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Majorization of C_0-semigroups in ordered Banach spaces

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Abstract. We give criteria for domination of strongly continuous semigroups in ordered Banach spaces that are not necessarily lattices, and thus obtain generalizations of certain results known in the lattice case. We give applications to semigroups generated by differential operators in function spaces which are not lattices.

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1. Introduction

Let E be a real Banach space, ordered by a cone K. A cone K is a closed convex subset of E with $\lambda K \subseteq K$ $(\lambda \geq 0)$, and $K \cap (-K) = \{0\}$. As usual $x \leq y :\iff y - x \in K$. Let $A : D(A) \to E$ and $B : D(B) \to E$ be generators of C_0-semigroups $T(t)$ and $S(t)$ $(t \geq 0)$, respectively.

In this paper we will give conditions on A and B such that S is a majorant of T, that is

$$-S(t)y \leq T(t)x \leq S(t)y \quad (t \geq 0, -y \leq x \leq y).$$

If E is a Banach lattice then this is equivalent to the following domination property of the semigroups

$$|T(t)f| \leq S(t)|f| \quad (t \geq 0, f \in E).$$

Domination of semigroups in Banach lattices has been studied and characterized in [5, C-II 4.]. For operators A and B given by forms in a Hilbert space we refer to characterizations in terms of forms in [6].

The main point in this paper is that we do not assume E to be a Banach lattice. The results thus apply in more general situations, on the other hand they cannot involve any of the additional structural features a Banach lattice provides. For lattices, the idea to obtain domination via a certain cone invariance property in the product space (cf. Theorem 1) has been used in [6].

2. Quasimonotone increasing operators

For a cone $K \subseteq E$ the subset

$$K^* = \{\varphi \in E^* : \ \varphi(x) \geq 0 \ (x \geq 0)\},$$

in the space of all continuous linear functionals E^*, is called the dual wedge.

Let D be a subspace of E. A linear operator $C : D \to E$ is *quasimonotone increasing*, in the sense of Volkmann [8], if

$$x \in K \cap D,\ \varphi \in K^*,\ \varphi(x) = 0 \implies \varphi(Cx) \geq 0.$$

Let $A : D(A) \to E$, and $B : D(B) \to E$ be linear mappings ($D(A), D(B)$ subspaces of E), and let $D := D(A) \cap D(B)$. We say that B is a *majorant* of A, if

1. $B - A : D \to E$ is increasing;
2. $B + A : D \to E$ is quasimonotone increasing.

Consider $E \times E$ ordered by the cone

$$K_0 = \{(x, y) \in E \times E : -y \leq x \leq y\},$$

and let $H : D(A) \times D(B) \to E \times E$ be defined as

$$H(x, y) = (Ax, By).$$

Then we have

Theorem 1. *Let B be a majorant of A. Then $H : D \times D \to E \times E$ is quasimonotone increasing (with respect to K_0).*

Proof: Let $\psi \in K_0^*$ and set

$$\varphi_1(u) = \frac{\psi(-u, u)}{2},\ \varphi_2(v) = \frac{\psi(v, v)}{2} \quad (u, v \in E).$$

Then, obviously $\varphi_1, \varphi_2 \in K^*$, and

$$\psi(x, y) = \varphi_1(y - x) + \varphi_2(y + x) \quad ((x, y) \in E \times E).$$

Hence, from

$$x, y \in D, \quad -y \leq x \leq y, \quad \psi(x, y) = 0,$$

we get

$$\varphi_1(y - x) = 0, \quad \varphi_2(y + x) = 0.$$

Since B is a majorant of A we have

$$(B - A)(y - x) \geq 0, \quad (B - A)(y + x) \geq 0,$$
$$\varphi_1((B + A)(y - x)) \geq 0, \quad \varphi_2((B + A)(y + x)) \geq 0.$$

In particular

$$\varphi_1((B - A)(y + x) + (B + A)(y - x)) = 2\varphi_1(By - Ax) \geq 0,$$
$$\varphi_2((B - A)(y - x) + (B + A)(y + x)) = 2\varphi_2(By + Ax) \geq 0.$$

Therefore

$$\psi(H(x, y)) = \psi(Ax, By) = \varphi_1(By - Ax) + \varphi_2(By + Ax) \geq 0,$$

thus $H : D \times D \to E \times E$ is quasimonotone increasing. $\square$

3. Comparison results

Let A, B and D be as in Section 2. In the sequel we assume that K has nonempty interior $\operatorname{Int} K$, and that $D \cap (\operatorname{Int} K) \neq \emptyset$. We fix $p \in D \cap (\operatorname{Int} K)$. Note that

$$(0, p) \in (D \times D) \cap (\operatorname{Int} K_0).$$

Theorem 2. *Let B be a majorant of A, and let $x, y \in C^1([0,T), E)$ satisfy*

1. *$x(t), y(t) \in D$ $(t \in [0,T))$;*
2. *$x'(t) = A(x(t))$, $y'(t) = B(y(t))$ $(t \in [0,T))$;*
3. *$-y(0) \leq x(0) \leq y(0)$.*

Then $-y(t) \leq x(t) \leq y(t)$ $(t \in [0,T))$.

Proof: According to Theorem 1 $H : D \times D \to E$ is quasimonotone increasing. We have $(D \times D) \cap (\operatorname{Int} K_0) \neq \emptyset$ and $(x(0), y(0)) \in (D \times D) \cap K_0$. Application of Theorem 1 in [3] proves $(x(t), y(t)) \in K_0$ $(t \in [0,T))$. □

Now, assume in addition that K is *normal*, i.e., that there exists $\gamma > 0$ such that $0 \leq x \leq y \Rightarrow \|x\| \leq \gamma \|y\|$, and that A and B are generators of C_0-semigroups $T(t)$ and $S(t)$ $(t \geq 0)$, respectively.

Let E be equivalently normed by the Minkowski functional $\|\cdot\|$ of the order interval $[-p, p]$. In particular, K_0 is normal since K is normal, and $E \times E$ can be equivalently normed by the Minkowski functional $|||\cdot|||$ of the order interval $[-(0,p), (0,p)]$.

Under these assumptions we prove the following comparison results:

Theorem 3. *Let B be a majorant of A, and let D be invariant under both semigroups $(T(\cdot))$ and $(S(\cdot))$. For any $x, y \in \overline{D}$ with $-y \leq x \leq y$ we have*

$$-S(t)y \leq T(t)x \leq S(t)y \quad (t \geq 0).$$

Theorem 4. *Let B be a majorant of A, let $D(A) \subseteq D(B)$, and let $S(t)p \in D(A)$ $(t \geq 0)$. Then*

$$-\|x\| S(t)p \leq T(t)x \leq \|x\| S(t)p \quad (t \geq 0)$$

for all $x \in E$.

Proof of Theorem 3: Let $n \in \mathbb{N}$. Since $x, y \in \overline{D}$ we can choose $x_n, z_n \in D$ such that

$$\|x_n - x\| \leq \frac{1}{3n}, \quad \|z_n - y\| \leq \frac{1}{3n},$$

and we set $y_n = z_n + (2/(3n))p \in D$. According to the properties of the chosen norm we have

$$\begin{aligned} -y_n = -z_n - \frac{2}{3n}p \leq -y - \frac{1}{3n}p \leq x - \frac{1}{3n}p \leq x_n \\ \leq x + \frac{1}{3n}p \leq y + \frac{1}{3n}p \leq z_n + \frac{2}{3n}p = y_n, \end{aligned}$$

in particular $(x_n, y_n) \in (D \times D) \cap K_0$, and

$$\|y_n - y\| \le \|z_n - y\| + \frac{2}{3n} \le \frac{1}{n}\,.$$

We set $w_n(t) = (T(t)x_n, S(t)y_n)$ $(t \ge 0)$. Then $w_n(t) \in D \times D$ for $t \ge 0$ (by the invariance assumption) and $w_n \in C^1([0,\infty), E \times E)$. According to Theorem 2 $w_n(t) \in K_0$ $(t \ge 0)$. For $n \to \infty$ we obtain

$$(T(t)x, S(t)y) \in K_0 \quad (t \ge 0)$$

by closedness of K_0. □

Proof of Theorem 4: Fix $x \in E$. Let $n \in \mathbb{N}$, and choose $x_n \in D(A)$ such that $\|x - x_n\| \le 1/n$. Set $w_n(t) = (T(t)x_n, \|x_n\|S(t)p)$ $(t \ge 0)$. Then,

$$w_n(t) \in D(A) \times D(A) = D \times D,$$

$w_n \in C^1([0,\infty), E \times E)$, $w_n'(t) = H(w_n(t))$ $(t \ge 0)$, and $w_n(0) \in K_0$ since

$$-\|x_n\|p \le x_n \le \|x_n\|p,$$

according to the properties of the chosen norm. Again Theorem 2 gives $w_n(t) \in K_0$ $(t \ge 0)$, and for $n \to \infty$ we obtain

$$(T(t)x, \|x\|S(t)p) \in K_0 \quad (t \ge 0).$$

□

4. One-sided estimates

Let $m_+ : E \times E \to \mathbb{R}$ denote the following directional derivative of $\|\cdot\|$, compare [4]:

$$m_+[x,y] = \lim_{h \to 0+} \frac{\|x + hy\| - \|x\|}{h}.$$

Theorem 5. *Under the assumptions of Theorem 3 and $\overline{D} = E$*

$$m_+[x, Ax] \le m_+[p, Bp]\|x\| \quad (x \in D(A)).$$

Proof: For the norm $|||\cdot|||$ on $E \times E$ it is easy to check that

$$|||(x,y)||| = \max\{\|y - x\|, \|y + x\|\}.$$

Let $M_+ : E^2 \times E^2 \to \mathbb{R}$ be the directional derivative with respect to this norm. According to Theorem 3, H is the generator of a positive C_0-semigroup. Theorem 1 in [1] proves

$$M_+[(x,y),(Ax,By)] \le M_+[(0,p),(0,Bp)]\, |||(x,y)||| \quad ((x,y) \in D(A) \times D(B)).$$

But

$$\begin{aligned} M_+[(0,p),(0,Bp)] &= \lim_{h\to 0+} \frac{|||(0,p)+h(0,Bp)||| - |||(0,p)|||}{h} \\ &= \lim_{h\to 0+} \frac{\|p+hBp\| - \|p\|}{h} = m_+[p,Bp], \end{aligned}$$

and

$$\begin{aligned} M_+[(x,0),(Ax,0)] &= \lim_{h\to 0+} \frac{|||(x,0)+h(Ax,0)||| - |||(x,0)|||}{h} \\ &= \lim_{h\to 0+} \frac{\|x+hAx\| - \|x\|}{h} = m_+[x,Ax] \quad (x \in D(A)). \end{aligned}$$

Hence

$$m_+[x,Ax] \le m_+[p,Bp]|||(x,0)||| = m_+[p,Bp]\|x\| \quad (x \in D(A)).$$

□

As an immediate consequence we have:

Corollary 1. *Under the assumptions of Theorem* 5

$$\|T(t)x\| \le \exp(tm_+[p,Bp])\|x\| \quad (t \in [0,\infty),\ x \in E).$$

5. Examples

First we consider Schrödinger type operators in a space of vector-valued functions.

Example 1. Let $\mathbb{R}^3$ be ordered by the ice cream cone

$$C := \{(x,y,z) : \sqrt{x^2+y^2} \le z\}$$

and let $E := BUC(\mathbb{R},\mathbb{R}^3)$ be ordered by the cone

$$K := \{f \in E : f(s) \in C\ (s \in \mathbb{R})\}.$$

The constant function $p : s \mapsto (0,0,1)$ belongs to $\mathrm{Int}(K)$, and K is normal. Since $(\mathbb{R}^3, C)$ is not a lattice, also (E, K) is not a lattice.

The second derivative $L := d^2/ds^2$ with domain $D(L) = BUC^2(\mathbb{R}, \mathbb{R}^3)$ is quasimonotone increasing in E: Clearly, d/ds with domain $BUC^1(\mathbb{R}, \mathbb{R}^3)$ is the generator of the translation group on E, which is positive. The Gaussian semigroup, whose generator is L, is obtained by convolution with a positive kernel, hence also positive. Consequently, L is quasimonotone increasing.

Now let $V \in BUC(\mathbb{R}, \mathbb{R}^{3\times 3})$ be positive in the sense that $V(s)C \subseteq C$ for all $s \in \mathbb{R}$. We denote the induced operator $E \to E, f \mapsto Vf$ also by V. Observe that $V : E \to E$ is bounded and increasing.

Now take another function $U \in BUC(\mathbb{R}, \mathbb{R}^{3\times 3})$. Then the operator $U : E \to E, f \mapsto Uf$ is bounded. Assume that V is a majorant for U, i.e., $V - U : E \to E$ is increasing and $V + U : E \to E$ is quasimonotone increasing. Let $A := L + U$ and $B := L + V$. Since U and V are bounded on E, we have $D(A) = D(B) = D(L) = BUC^2(\mathbb{R}, \mathbb{R}^3)$, and A and B are generators of C_0-semigroups on E. Moreover, B is a majorant for A since $B - A = V - U$ is increasing and $B + A = 2L + V + U$ is quasimonotone increasing. Since $D = D(A) = D(B)$ is invariant under both semigroups, we obtain by Theorem 3, for any $f, g \in E$ with $-g \le f \le g$, that

$$(*) \qquad -e^{t(L+V)}g \le e^{t(L+U)}f \le e^{t(L+V)}g \quad (t \ge 0).$$

We may also apply Theorems 4 and 5 and Corollary 1 to the function $p : s \mapsto (0,0,1)$.

We discuss a few special cases.

(a) $U = 0$: Then V is a majorant of U and we obtain

$$-e^{t(L+V)}g \le e^{tL}f \le e^{t(L+V)}g \quad (t \ge 0)$$

for any $f, g \in E$ with $-g \le f \le g$.

(b) $V = 0$: Then V is a majorant of U if and only if $-U$ is increasing and U is quasimonotone increasing. This holds if and only if, for each $s \in \mathbb{R}$, $-U(s)C \subseteq C$ and $U(s)$ is quasimonotone increasing with respect to C. The latter holds if and only if U is of the form $U(s) = -u(s)I$ where $I \in \mathbb{R}^{3\times 3}$ is the identity and $u \in BUC(\mathbb{R}, \mathbb{R})$ satisfies $u(s) \ge 0$ for $s \in \mathbb{R}$. In this situation we obtain

$$-e^{tL}g \le e^{t(L-u(\cdot)I)}f \le e^{tL}g \quad (t \ge 0)$$

for any $f, g \in E$ with $-g \le f \le g$.

(c) $-V \le U \le V$: This condition means that both $V - U$ and $V + U$ are increasing and thus implies that V is a majorant of U. Hence $(*)$ holds for all $f, g \in E$ with $-g \le f \le g$.

Remark. Example 1 can also be done for the Laplacian Δ on $\mathbb{R}^n$ in place of L on $\mathbb{R}$, taking as space $E_n := BUC(\mathbb{R}^n, \mathbb{R}^3)$ and as cone

$$K_n := \{f \in E_n : f(s) \in C \ (s \in \mathbb{R}^n)\}.$$

The results should be compared to the corresponding ones for Schrödinger operators on $BUC(\mathbb{R}^n, \mathbb{R})$. If $v \in BUC(\mathbb{R}^n, \mathbb{R})$ is positive, i.e. $v(s) \geq 0$ for all $s \in \mathbb{R}^n$, then we have, for all $t \geq 0$,

$$|e^{t\Delta} f| \leq e^{t(\Delta+v)}|f|, \qquad |e^{t(\Delta-v)} f| \leq e^{t\Delta}|f|.$$

While the first estimate holds on E_n for increasing $V \in BUC(\mathbb{R}^n, \mathbb{R}^{3\times 3})$ without restriction (cf. (a)), we need a restriction (cf. (b)) for the second estimate.

In certain spaces, in particular in Hilbert spaces, the Trotter product formula may be applied to obtain domination results for semigroups. In general, however, the Trotter product formula is not applicable (cf. [2]).

In the next example we give an application to a coupled system.

Example 2. Let E and L be as in Example 1 and let $D := d/ds$ with domain $BUC^1(\mathbb{R}, \mathbb{R}^3)$. Then $A_0 := \begin{pmatrix} L & 0 \\ 0 & D \end{pmatrix}$ with product domain is quasimonotone increasing in $E \times E$ with respect to $K \times K$. For $j = 1, 2$ let $U_j, V_j \in BUC(\mathbb{R}, \mathbb{R}^{3\times 3})$, such that $-V_j(s) \leq U_j(s) \leq V_j(s)$ for all $s \in \mathbb{R}$. Then $B := \begin{pmatrix} L & V_1 \\ V_2 & D \end{pmatrix}$ is a majorant of $A := \begin{pmatrix} L & U_1 \\ U_2 & D \end{pmatrix}$ and $D(A) = D(B) = D(A_0)$. Theorem 3 applies and yields domination of e^{tA} by e^{tB}.

Remark. In the Banach lattice situation Example 2 corresponds to the fact that the semigroup e^{tA} is dominated by the semigroup whose generator is $\begin{pmatrix} L & |U_1| \\ |U_2| & D \end{pmatrix}$ (cf. [7] where this operator is related to the modulus semigroup of e^{tA}).

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