How non-symmetric can a copula be?

Erich Peter Klement, Radko Mesiar

Abstract. A two-place function measuring the degree of non-symmetry for (quasi-)copulas is considered. We construct copulas which are maximally non-symmetric on certain subsets of the unit square. It is shown that there is no copula (and no quasi-copula) which is maximally non-symmetric on the whole unit square.

Keywords: copula, quasi-copula, symmetry, opposite diagonal
Classification: Primary 62H05; Secondary 62E10

1. Introduction

Copulas (first mentioned in [11], for an excellent survey see [9]) and quasi-copulas (introduced in [1] and conveniently characterized in [4]) play a key role in the analysis of bivariate distribution functions with given marginals. The basic result in this context is Sklar's Theorem ([11], [12]) showing that the joint distribution of a random vector and the corresponding marginal distributions are linked by some copula: if (X, Y) is a random vector, $F_X, F_Y : [-\infty, \infty] \to [0, 1]$ are its marginal distribution functions, then $H_{XY} : [-\infty, \infty]^2 \to [0, 1]$ is a joint distribution of (X, Y) if and only if there is a two-dimensional copula C_{XY} such that for all $(x, y) \in [-\infty, \infty]^2$ we have

$$H_{XY}(x, y) = C_{XY}(F_X(x), F_Y(y)).$$

Moreover, if F_X and F_Y are continuous then C_{XY} is unique, otherwise C_{XY} is uniquely determined only on $\mathrm{Ran}(F_X) \times \mathrm{Ran}(F_Y)$.

Recall that a (*two-dimensional*) *copula* is a function $C : [0, 1]^2 \to [0, 1]$ such that $C(0, x) = C(x, 0) = 0$ and $C(1, x) = C(x, 1) = x$ for all $x \in [0, 1]$, and C is 2-increasing, i.e., for all $x_1, x_2, y_1, y_2 \in [0, 1]$ with $x_1 \le x_2$ and $y_1 \le y_2$ we have

$$C(x_1, y_1) + C(x_2, y_2) \ge C(x_1, y_2) + C(x_2, y_1).$$

A *quasi-copula* is a function $Q : [0, 1]^2 \to [0, 1]$ such that $Q(0, x) = Q(x, 0) = 0$ and $Q(1, x) = Q(x, 1) = x$ for all $x \in [0, 1]$, Q is non-decreasing (in each component), and Q is 1-Lipschitz, i.e., for all $x_1, x_2, y_1, y_2 \in [0, 1]$

$$|Q(x_1, y_1) - Q(x_2, y_2)| \le |x_1 - x_2| + |y_1 - y_2|.$$

Obviously, each copula is a quasi-copula, but not vice versa. Each copula C satisfies

$$(1.1) \qquad\qquad W \leq C \leq M,$$

where the Fréchet-Hoeffding lower and upper bounds W and M are given by $W(x,y) = \max(x+y-1,0)$ and $M(x,y) = \min(x,y)$, respectively, and the same holds for quasi-copulas.

In general, a copula is neither symmetric (commutative) nor associative (see [8]), and it is well-known that each associative copula is also symmetric and, consequently, a (continuous) triangular norm [6], [10] (again the converse does not necessarily hold).

There is a close relationship between symmetric copulas and interchangeable random variables X and Y (where the random vectors (X,Y) and (Y,X) are identically distributed). Clearly, two interchangeable random variables X and Y must be identically distributed, i.e., have a common univariate distribution function, and for identically distributed random variables X and Y their interchangeability is equivalent to the symmetry of their copula C_{XY} (see [9, Theorem 2.7.4]).

As a consequence, for exchangeable random variables X and Y with copula C, the symmetry of C implies $C(y,x) = C(x,y)$. In general (i.e., for non-exchangeable random variables X and Y) this is no more true, but any estimate of the value $C(y,x)$ by means of $C(x,y)$ will be helpful when modelling bivariate statistical data, especially in order to exclude irrelevant models.

Therefore, we are interested in "how non-symmetric" a copula can be, and we construct copulas which are "maximally" non-symmetric on certain distinguished subsets of the unit square. Finally we show that no copula (and no quasi-copula) can be "maximally" non-symmetric on the whole unit square.

2. Degree of non-symmetry

Given a copula C, the function $d_C : [0,1]^2 \to [0,1]$ defined by

$$d_C(x,y) = |C(x,y) - C(y,x)|$$

provides a "measure" of its non-symmetry at each point of the unit square $[0,1]^2$, and its Chebyshev norm $\|d_C\|_\infty$ given by

$$\|d_C\|_\infty = \sup\{d_C(x,y) \mid (x,y) \in [0,1]^2\}$$

can be viewed as the *degree of non-symmetry* of C. Obviously, for each copula C the function d_C vanishes on the boundary as well as on the diagonal $\{(x,x) \mid x \in [0,1]\}$ of $[0,1]^2$. Also, a copula C is symmetric if and only if $\|d_C\|_\infty = 0$.

Example 2.1. The copula C given by $C(x, y) = xy - x^3 y(1 - x)(1 - y)$ is non-symmetric, and we obtain $d_C(x, y) = xy(1 - x)(1 - y)|x^2 - y^2|$. A simple computation then yields $\|d_C\|_\infty = d_C(0.3418922, 0.7768102) = 0.0189801$.

In order to find out the maximal degree of non-symmetry of copulas consider the function $d^*: [0, 1]^2 \to [0, 1]$ defined by

$$d^* = \sup\{d_C \mid C \text{ is a copula}\}.$$

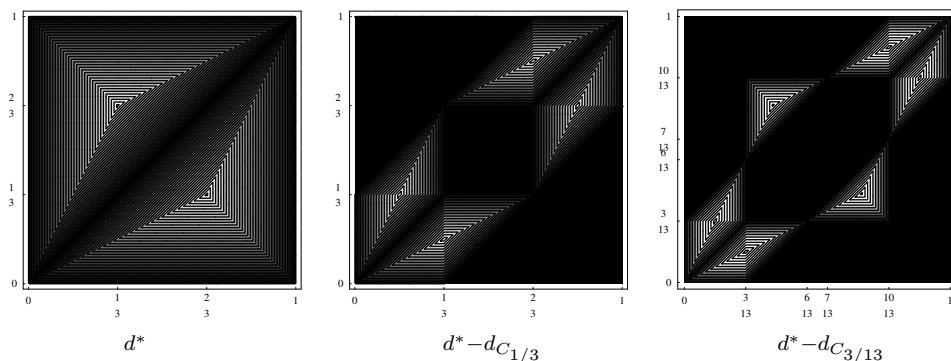$$d^* \qquad\qquad d^* - d_{C_{1/3}} \qquad\qquad d^* - d_{C_{3/13}}$$

FIGURE 1: Maximal non-symmetry

We now compute the function d^* (see Figure 1 left) and show that for each point $(x_0, y_0) \in [0, 1]^2$ we can find a copula C such that d_C and d^* coincide on two straight line segments containing the points (x_0, y_0) and (y_0, x_0).

Proposition 2.2.

   (i) *For all $(x, y) \in [0, 1]^2$ we have $d^*(x, y) = \min(|x - y|, x, y, 1 - x, 1 - y)$.*
   (ii) *For each $\lambda \in [0, 1]$ the function $C_\lambda: [0, 1]^2 \to [0, 1]$ given by*

$$C_\lambda(x, y) = \max(M(x, y - \lambda), W(x, y))$$

   is a copula such that we have $d_{C_\lambda}(x, y) = d^*(x, y)$ for all $(x, y) \in [0, 1]^2$ with $|x - y| = \lambda$.

PROOF: Let C be a copula and assume, without loss of generality, $x \leq y$ and $C(x, y) \leq C(y, x)$. Then the monotonicity of C yields $C(x, y) \leq C(y, x) \leq C(y, y)$ which, together with (1.1) and the fact that C is 1-Lipschitz, implies $d_C(x, y) \leq \min(|x - y|, M(x, y) - W(x, y))$. A simple computation shows that the latter expression coincides with $\min(|x - y|, x, y, 1 - x, 1 - y)$, i.e., for all $(x, y) \in [0, 1]^2$

$$d_C(x, y) \leq \min(|x - y|, x, y, 1 - x, 1 - y).$$

Now fix an arbitrary point $(x_0, y_0) \in [0,1]^2$ and put $\lambda = |x_0 - y_0|$. If we can show that C_λ in (ii) is a copula satisfying

$$(2.1) \qquad d_{C_\lambda}(x_0, y_0) = \min(\lambda, x_0, y_0, 1 - x_0, 1 - y_0)$$

this will complete the proof of (i).

Since C_λ is a shuffle of M it is a copula (see [9]). Note that for each $(x,y) \in [0,1]^2$

$$d_{C_\lambda}(x,y) = \min(\max(\min(x - \lambda, y, 1 - x, 1 - \lambda - y),$$
$$\min(y - \lambda, x, 1 - y, 1 - \lambda - x), 0), |x - y|, \lambda).$$

Then the verification of (2.1) is a matter of simple but tedious checking of all possible cases. Since λ only depends on $|x_0 - y_0|$, the proof of (ii) is complete, too.   □

An immediate consequence of Proposition 2.2 is the following:

Corollary 2.3. *For each copula C and each $(x,y) \in [0,1]^2$ we have:*

$$C(y,x) \in [\max(W(y,x), C(x,y) - |x - y|), \min(M(y,x), C(x,y) + |x - y|)].$$

Observe that the estimate for $C(y,x)$ in Corollary 2.3 is better than the estimate derived from the Fréchet-Hoeffding bounds W and M: if for a copula C we have $C(0.5, 0.6) = 0.3$ then the Fréchet-Hoeffding bounds imply $C(0.6, 0.5) \in [0.1, 0.5]$, whereas Corollary 2.3 tells us $C(0.6, 0.5) \in [0.2, 0.4]$.

Although copulas form a proper subclass of the class of quasi-copulas, the fact that we did not need the 2-increasingness of copulas implies:

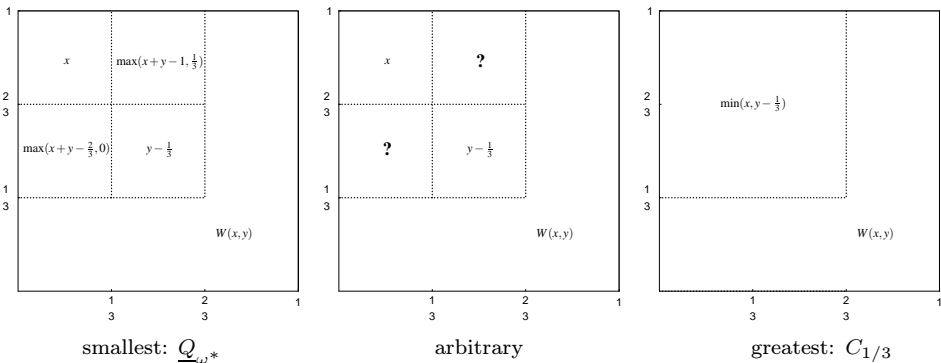**Corollary 2.4.** *We also have $d^* = \sup\{d_Q \mid Q \text{ is a quasi-copula}\}$.*

FIGURE 2: Copulas with opposite diagonal ω^*

Some straightforward calculations show that the maximal value of d^* equals $\frac{1}{3}$ and that there is indeed a copula, namely, $C_{1/3}$ (see Figure 2 right) such that $d_{C_{1/3}}$ attains this maximal value in the points $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ (see Figure 1 center):

Corollary 2.5.

(i) *For each* $\lambda \in \left[0, \frac{1}{3}\right]$ *we have*

$$\|d_{C_\lambda}\|_\infty = d_{C_\lambda}(\lambda, 1 - \lambda) = \lambda.$$

(ii) *In particular, we have*

$$\|d^*\|_\infty = d^*(\tfrac{1}{3}, \tfrac{2}{3}) = \tfrac{1}{3} = d_{C_{1/3}}(\tfrac{1}{3}, \tfrac{2}{3}) = \|d_{C_{1/3}}\|_\infty.$$

Example 2.6. From the proof of Proposition 2.2 it follows that in the class $\mathcal{S}_M$ of shuffles of M (see [9]) for each $(x_0, y_0) \in [0, 1]^2$ we can find a copula $C \in \mathcal{S}_M$ such that $d_C(x_0, y_0) = d^*(x_0, y_0)$. For other well-known classes of copulas this does not hold:

(i) Evidently, for each associative (and, consequently, for each Archimedean) copula C the function d_C vanishes on the whole unit square $[0, 1]^2$.

(ii) In the class $\mathcal{A}$ of maximum attractors [2] (compare also [3], [7]) we obtain

$$\sup\{\|d_C\|_\infty \mid C \in \mathcal{A}\} = \tfrac{1}{5} \cdot (\tfrac{4}{5})^4.$$

This extremal value is attained in the points $(\frac{1}{3}, \frac{2}{3})$ and $(\frac{2}{3}, \frac{1}{3})$ by the function d_{C_A}, where the maximum attractor C_A is given by

$$C_A(x, y) = (xy)^{A\left(\frac{\log x}{\log(xy)}\right)}$$

and the dependence function $A \colon [0, 1] \to [0, 1]$ by

$$A(x) = \begin{cases} 1 - x & \text{if } x \in \left[0, \frac{1}{3}\right], \\ \frac{x+1}{2} & \text{otherwise.} \end{cases}$$

Example 2.7. Although for the copula $C_{1/3}$ we know that $d_{C_{1/3}}$ attains the maximal value of d^* in $(\frac{1}{3}, \frac{2}{3})$, there are other members of the family $(C_\lambda)_{\lambda \in [0,1]}$ such that the area of the subset of $[0, 1]^2$ on which d_{C_λ} and d^* coincide is greater. In general, for $\lambda \in [0, 1]$ the area of the subset of $[0, 1]^2$ on which d_{C_λ} and d^* coincide equals $(1 - \lambda)^2 + (\max(1 - 2\lambda, 0))^2 - 2(\max(1 - 3\lambda, 0))^2$, assuming its maximal value $\frac{9}{13}$ for $\lambda = \frac{3}{13}$ (see Figure 1 right).

3. Non-symmetry and opposite diagonal section

A closer look at the copula $C_{1/3}$ shows that the functions $d_{C_{1/3}}$ and d^* coincide on $\left[\frac{1}{3}, \frac{2}{3}\right]^2 \cup \{(x,y) \in [0,1]^2 \mid |x-y| \geq \frac{1}{3}\}$ (see Figure 1 center). This means, in particular, that we have $d_{C_{1/3}}(x, 1-x) = d^*(x, 1-x)$ for all $x \in [0,1]$, i.e., $C_{1/3}$ is "maximally non-symmetric" on the whole opposite diagonal $\{(x, 1-x) \mid x \in [0,1]\}$ of the unit square $[0,1]^2$ (note that $C_{1/3}$ is the only copula in the family $(C_\lambda)_{\lambda \in [0,1]}$ with this property).

From [5] we know that, for a given copula C, its opposite diagonal section $\omega_C : [0,1] \to [0,1]$ defined by $\omega_C(x) = C(x, 1-x)$ must be a 1-Lipschitz function satisfying $\omega_C(0) = \omega_C(1) = 0$.

Therefore, if for some copula C we require $d_C(x, 1-x) = d^*(x, 1-x)$ for all $x \in [0,1]$, the only possibilities are either $\omega_C = \omega^*$ or $\omega_C = \omega_1$, where the functions $\omega^*, \omega_1 : [0,1] \to [0,1]$ are given by

$$\omega^*(x) = \max(\min(x, \tfrac{2}{3} - x), 0),$$
$$\omega_1(x) = \max(\min(1-x, x - \tfrac{1}{3}), 0).$$

However, if for some (necessarily non-symmetric) copula C we have $\omega_C = \omega^*$ then for the copula C_1 defined by $C_1(x,y) = C(y,x)$ we have $\omega_{C_1} = \omega_1$. This means that we can restrict our considerations to copulas C satisfying $\omega_C = \omega^*$.

From [5, Proposition 7.3] it follows that $C_{1/3}$ is just the greatest copula with opposite diagonal section ω^*. Moreover, because of [5, Proposition 6.5(ii)] the smallest quasi-copula $\underline{Q}_{\omega^*}$ with opposite diagonal section ω^* is given by

$$\underline{Q}_{\omega^*}(x,y) = \begin{cases} x & \text{if } (x,y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{2}{3}, 1\right], \\ \max(x + y - \frac{2}{3}, 0) & \text{if } (x,y) \in \left[0, \frac{1}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right[, \\ \max(x + y - 1, \frac{1}{3}) & \text{if } (x,y) \in \left]\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{2}{3}, 1\right], \\ y - \frac{1}{3} & \text{if } (x,y) \in \left]\frac{1}{3}, \frac{2}{3}\right] \times \left[\frac{1}{3}, \frac{2}{3}\right[, \\ W(x,y) & \text{otherwise.} \end{cases}$$

For our special opposite diagonal section ω^*, the quasi-copula $\underline{Q}_{\omega^*}$ turns out to be a copula since it is again a shuffle of M (see Figure 2 left).

With these preliminary considerations, we are able to show:

Proposition 3.1. *There is no copula C such that $d_C = d^*$.*

PROOF: Suppose that C is a copula such that $d_C = d^*$. Then, in particular, d_C and d^* must coincide on the opposite diagonal, i.e., we must have either $\omega_C = \omega^*$ or $\omega_C = \omega_1$. Assume without loss of generality that $\omega_C = \omega^*$. Since $\underline{Q}_{\omega^*}$ and $C_{1/3}$ are the smallest and greatest copula with opposite diagonal section ω^*, it

follows immediately that each copula C with $\omega_C = \omega^*$ coincides with $\underline{Q}_{\omega^*}$ and $C_{1/3}$ on

$$[0,1]^2 \setminus \left(\left(\left]0, \tfrac{1}{3}\right[\times \left]\tfrac{1}{3}, \tfrac{2}{3}\right[\right) \cup \left(\left]\tfrac{1}{3}, \tfrac{2}{3}\right[\times \left]\tfrac{2}{3}, 1\right[\right) \right)$$

(see Figure 2 center — the question marks indicate the regions where C is not uniquely determined by the lower and upper bounds $\underline{Q}_{\omega^*}$ and $C_{1/3}$). As a consequence, C coincides with the symmetric copula W on the set $\left[0, \tfrac{1}{3}\right]^2 \cup \left[\tfrac{2}{3}, 1\right]^2$, implying that d_C vanishes on this set. Since d^* vanishes only on the boundary and the diagonal of $[0,1]^2$ this shows that for no copula C the equality $d_C = d^*$ can hold. $\qquad\square$

Since again the 2-increasingness of copulas was not used in our argument, we also have shown:

Corollary 3.2. *There is no quasi-copula Q such that $d_Q = d^*$.*

Example 3.3. Clearly, for each symmetric (quasi-)copula C the value $\|d^* - d_C\|_\infty$ attains its maximum $\tfrac{1}{3}$. For the family $(C_\lambda)_{\lambda \in [0,1]}$ of copulas considered in Proposition 2.2(ii) we obtain $\|d^* - d_{C_\lambda}\|_\infty = \min(\max(\tfrac{1}{3} - \lambda, \tfrac{\lambda}{2}), \tfrac{1}{3})$. This value is minimal for $\lambda = \tfrac{2}{9}$, and we get $\|d^* - d_{C_{2/9}}\|_\infty = \tfrac{1}{9}$. Observe, however, that also for $\lambda \in \left[\tfrac{2}{3}, 1\right[$ we get the maximal value $\|d^* - d_{C_\lambda}\|_\infty = \tfrac{1}{3}$, although the corresponding copulas C_λ are non-symmetric.

Note added in proof: Similar results were obtained independently by R.B. Nelsen (Extremes of nonexchangeability, Statist. Papers, to appear).

References

[1] Alsina C., Nelsen R.B., Schweizer B., *On the characterization of a class of binary operations on distribution functions*, Statist. Probab. Lett. **17** (1993), 85–89.

[2] Capéraà P., Fougères A.-L., Genest C., *A nonparametric estimation procedure for bivariate extreme value copulas*, Biometrika **84** (1997), 567–577.

[3] Capéraà P., Fougères A.-L., Genest C., *Bivariate distributions with given extreme value attractor*, J. Multivariate Anal. **72** (2000), 30–49.

[4] Genest C., Quesada Molina J.J., Rodríguez Lallena J.A., Sempi C., *A characterization of quasi-copulas*, J. Multivariate Anal. **69** (1999), 193–205.

[5] Klement E.P., Kolesárová A., *Extension to copulas and quasi-copulas as special 1-Lipschitz aggregation operators*, Kybernetika (Prague) **41** (2005), 329–348.

[6] Klement E.P., Mesiar R., Pap E., *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[7] Klement E.P., Mesiar R., Pap E., *Archimax copulas and invariance under transformations*, C.R. Math. Acad. Sci. Paris **340** (2005), 755-758.

[8] Mikusiński P., Taylor M.D., *A remark on associative copulas*, Comment. Math. Univ. Carolinae **40** (1999), 789–793.

[9] Nelsen R.B., *An introduction to copulas*, Lecture Notes in Statistics 139, Springer, New York, 1999.

[10] Schweizer B., Sklar A., *Probabilistic Metric Spaces*, North-Holland, New York, 1983.

[11] Sklar A., *Fonctions de répartition à n dimensions et leurs marges*, Publ. Inst. Statist. Univ. Paris **8** (1959), 229–231.

[12] Sklar A., *Random variables, joint distribution functions, and copulas*, Kybernetika (Prague) **9** (1973), 449–460.

Department of Knowledge-Based Mathematical Systems, Johannes Kepler University, Linz, Austria

E-mail: ep.klement@jku.at

Department of Mathematics and Descriptive Geometry, Faculty of Civil Engineering, Slovak University of Technology, Bratislava, Slovakia
and
Institute of Theory of Information and Automation, Czech Academy of Sciences, Prague, Czech Republic

E-mail: mesiar@math.sk