The family of I-density type topologies

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Abstract. We investigate a family of topologies introduced similarly as the I-density topology. In particular, we compare these topologies with respect to inclusion and we look for conditions under which these topologies are identical.

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We use here a standard notation. Let $\mathbb{N}$ be the set of all positive integers, $\mathcal{B}$ the family of subsets of the real line having the Baire property and I the σ-ideal of meager sets. For every set A and $x, t \in \mathbb{R}$, we set $A + x = \{a + x; a \in A\}$ and $t \cdot A = \{t \cdot a; a \in A\}$, where χ_A is the characteristic function of A and A' the complement of A.

Let S be the family of all nondecreasing and unbounded sequences of positive real numbers. Every sequence $\{s_n\}_{n\in\mathbb{N}} \in S$ is denoted by $\langle s\rangle$.

Let us recall the notion of an I-density point of a set $A \in \mathcal{B}$ ([PWW1]). The point 0 is an I-density point of a set $A \in \mathcal{B}$ if for every sequence $\{t_n\}_{n\in\mathbb{N}} \in S$ there exists a subsequence $\{t_{n_p}\}_{p\in\mathbb{N}}$ such that $\chi_{(t_{n_p}\cdot A)\cap[-1,1]} \underset{p\to\infty}{\longrightarrow} 1$ I-a.e. on $[-1, 1]$.

Based on the observation that starting from another fixed sequence different results can be obtained, the notion of an I-density point connected with a fixed sequence from the family S has been introduced in [HH].

Definition 1. Let $\langle s\rangle \in S$. The point 0 is an $\langle s\rangle$-I-density point of a set $A \in \mathcal{B}$ if for every subsequence $\{s_{n_m}\}_{m\in\mathbb{N}} \subset \langle s\rangle$ there exists a subsequence $\{s_{n_{m_p}}\}_{p\in\mathbb{N}}$ such that $\chi_{(s_{n_{m_p}}\cdot A)\cap[-1,1]} \underset{p\to\infty}{\longrightarrow} 1$ I-a.e. on $[-1, 1]$.

A point $x \in \mathbb{R}$ is an $\langle s\rangle$-I-density point of A if 0 is an $\langle s\rangle$-I-density point of the set $A - x$.

A point $x \in \mathbb{R}$ is an $\langle s\rangle$-I-dispersion point of A if x is an $\langle s\rangle$-I-density point of A'.

We can define one-sided $\langle s\rangle$-I-density points in the natural way.

For any $\langle s\rangle \in S$ and $A \in \mathcal{B}$, putting

$$\Phi_{\langle s\rangle I}(A) = \{x \in \mathbb{R}; x \text{ is an } \langle s\rangle\text{-}I\text{-density point of } A\}$$

we get that $\Phi_{\langle s\rangle I} : \mathcal{B} \to \mathcal{B}$ is a lower density operator (see [HH]).

Applying this operator we define for every fixed sequence $\langle s\rangle$ the topology $\mathcal{T}_{\langle s\rangle I} = \{A \in \mathcal{B}; A \subset \Phi_{\langle s\rangle I}(A)\}$, which fulfils the inclusion: $\mathcal{T}_I \subset \mathcal{T}_{\langle s\rangle I}$, where $\mathcal{T}_I$ denotes the I-density topology ([HH]).

The main aim of this paper is to compare topologies connected with different sequences.

First of all, if $\langle s\rangle$ is the sequence of all natural numbers then $\mathcal{T}_{\langle s\rangle I} = \mathcal{T}_I$ ([PWW1]).

Now we state the main results.

Let $S_0 = \{\langle s\rangle \in \mathcal{S} : \liminf_{n\to\infty} \frac{s_n}{s_{n+1}} = 0\}$.

Theorem 1. *Let $\langle s\rangle \in S$. Then $\mathcal{T}_{\langle s\rangle I} = \mathcal{T}_I$ if and only if $\langle s\rangle \in S \setminus S_0$.*

Theorem 2. *Let $\langle s\rangle, \langle t\rangle \in S_0$ and $\lim_{m\to\infty} \frac{s_m}{t_m} = \alpha \in (0, +\infty)$. Then $\mathcal{T}_{\langle s\rangle I} = \mathcal{T}_{\langle t\rangle I}$ if and only if $\alpha = 1$.*

Before presenting the proofs we need some properties of our topologies.

Properties.

(1) *Let $\langle s\rangle, \langle t\rangle \in \mathcal{S}$. Then $\mathcal{T}_{\langle s\rangle I} = \mathcal{T}_{\langle t\rangle I}$ if and only if $\Phi_{\langle s\rangle I}(A) = \Phi_{\langle t\rangle I}(A)$ for every $A \in \mathcal{B}$.*

(2) *Let $\langle s\rangle \in \mathcal{S}$ and $1 \le \alpha < \infty$. Then $\mathcal{T}_{\langle s\rangle I} \subset \mathcal{T}_{\langle \alpha s\rangle I}$, where $\langle \alpha s\rangle = \{\alpha s_n\}_{n\in\mathbb{N}}$.*

(3) *Let $\langle s\rangle \in \mathcal{S}$. Then for an arbitrary subsequence $\langle s'\rangle \subset \langle s\rangle$ we have $\mathcal{T}_{\langle s\rangle I} \subset \mathcal{T}_{\langle s'\rangle I}$.*

(4) *Let $\langle s\rangle \in \mathcal{S}$. If for any subsequence of the sequence of all natural numbers $\langle n'\rangle \subset \{n\}_{n\in\mathbb{N}}$ there exists a subsequence $\langle n''\rangle \subset \langle n'\rangle$ such that $\mathcal{T}_{\langle s\rangle I} \subset \mathcal{T}_{\langle n''\rangle I}$, then $\mathcal{T}_{\langle s\rangle I} \subset \mathcal{T}_I$.*

(5) $\forall \langle s\rangle \in S \ \ \forall x \in \mathbb{R} \ \ \forall A \in \mathcal{B} \ \ (A \in \mathcal{T}_{\langle s\rangle I} \implies A + x \in \mathcal{T}_{\langle s\rangle I})$.

(6) $\forall \langle s\rangle \in S \ \ \forall A \in \mathcal{B} \ \ (A \in \mathcal{T}_{\langle s\rangle I} \implies -A \in \mathcal{T}_{\langle s\rangle I})$.

(7) $\forall \langle s\rangle \in S \ \ \forall |m| \ge 1 \ \ \forall A \in \mathcal{B} \ \ (A \in \mathcal{T}_{\langle s\rangle I} \implies m \cdot A \in \mathcal{T}_{\langle s\rangle I})$.

(8) $\forall \langle s\rangle \in S_0 \ \ \exists A \in \mathcal{B} \ \ \forall |m| < 1 \ (A \in \mathcal{T}_{\langle s\rangle I} \wedge m \cdot A \notin \mathcal{T}_{\langle s\rangle I})$.

The first four are simple consequences of the definitions and properties of lower densities. We want only to show one implication from (1) (the inverse is obvious).

Proof of (1): Let $\langle s\rangle, \langle t\rangle \in S$. We assume that $\mathcal{T}_{\langle s\rangle I} = \mathcal{T}_{\langle t\rangle I}$ and there exists a set $A \in \mathcal{B}$ such that $\Phi_{\langle s\rangle I}(A) \neq \Phi_{\langle t\rangle I}(A)$, for example $\Phi_{\langle s\rangle I}(A) \not\subseteq \Phi_{\langle t\rangle I}(A)$. Since $\Phi_{\langle t\rangle I}(A) \in \mathcal{T}_{\langle t\rangle I} = \mathcal{T}_{\langle s\rangle I}$, by definition of $\mathcal{T}_{\langle s\rangle I}$ we have $\Phi_{\langle t\rangle I}(A) \subset \Phi_{\langle s\rangle I}(\Phi_{\langle t\rangle I}(A))$ which is equal to $\Phi_{\langle s\rangle I}(A)$ because $\Phi_{\langle t\rangle I}(A)$ is equivalent to A (the Lebesgue Density Theorem works here), so we get a contradiction. $\square$

The next four properties have been already published ([HH], [H]). A justification of (5)–(7) is again easy so we can omit it. We want only to sketch the proof of the last one.

Proof of (8): Let $\langle s\rangle \in S_0$. Then there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}$ of $\langle s\rangle$ such that $\lim_{k\to\infty} \frac{s_{n_k}}{s_{n_k+1}} = 0$.

Put $X = \bigcup_{j=1}^{\infty}[\frac{1}{s_{n_j+1}}, \frac{1}{\sqrt{s_{n_j}\cdot s_{n_j+1}}}]$. Then 0 is an $\langle s\rangle$-I-dispersion point of a set X. Defining $Y = -X \cup X$ we have $A = \{0\} \cup (\mathbb{R} \setminus Y) \in \mathcal{T}_{\langle s\rangle I}$.

For $m = 0$ it is obvious that $m \cdot A \notin \mathcal{T}_{\langle s\rangle I}$.

Now we want to show that 0 is not a right $\langle s\rangle$-I-dispersion point of the set $m\cdot X$ for $m \in (-1,1)\setminus\{0\}$. There is no loss of generality in assuming that $m \in (0,1)$. We can find $k_0 \in \mathbb{N}$ such that for any $k > k_0$ we have $\sqrt{\frac{s_{n_k}}{s_{n_k+1}}} < m$. Then 0 is not a right $\langle s\rangle$-I-dispersion point of the set $m\cdot\bigcup_{j=k_0}^{\infty}[\frac{1}{s_{n_j+1}}, \frac{1}{\sqrt{s_{n_j}\cdot s_{n_j+1}}}]$, so neither of the set $m\cdot X$. Hence $m \cdot A = \{0\} \cup (\mathbb{R} \setminus m\cdot Y) \notin \mathcal{T}_{\langle s\rangle I}$.

For details see [HH]. □

Proof of Theorem 1: Sufficiency. Since $\mathcal{T}_I \subset \mathcal{T}_{\langle s\rangle I}$ for every sequence $\langle s\rangle \in \mathcal{S}$, it is enough to show the inclusion: $\mathcal{T}_{\langle s\rangle I} \subset \mathcal{T}_I$.

Let $\langle s\rangle \in S \setminus S_0$. We denote $\liminf_{k\to\infty} \frac{s_k}{s_{k+1}}$ by λ, so $\lambda > 0$.

Let $\langle n'\rangle = \{n_j\}_{j\in\mathbb{N}}$ denote an arbitrary sequence of natural numbers, $\langle n'\rangle \in S$. Then there exists $j_0 \in \mathbb{N}$ such that for each $j \ge j_0$, $j \in \mathbb{N}$, there exists $k_j \in \mathbb{N}$ which fulfils the condition $s_{k_j} \le n_j \le s_{k_j+1}$. There is no loss of generality in assuming that $j_0 = 1$. Now we choose a subsequence $\{n_{j_l}\}_{l\in\mathbb{N}}$ from the sequence $\{n_j\}_{j\in\mathbb{N}}$ such that each interval $[s_{k_{j_l}}, s_{k_{j_l}+1}]$ contains only one term of the sequence $\{n_{j_l}\}_{l\in\mathbb{N}}$. Since $s_{k_{j_l}} \le n_{j_l} \le s_{k_j+1}$ for each $l \in \mathbb{N}$, we have

$$1 \le \frac{n_{j_l}}{s_{k_{j_l}}} \le \frac{s_{k_{j_l}+1}}{s_{k_{j_l}}}$$

and

$$\begin{aligned} 1 \le \limsup_{l\to\infty} \frac{n_{j_l}}{s_{k_{j_l}}} &\le \limsup_{l\to\infty} \frac{s_{k_{j_l}+1}}{s_{k_{j_l}}} \\ &= 1/\liminf_{l\to\infty} \frac{s_{k_{j_l}}}{s_{k_{j_l}+1}} \le 1/\liminf_{k\to\infty} \frac{s_k}{s_{k+1}} = \frac{1}{\lambda} < +\infty. \end{aligned}$$

Therefore there exists a subsequence $\{\frac{n_{j_{l_p}}}{s_{k_{j_{l_p}}}}\}_{p\in\mathbb{N}} \subset \{\frac{n_{j_l}}{s_{k_{j_l}}}\}_{l\in\mathbb{N}}$ tending to α, where $1 \le \alpha < \infty$. Then $\lim_{p\to+\infty} \frac{n_{j_{l_p}}}{\alpha\cdot s_{k_{j_{l_p}}}} = 1$. Using the notation:

$$\langle n''\rangle = \{n_{j_{l_p}}\}_{p\in\mathbb{N}} \quad \text{and} \quad \langle s''\rangle = \{s_{k_{j_{l_p}}}\}_{p\in\mathbb{N}}$$

we obtain (by Theorem 2, which will be proved later) the equality of topologies

$$\mathcal{T}_{\langle n''\rangle I}=\mathcal{T}_{\langle \alpha s''\rangle I}.$$

Furthermore, by Properties (2) and (3), we have

$$\mathcal{T}_{\langle s\rangle I}\subset\mathcal{T}_{\langle s'\rangle I}\subset\mathcal{T}_{\langle s''\rangle I}\subset\mathcal{T}_{\langle \alpha s'\rangle I}=\mathcal{T}_{\langle n''\rangle I}.$$

Property (4) now yields $\mathcal{T}_{\langle s\rangle I}\subset\mathcal{T}_I$ which is the desired conclusion. □

Necessity of the condition $\langle s\rangle\in S\setminus S_0$ has been already stated in [HH]. We repeat here the proof. We want to show that if $\langle s\rangle\in S_0$ then $\mathcal{T}_{\langle s\rangle I}\not\subseteq\mathcal{T}_I$.

From our assumption there exists a subsequence $\{s_{n_k}\}_{k\in\mathbb{N}}\subset\{s_n\}_{n\in\mathbb{N}}$ such that $\lim_{k\to\infty}\frac{s_{n_k}}{s_{n_k+1}}=0$. We can assume that the sequence $\{\frac{s_{n_k}}{s_{n_k+1}}\}_{k\in\mathbb{N}}$ is decreasing (if necessary we can choose a subsequence).

Let

$$A=\bigcup_{j=1}^{\infty}\left[\frac{1}{s_{n_j+1}},\frac{1}{\sqrt{s_{n_j}\cdot s_{n_j+1}}}\right].$$

We will show that 0 is a right $\langle s\rangle$-I-dispersion point of the set A, it means that for each subsequence $\{s_{n_m}\}_{m\in\mathbb{N}}\subset\{s_n\}_{n\in\mathbb{N}}$ there exists a subsequence $\{s_{n_{m_p}}\}_{p\in\mathbb{N}}$ such that $\chi_{(s_{n_{m_p}}\cdot A)\cap[0,1]}\underset{p\to\infty}{\longrightarrow}0$ I-a.e. on $[0,1]$. Let $j(l)=\min\{j\in\mathbb{N}:l<n_j+1\}$. We observe that

$$\begin{aligned}
&\left(s_{n_m}\cdot\bigcup_{j=1}^{\infty}\left[\frac{1}{s_{n_j+1}},\frac{1}{\sqrt{s_{n_n}\cdot s_{n_j+1}}}\right]\right)\cap[0,1]\\
&=\left(s_{n_m}\cdot\bigcup_{j=j(n_m)}^{\infty}\left[\frac{1}{s_{n_j+1}},\frac{1}{\sqrt{s_{n_j}\cdot s_{n_j+1}}}\right]\right)\cap[0,1]\\
&\subset\left(s_{n_m}\cdot\left[0,\frac{1}{\sqrt{s_{n_{j(n_m)}}\cdot s_{n_{j(n_m)}+1}}}\right]\right)\cap[0,1]\\
&=\left[0,\frac{s_{n_m}}{\sqrt{s_{n_{j(n_m)}}\cdot s_{n_{j(n_m)}+1}}}\right]\cap[0,1]\subset\left[0,\frac{s_{n_{j(n_m)}}}{\sqrt{s_{n_{j(n_m)}}\cdot s_{n_{j(n_m)}+1}}}\right]\cap[0,1]\\
&=\left[0,\sqrt{\frac{s_{n_{j(n_m)}}}{s_{n_{j(n_m)+1}}}}\right]\cap[0,1].
\end{aligned}$$

Since $\limsup_m\left[0,\sqrt{\frac{s_{n_{j(n_m)}}}{s_{n_{j(n_m)+1}}}}\right]=\{0\}$, we have $\chi_{(s_{n_m}\cdot A)\cap[0,1]}\underset{m\to\infty}{\longrightarrow}0$ I-a.e. on [0,1], so 0 is an $\langle s\rangle$-I-dispersion point of the set $\widetilde{A}=-A\cup A$.

Let $B = (0, \frac{1}{s_{n_1}}) \setminus A$ and $\widetilde{B} = -B \cup B \cup \{0\}$. Then $\widetilde{B} \in \mathcal{T}_{\langle s\rangle I}$. Of course $B = \bigcup_{j=1}^{\infty} \left(\frac{1}{\sqrt{s_{n_j} \cdot s_{n_j+1}}}, \frac{1}{s_{n_j}}\right)$.

We will show that 0 is not a right I-density point of the set B, it means that there exists a sequence $\{t_k\}_{k\in\mathbb{N}} \in S$ such that for each subsequence $\{t_{k_p}\}_{p\in\mathbb{N}} \subset \{t_k\}_{k\in\mathbb{N}}$, the convergence $\chi_{(t_{k_p}\cdot B)\cap[0,1]} \xrightarrow[p\to\infty]{} 1$ I a.e. does not hold. Let $t_k = \sqrt{s_{n_k} \cdot s_{n_k+1}}$ for $k \in \mathbb{N}$. Observe that

$$
\begin{aligned}
(t_k \cdot B) \cap [0,1] &= \left(\sqrt{s_{n_k} \cdot s_{n_k+1}} \cdot \bigcup_{j=1}^{\infty} \left(\frac{1}{\sqrt{s_{n_j} \cdot s_{n_j+1}}}, \frac{1}{s_{n_j}}\right)\right) \cap [0,1] \\
&= \left(\sqrt{s_{n_k} \cdot s_{n_k+1}} \cdot \bigcup_{j=k+1}^{\infty} \left(\frac{1}{\sqrt{s_{n_j} \cdot s_{n_j+1}}}, \frac{1}{s_{n_j}}\right)\right) \cap [0,1] \\
&\subset \left(\sqrt{s_{n_k} \cdot s_{n_k+1}} \cdot \left[0, \frac{1}{s_{n_{k+1}}}\right]\right) \cap [0,1] \subset \left[0, \frac{\sqrt{s_{n_k} \cdot s_{n_k+1}}}{s_{n_k+1}}\right] \cap [0,1] \\
&= \left[0, \sqrt{\frac{s_{n_k}}{s_{n_k+1}}}\right] \cap [0,1].
\end{aligned}
$$

Since $\limsup_k [0, \sqrt{\frac{s_{n_k}}{s_{n_k+1}}}] = \{0\}$, we have $\chi_{t_k\cdot B\cap[0,1]}(x) \xrightarrow[k\to\infty]{} 0$ for $x \in (0,1]$. Therefore $\widetilde{B} \notin \mathcal{T}_I$. □

Corollary 1. *For every sequence $\langle s\rangle \in S \setminus S_0$ and for every sequence $\langle t\rangle \in S_0$, $\mathcal{T}_{\langle s\rangle I} \subsetneq \mathcal{T}_{\langle t\rangle I}$.*

Now we can add one more property.

Corollary 2. *For every sequence $\langle s\rangle \in S \setminus S_0$ and for every $m \in \mathbb{R} \setminus \{0\}$, if $A \in \mathcal{T}_{\langle s\rangle I}$ then $m \cdot A \in \mathcal{T}_{\langle s\rangle I}$.*

For the proof of Theorem 2 we need two lemmas.

Lemma 1 ([PWW2])**.** *Let A be an open set and let the sequences $\{i_n\}_{n\in\mathbb{N}}$ and $\{j_n\}_{n\in\mathbb{N}}$ have the following properties: $i_n > 0$, $j_n > 0$ for each $n \in \mathbb{N}$, $\lim_{n\to\infty} i_n = +\infty$, $\lim_{n\to\infty} j_n = +\infty$, $\lim_{n\to\infty} \frac{j_n}{i_n} = 1$ and let $\chi_{(i_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0$ I-a.e. on $[-1,1]$. Then also $\chi_{(j_n\cdot A)\cap[-1,1]} \xrightarrow[n\to\infty]{} 0$ I-a.e. on $[-1,1]$.*

In Lemma 2 we state an equivalent condition for being an $\langle s\rangle$-I-dispersion point of an open set. The idea was motivated by [Ł].

Lemma 2. *Let $\langle s\rangle \in \mathcal{S}$. The point 0 is a right-hand $\langle s\rangle$-I-dispersion point of an open set G if and only if, for every natural number n, there exist a natural* *number k and a real number $\delta > 0$ such that for each $m \in \mathbb{N}$ such that $\frac{1}{s_m} < \delta$ and for each $i \in \{1, \dots, n\}$, there exists a natural number $j \in \{1, \dots, k\}$ such that*

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk} \right) \cdot \frac{1}{s_m}, \quad \left(\frac{i-1}{n} + \frac{j}{nk} \right) \cdot \frac{1}{s_m} \right) = \emptyset.$$

Proof: We shall first prove the necessity for $\langle s \rangle$-I-dispersion. Assume that 0 is a right-hand $\langle s \rangle$-I-dispersion point of the open set G and suppose the assertion of the lemma is false. Then we could find a natural number n_0 such that, for each $k \in \mathbb{N}$ and $\delta_k = \frac{1}{k}$, there exist $m_k \in \mathbb{N}$ such that $k < s_{m_k}$ and $i_k \in \{1, \dots, n_0\}$ such that, for each $j \in \{1, \dots, k\}$

$$G \cap \left(\left(\frac{i_k - 1}{n_0} + \frac{j-1}{n_0 k} \right) \cdot \frac{1}{s_{m_k}}, \quad \left(\frac{i_k - 1}{n_0} + \frac{j}{n_0 k} \right) \cdot \frac{1}{s_{m_k}} \right) \neq \emptyset.$$

Since i_k is chosen from a finite set, there exists a subsequence $\{s_{m_{k_l}}\}_{l \in \mathbb{N}} \subset \{s_{m_k}\}_{k \in \mathbb{N}}$ such that the number i_{k_l} is common for all l. For simplicity we denote it by i_0 and the chosen subsequence by $\{s_{m_k}\}_{k \in \mathbb{N}}$. Let $\{s_{m_{k_z}}\}_{z \in \mathbb{N}}$ be any subsequence of $\{s_{m_k}\}_{k \in \mathbb{N}}$. For every natural number $p \in \mathbb{N}$ the set $\bigcup_{z=p}^{\infty}((s_{m_{k_z}} \cdot G) \cap (\frac{i_0-1}{n_0}, \frac{i_0}{n_0}))$ is open and dense on $[\frac{i_0-1}{n_0}, \frac{i_0}{n_0}]$, so

$$\bigcap_{p=1}^{\infty} \bigcup_{z=p}^{\infty} \left((s_{m_{k_z}} \cdot G) \cap \left[\frac{i_0 - 1}{n_0}, \frac{i_0}{n_0} \right] \right)$$

is residual on $[\frac{i_0-1}{n_0}, \frac{i_0}{n_0}]$. Consequently

$$\limsup_z \left((s_{m_{k_z}} \cdot G) \cap [-1,1] \right) \supset \bigcap_{p=1}^{\infty} \bigcup_{z=p}^{\infty} \left((s_{m_{k_z}} \cdot G) \cap \left[\frac{i_0 - 1}{n_0}, \frac{i_0}{n_0} \right] \right) \notin I.$$

Hence there exists a sequence $\{s_{m_k}\}_{k \in \mathbb{N}}$ such that for each subsequence $\{s_{m_{k_z}}\}_{z \in \mathbb{N}} \subset \{s_{m_k}\}_{k \in \mathbb{N}}$, $\limsup_z((s_{m_{k_z}} \cdot G) \cap [-1,1])$ is a not a meager set. This contradicts our assumption that 0 is an $\langle s \rangle$-I-dispersion point of G.

Now assume that the condition from our lemma is true and our goal is to show that 0 is a right-hand $\langle s \rangle$-I-dispersion point of G.

Let $\{s_{m_p}\}_{p \in \mathbb{N}}$ be an arbitrary subsequence of $\langle s \rangle$. The subsequence of $\{s_{m_p}\}_{p \in \mathbb{N}}$ will be defined by induction. For $n = 1$ there exist $k_1 \in \mathbb{N}$ and $\delta_1 > 0$ such that for each $m \in \mathbb{N}$ for which $\frac{1}{s_m} < \delta_1$ and for $i = 1$ there exists $j = j(s_m, 1) \in \{1, \dots, k_1\}$ such that

$$G \cap \left(\frac{j-1}{k_1} \cdot \frac{1}{s_m}, \frac{j}{k_1} \cdot \frac{1}{s_m} \right) = \emptyset.$$

Let $\{s_{m_{\alpha_1(z)}}\}_{z\in\mathbb{N}}$ be a subsequence of $\{s_{m_p}\}_{p\in\mathbb{N}}$ such that for each $z \in \mathbb{N}$ we have $\frac{1}{s_{m_{\alpha_1(z)}}} < \delta_1$ and the number $j(s_{m_{\alpha_1(z)}}, 1) = j_{11}$ is common for all $z \in \mathbb{N}$. Put $s_{m_{p_1}} = s_{m_{\alpha_1(1)}}$.

Assume the sequence $\{s_{m_{\alpha_{n-1}(z)}}\}_{z\in\mathbb{N}}$ and $s_{m_{p_{n-1}}} = s_{m_{\alpha_{n-1}(1)}}$ to be defined. For a natural number n there exist k_n and $\delta_n > 0$ such that for each $m \in \mathbb{N}$ for which $\frac{1}{s_m} < \delta_n$ and for $i \in \{1 \dots n\}$ there exists $j = j(s_m, i) \in \{1, \dots, k_n\}$ such that

$$G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{n \cdot k_n} \right) \cdot \frac{1}{s_m}, \quad \left(\frac{i-1}{n} + \frac{j}{n \cdot k_n} \right) \cdot \frac{1}{s_m} \right) = \emptyset.$$

Let $\{s_{m_{\alpha_n(z)}}\}_{z\in\mathbb{N}}$ be a subsequence of $\{s_{m_{\alpha_{n-1}(z)}}\}_{z\in\mathbb{N}}$ such that for each $z \in \mathbb{N}$ we have $\frac{1}{s_{m_{\alpha_n(z)}}} < \delta_n$ and $j = (s_{m_{\alpha_n(z)}}, 1) = j_{n1,\dots,}\ j(s_{m_{\alpha_n(z)}}, n) = j_{nn}$ are common for all $z \in \mathbb{N}$. Put $s_{m_{p_n}} = s_{m_{\alpha_n(1)}}$. We proceed by induction.

The task is now to show that $\{x : \chi_{(s_{m_{p_n}} \cdot G)\cap[0,1]} \not\to 0\} \in I$. Let $(a, b) \subset [0, 1]$. Then there exist a natural number n_0 and $i_0 \in \{1, \dots, n_0\}$ such that $[\frac{i_0-1}{n_0}, \frac{i_0}{n_0}] \subset (a, b)$.

We shall consider a sequence $\{s_{m_{\alpha_{n_0}(z)}}\}_{z\in\mathbb{N}}$ and a natural number k_{n_0} corresponding to n_0. Then for each $n \geq n_0$ $\ s_{m_{p_n}} \in \{s_{m_{\alpha_{n_0}(z)}}\}_{z\in\mathbb{N}}$. Hence for each $n \geq n_0$ there exists $j = j_{n_0 i_0}$ such that

$$G \cap \left(\left(\frac{i_0 - 1}{n_0}, \frac{j-1}{n_0 k_{n_0}} \right) \cdot \frac{1}{s_{m_{p_n}}}, \left(\frac{i_0 - 1}{n_0} + \frac{j}{n_0 k_{n_0}} \right) \cdot \frac{1}{s_{m_{p_n}}} \right) = \emptyset.$$

Let

$$(c, d) = \left(\frac{i_0 - 1}{n_0} + \frac{j-1}{n_0 k_{n_0}}, \frac{i_0 - 1}{n_0} + \frac{j}{n_0 k_{n_0}} \right).$$

Then $(c, d) \subset (a, b)$ and for each $n \geq n_0$ we have

$$\emptyset = G \cap \left(c \cdot \frac{1}{s_{m_{p_n}}}, d \cdot \frac{1}{s_{m_{p_n}}} \right) = \frac{1}{s_{m_{p_n}}} \left(\left(s_{m_{p_n}} \cdot G \right) \cap (c, d) \right),$$

so

$$(c, d) \subset [0, 1] \setminus \left(\left(s_{m_{p_n}} \cdot G \right) \cap [0, 1] \right).$$

Therefore

$$(c, d) \subset \bigcup_{n=1}^{\infty} \bigcap_{n=r}^{\infty} [0, 1] \setminus \left(\left(s_{m_{p_r}} \cdot G \right) \cap [0, 1] \right)$$

and $\limsup_r ((s_{m_{p_r}} \cdot G) \cap [0, 1])$ is nowhere dense. Thus

$$\chi_{(s_{m_{p_r}} \cdot G)\cap[0,1]} \xrightarrow[r\to\infty]{} 0 \quad I \ a.e.$$

which completes the proof. □

Proof of Theorem 2: Let $\langle s\rangle, \langle t\rangle \in S$ and $\lim_{m\to\infty} \frac{s_m}{t_m} = 1$. Then using Lemma 1 we get immediately the equality of topologies.

Now, let $\langle s\rangle, \langle t\rangle \in S_0$ and $\lim_{m\to\infty} \frac{s_m}{t_m} = \alpha \in (0, +\infty)$. Let us suppose that $0 < \alpha < 1$. We can assume that $\frac{s_m}{t_m} > \frac{1}{2}\alpha$ for all $m \in \mathbb{N}$. We want to show that $\mathcal{T}_{\langle s\rangle I} \neq \mathcal{T}_{\langle t\rangle I}$.

From the proof of Property (8) it follows that there exists a set Y, which is a countable sum of closed intervals, such that $\{0\} \cup (\mathbb{R} \setminus Y) \in \mathcal{T}_{\langle t\rangle I}$ and 0 is not a $\langle t\rangle$-I-density point of the set $\mathbb{R} \setminus \alpha Y$, which is equivalent to the fact that 0 is not an $\langle \alpha t\rangle$-I-dispersion point of the set Y, so neither of the set $G = \operatorname{int} Y$ since $Y \setminus \operatorname{int} Y \in I$.

It suffices to show that 0 is not an $\langle s\rangle$-I-dispersion point of the set G, because it means that 0 is not an $\langle s\rangle$-I-dispersion point of Y, so $\{0\} \cup (\mathbb{R} \setminus Y) \notin \mathcal{T}_{\langle s\rangle I}$.

For convenience we restrict our consideration to the right-hand case and suppose, contrary to our claim, that 0 is a right-hand $\langle s\rangle$-I-dispersion point of the open set G. By Lemma 2 we know that

(∗) for every natural number n there exist a natural number k and a real number $\delta > 0$ such that for every natural m satisfying $\frac{1}{s_m} < \delta$ and for each $i \in \{1, \ldots, n\}$ there exists a natural number $j \in \{1, \ldots, k\}$ such that $G \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) \cdot \frac{1}{s_m}, \left(\frac{i-1}{n} + \frac{j}{nk}\right) \cdot \frac{1}{s_m}\right) = \emptyset$.

We shall show that

for every natural number N there exist a natural number K and a real number $\Delta > 0$ such that for every natural m satisfying the inequality $\frac{1}{\alpha t_m} < \Delta$ and for each $\tilde{i} \in \{1, \ldots, N\}$ there exists a natural number $\tilde{j} \in \{1, \ldots, K\}$ such that $Y \cap \left(\left(\frac{\tilde{i}-1}{N} + \frac{\tilde{j}-1}{NK}\right) \cdot \frac{1}{\alpha t_m}, \left(\frac{\tilde{i}-1}{N} + \frac{\tilde{j}}{NK}\right) \cdot \frac{1}{\alpha t_m}\right) = \emptyset$.

Consider an arbitrary natural number N. Applying (∗) for $n = N$ we choose $k \in N$ and $\delta > 0$ satisfying (∗). Since, by assumption, $\frac{s_n}{\alpha t_n}$ tends to 1, it follows that

(∗∗) for every $\epsilon > 0$ there exists a natural number n_ϵ such that for every $n > n_\epsilon$ we have an inequality $|\frac{s_n - \alpha t_n}{\alpha t_n}| < \epsilon$.

Set $K = 3k$ and we fix $\Delta > 0$ such that

(1) $\Delta < \frac{\delta}{2}$

and

(2) for every $m \in \mathbb{N}$, if $\frac{1}{s_m} < 2\Delta$ then $m > n_\epsilon$, where $\epsilon = \frac{1}{2NK}$.

Therefore for every $m \in \mathbb{N}$ such that $\frac{1}{\alpha t_m} < \Delta$ we have $\frac{1}{s_m} < 2\Delta < \delta$ (since

$\frac{1}{\alpha t_m} > \frac{1}{2s_m}$), so by (2) and $(**)$ the following inequality holds:

$$\left|\frac{s_m - \alpha t_m}{\alpha t_m}\right| < \frac{1}{2NK}\,.$$

Fix an arbitrary $\tilde{i} \in \{1, \ldots, N\}$. From $(*)$ for $i = \tilde{i}$ there exists a natural number $j \in \{1, \ldots, k\}$ such that

$$Y \cap \left(\left(\frac{i-1}{n} + \frac{j-1}{nk}\right) \cdot \frac{1}{s_m},\ \left(\frac{i-1}{n} + \frac{j}{nk}\right) \cdot \frac{1}{s_m}\right) = \emptyset.$$

To obtain a contradiction, suppose that for every $\tilde{j} \in \{1, \ldots, K\}$ the set Y has common points with the interval $((\frac{i-1}{N} + \frac{\tilde{j}-1}{NK}) \cdot \frac{1}{\alpha t_m}, (\frac{i-1}{N} + \frac{\tilde{j}}{NK}) \cdot \frac{1}{\alpha t_m})$, so for every $\tilde{j} \in \{1, \ldots, K\}$ there exists $y \in G$ such that $y \in (\frac{i-1}{n} + \frac{\tilde{j}-1}{3nk}, \frac{i-1}{n} + \frac{\tilde{j}}{3nk}) \cdot \frac{1}{\alpha t_m}$, it means $y \in (0, \frac{1}{\alpha t_m})$ and $y \cdot \alpha t_m \in (\frac{i-1}{n} + \frac{\tilde{j}-1}{3nk}, \frac{i-1}{n} + \frac{\tilde{j}}{3nk})$. From $(*)$ we see that there exists a number $j \in \{1, \ldots, n\}$ such that for any $y \in Y$ the point $y \cdot s_m$ does not belong to the interval $(\frac{i-1}{n} + \frac{j-1}{nk}, \frac{i-1}{n} + \frac{j}{nk})$. But for $\tilde{j} = 3j - 1$ there exists a point $y \in Y$ such that $y \cdot \alpha t_m \in (\frac{i-1}{n} + \frac{3j-2}{3nk}, \frac{i-1}{n} + \frac{3j-1}{3nk})$. Simultaneously $|y \cdot \alpha t_m - y \cdot s_m| = |y \cdot (\alpha t_m - s_m)| < \frac{1}{\alpha t_m}|\alpha t_m - s_m| < \frac{1}{2NK} = \frac{1}{6nk}$, hence $y \cdot s_m \in \{\frac{i-1}{n} + \frac{3j-3}{3nk}, \frac{i-1}{n} + \frac{3j}{3nk}) = (\frac{i-1}{n} + \frac{j-1}{nk}, \frac{i-1}{n} + \frac{j}{nk})$. This contradiction completes the proof. $\square$

By Theorem 1 it is obvious that for sequences belonging to $S \setminus S_0$ we can have the same topology even if the sequences considered do not satisfy the condition $\lim_{n\to\infty} \frac{s_n}{t_n} = 1$.

The following theorems show more properties of the family of I-density type topologies.

Theorem 3. *For every sequence $\langle t\rangle \in S_0$ there exists a sequence $\langle s\rangle \in S_0$ such that $\mathcal{T}_{\langle s\rangle I} \subsetneq \mathcal{T}_{\langle t\rangle I}$.*

Proof: Let $\langle t\rangle \in S_0$. Then set $\alpha \in (0, 1)$ and let $\langle s\rangle = \langle \alpha t\rangle$. Then $\langle s\rangle \in S_0$ and $\lim_{n\to\infty} \frac{s_n}{t_n} = \alpha \neq 1$, so by Theorem 2 $\mathcal{T}_{\langle t\rangle I} \neq \mathcal{T}_{\langle s\rangle I}$ and by Property (2) $\mathcal{T}_{\langle s\rangle I} \subseteq \mathcal{T}_{\langle t\rangle I}$. $\square$

Theorem 4. *For every sequence $\langle t\rangle \in S$ there exists a sequence $\langle s\rangle \in S$ such that $\mathcal{T}_{\langle t\rangle I} \subsetneq \mathcal{T}_{\langle s\rangle I}$.*

Proof: If $\langle t\rangle \in S \setminus S_0$ then $\mathcal{T}_{\langle t\rangle I} = \mathcal{T}_I$ and it is sufficient to take an arbitrary sequence $\langle s\rangle \in S_0$. Let us assume that $\langle t\rangle \in S_0$. We define $\langle s\rangle = \langle \alpha t\rangle$, where $\alpha \in \mathbb{R}$ and $\alpha > 1$. Then by Property (2), $\mathcal{T}_{\langle t\rangle I} \subset \mathcal{T}_{\langle s\rangle I}$ and from Theorem 2 it follows that $\mathcal{T}_{\langle t\rangle I} \neq \mathcal{T}_{\langle s\rangle I}$. $\square$

Theorem 5. *There exist sequences $\langle s\rangle$, $\langle t\rangle \in S_0$ such that $\mathcal{T}_{\langle s\rangle I} \setminus \mathcal{T}_{\langle t\rangle I} \neq \emptyset$ and $\mathcal{T}_{\langle t\rangle I} \setminus \mathcal{T}_{\langle s\rangle I} \neq \emptyset$.*

Proof: Let $\langle s\rangle = \{(2n-1)!\}_{n\in\mathbb{N}}$, $\langle t\rangle = \{(2n)!\}_{n\in\mathbb{N}}$. Of course $\langle s\rangle, \langle t\rangle \in S_0$. Set $Y_1 = \bigcup_{k=1}^{\infty}(\frac{1}{(2k)!}, \frac{1}{(2k-1)!})$, $Y_2 = \bigcup_{k=2}^{\infty}(\frac{1}{(2k-1)!}, \frac{1}{(2k-2)!})$. We have $Y_1 \cap Y_2 = \emptyset$ and $[0,1] \setminus (Y_1 \cup Y_2) \in I$. Moreover

$$\begin{aligned}(t_n \cdot Y_1) \cap [0,1] &= \left((2n)! \cdot \bigcup_{k=1}^{\infty} \left(\frac{1}{(2k)!}, \frac{1}{(2k-1)!}\right)\right) \cap [0,1] \\ &= \left((2n)! \cdot \bigcup_{k=n+1}^{\infty} \left(\frac{1}{(2k)!}, \frac{1}{(2k-1)!}\right)\right) \cap [0,1] \\ &\subset \left((2n)! \cdot \left[0, \frac{1}{(2n+1)!}\right)\right) \cap [0,1] \\ &= \left[0, \frac{(2n)!}{(2n+1)!}\right) \cap [0,1] = \left[0, \frac{1}{2n+1}\right)\end{aligned}$$

and, of course, for any subsequence $\{t_{n_p}\}_{p\in\mathbb{N}} \subset \langle t\rangle$, $(t_{n_p} \cdot Y_1) \cap [0,1] \subset [0, \frac{1}{2n_p+1})$. It follows that $\limsup_p (t_{n_p} \cdot Y_1) \cap [0,1] = \{0\} \in I$, hence 0 is a right-hand $\langle t\rangle$-I-dispersion point of Y_1, which gives that it is a right-hand $\langle t\rangle$-I-density point of Y_2. Finally $Z_2 = (-Y_2) \cup \{0\} \cup Y_2 \in \mathcal{T}_{\langle t\rangle I}$.

In the same manner we can see that $(s_n \cdot Y_2) \cap [0,1] \subset [0, \frac{1}{2n})$ and conclude that $Z_1 = (-Y_1) \cup \{0\} \cup Y_1 \in \mathcal{T}_{\langle s\rangle I}$. We thus get $Z_1 \in \mathcal{T}_{\langle s\rangle I} \setminus \mathcal{T}_{\langle t\rangle I}$ and $Z_2 \in \mathcal{T}_{\langle t\rangle I} \setminus \mathcal{T}_{\langle s\rangle I}$. □

Theorem 6. *Let $\mathcal{T}^*$ be a topology generated by $\bigcup_{\langle s\rangle\in S} \mathcal{T}_{\langle s\rangle I}$. Then $\bigcup_{\langle s\rangle\in S} \mathcal{T}_{\langle s\rangle I} \neq \mathcal{T}^* = 2^{\mathbb{R}}$.*

Proof: It is immediate that $\bigcup_{\langle s\rangle\in S} \mathcal{T}_{\langle s\rangle I} \neq 2^{\mathbb{R}}$ because $\bigcup_{\langle s\rangle\in S} \mathcal{T}_{\langle s\rangle} \subset \mathcal{B}$. Our proof starts with the observation that if for every $x \in A$, where $A \in \mathcal{B}$, there exists a sequence $\langle s\rangle \in S$ such that $x \in \Phi_{\langle s\rangle I}(A)$ then $A \in \mathcal{T}^*$. Indeed, let $A \in \mathcal{B}$, $x \in A$ and $\langle s\rangle \in S$ be a sequence such that $x \in \Phi_{\langle s\rangle I}(A)$. Since $(\Phi_I(A) \triangle A) \in I$, we have $x \in \Phi_{\langle s\rangle I}(A \cap \Phi_I(A))$. Simultaneously $A \cap \Phi_I(A) \in \mathcal{T}_I \subset \mathcal{T}_{\langle s\rangle I}$. Therefore $(A \cap \Phi_I(A)) \cup \{x\} \in \mathcal{T}_{\langle s\rangle I} \subset \mathcal{T}^*$ and finally $A = \bigcup_{x\in A}((A \cap \Phi_I(A)) \cup \{x\}) \in \mathcal{T}^*$.

We next show that singletons are open in $\mathcal{T}^*$. Let $E = \bigcup_{n=1}^{\infty}(\frac{1}{a_n}, \frac{1}{b_n})$ where $a_n = (2n+1)!$, $b_n = (2n)!$ for $n \in \mathbb{N}$. Then $\langle a\rangle, \langle b\rangle \in S$. We claim that 0 is a right-hand $\langle a\rangle$-I-dispersion point of the set E, because $(a_n \cdot E) \cap [0,1] \subset (0, \frac{1}{2n+2})$ and hence $\chi_{(a_n \cdot E)\cap[0,1]} \underset{n\to\infty}{\longrightarrow} 0$ I a.e. on $[0,1]$ and so does each subsequence. Similarly 0 is a right-hand $\langle b\rangle$-I-density point of the set E, because $(b_n \cdot E) \cap [0,1] \supset (\frac{1}{2n+1}, 1)$ and hence $\chi_{(b_n \cdot E)\cap[0,1]} \underset{n\to\infty}{\longrightarrow} 1$ I-a.e. on $[0,1]$ and so does each subsequence.

Putting $A = E \cup \{0\} \cup (-E)$ we obtain $0 \in \Phi_{\langle b\rangle I}(A)$ and for the set $B = \bigcup_{n=1}^{\infty}((\frac{1}{b_{n+1}}, \frac{1}{a_n}) \cup (-\frac{1}{a_n}, -\frac{1}{b_{n+1}})) \cup \{0\}$ we have $0 \in \Phi_{\langle a\rangle I}(B)$, so by the above $A, B \in \mathcal{T}^*$. Therefore $\{0\} = A \cap B \in \mathcal{T}^*$. Since the topologies considered are invariant under translations, we have $\{x\} = (A+x) \cap (B+x) \in \mathcal{T}^*$ for any $x \in \mathbb{R}$, and finally $\mathcal{T}^* = 2^{\mathbb{R}}$. □

Theorem 7. *Let* $\mathcal{T} = \{T_{\langle s\rangle I}; \langle s\rangle \in S\} = \{\mathcal{T}_I\} \cup \{\mathcal{T}_{\langle s\rangle I}; \langle s\rangle \in S_0\}$. *Then* $\operatorname{card}(\mathcal{T}) = \mathfrak{c}$.

Proof: Obviously $\operatorname{card}(\mathcal{T}) \leq \mathfrak{c}$.

If $\langle s\rangle \in S_0$ then for every $\alpha > 0$ a sequence $\langle \alpha s\rangle \in S_0$. By Theorem 2 for every $\alpha, \beta > 0$, $\alpha \neq \beta$ we have $\mathcal{T}_{\langle \alpha s\rangle I} \neq \mathcal{T}_{\langle \beta s\rangle I}$ so $\operatorname{card}(\mathcal{T}) \geq \mathfrak{c}$. □

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