aleph_0}$ and there is a set $C \subset \mathbb{R}$ of cardinality less than continuum which is not Lebesgue null. Thus C is not Lebesgue measurable, since otherwise it would contain a Borel set of positive Lebesgue measure and therefore its cardinality would be that of the continuum (we may use, e.g., [5, Theorem 29.1] again). This gives (b) using the same argument as in the proof of (a). □

Acknowledgment. We would like to thank Ondřej Kalenda for his help in finding results on the existence of needed models of ZFC to get the intended conclusions of Proposition 3.2.

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(*Received* April 14, 2005, *revised* June 9, 2005)