Countable chains of distributive lattices as maximal semilattice quotients of positive cones of dimension groups

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Abstract. We construct a countable chain of Boolean semilattices, with all inclusion maps preserving the join and the bounds, whose union cannot be represented as the maximal semilattice quotient of the positive cone of any dimension group. We also construct a similar example with a countable chain of strongly distributive bounded semilattices. This solves a problem of F. Wehrung.

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Introduction

For a ring R, we denote by $\mathrm{FP}(R)$ the class of all finitely generated projective right R modules. We denote by $[A]$ the isomorphism class of a module $A \in \mathrm{FP}(R)$ and by $V(R)$ the monoid of all isomorphism classes of modules from $\mathrm{FP}(R)$, with the operation of addition defined by $[A] + [B] = [A \oplus B]$. If the ring R is von Neumann regular, then the monoid $V(R)$ satisfies the refinement property and the semilattice $\mathrm{Id}_c(R)$ of finitely generated two-sided ideals of R is isomorphic to the maximal semilattice quotient of $V(R)$ ([10, Proposition 4.6]). Modules $A, B \in \mathrm{FP}(R)$ are *stably equivalent*, if there exists $C \in \mathrm{FP}(R)$ such that $A \oplus C \simeq B \oplus C$. We denote by $[A]_s$ the stable equivalence class of $A \in \mathrm{FP}(R)$, and by $V_s(R)$ the quotient $\{[A]_s \mid A \in \mathrm{FP}(R)\}$ of $V(R)$ modulo the stable equivalence. We set $K_0(R) = \{[A]_s - [B]_s \mid A,\ B \in \mathrm{FP}(R)\}$ and we define $([A]_s - [B]_s) + ([C]_s - [D]_s) = [A \oplus C]_s - [B \oplus D]_s$. Then $K_0(R)$ is an abelian group equipped with a preorder determined by the positive cone $V_s(R)$.

If the ring R is unit-regular, then the equivalence and the stable equivalence of modules from $\mathrm{FP}(R)$ coincide, $V(R) = V_s(R)$, $K_0(R)$ is a partially ordered abelian group, and $\mathrm{Id}_c(R)$ is isomorphic to the maximal semilattice quotient of its positive cone $V(R)$. The monoid $V(R)$ satisfies the refinement property and it generates $K_0(R)$. If R is a direct limit of von Neumann regular rings whose primitive factors are artinian, in particular, if R is a locally matricial algebra (over

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a field), then $K_0(R)$ is also unperforated ([3, Theorem 15.12]), that is, $K_0(R)$ is a *dimension group* (see [4], [2]).

Our study of representations of distributive $(\vee, 0)$-semilattices in maximal semilattice quotients of dimension groups is motivated by the study of representations of distributive $(\vee, 0)$-semilattices as semilattices of two-sided ideals of locally matricial algebras. G.M. Bergman [1] proved that every countable distributive $(\vee, 0)$-semilattice is isomorphic to the join-semilattice of finitely generated ideals of some locally matricial algebra. By [5, Theorem 1.1], a dimension group of size at most $\aleph_1$ is isomorphic to $K_0(R)$ of some locally matricial algebra. It follows that a distributive $(\vee, 0)$-semilattices of size $\aleph_1$ is isomorphic to the semilattice of finitely generated ideals of a locally matricial algebra if and only if it is isomorphic to the maximal semilattice quotient of the positive cone of some dimension group (such a group, if it exists, can be always taken of size at most $\aleph_1$).

It follows from a direct construction in [11] that a distributive $(\vee, 0)$-semilattice is isomorphic to the semilattice of two sided ideals of a von Neumann regular ring. However the construction of F. Wehrung [12] gives an example of a distributive $(\vee, 0)$-semilattice of size $\aleph_1$ not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group, and therefore not isomorphic to the semilattice of finitely generated two-sided ideals of any locally matricial algebra. The key idea of his construction consists of the formulation of a semilattice property, denoted by $\mathrm{URP}_{\mathrm{sr}}$ ([12, Definition 4.2]), that is satisfied by the maximal semilattice quotient of the positive cone of any dimension group, and the construction of a distributive $(\vee, 0)$-semilattice S_{ω_1} of size $\aleph_1$ that does not satisfy this property. Further, he proved [12, Section 7] that a direct limit of a countable chain of distributive lattices and join-homomorphisms satisfies $\mathrm{URP}_{\mathrm{sr}}$ and formulated the following problem ([12, Problem 1]):

Problem 1. Let $S = \varinjlim_{n<\omega} D_n$ with all D_n-s being distributive lattices with zero and all transition maps being $(\vee, 0)$-homomorphisms. Does there exists a dimension group G such that $S \simeq \nabla(G^+)$?

We solve this problem by constructing a union of a countable chain of Boolean semilattices, resp. strongly distributive $(\vee, 0, 1)$-semilattices (such that all inclusions are $(\vee, 0, 1)$-homomorphisms), not isomorphic to the maximal semilattice quotient of any Riesz monoid in which every nonzero element is anti-idempotent, and therefore not isomorphic to the maximal semilattice quotient of the positive cone of any dimension group.

Basic concepts

All monoids are written additively. A commutative monoid M is equipped with *the algebraic preordering*: for all $a, b \in M$, $a \le b$ if $b = a + c$ for some $c \in M$. We say that an element e of a commutative monoid is *anti-idempotent* provided that $2ne \not\le ne$ (equivalently, $(n+1)e \not\le ne$), for every $n \in \mathbb{N}$.

The class of all $(\vee, 0)$-semilattices coincides with the class of all commutative monoids in which every element is idempotent. On the other hand, for every commutative monoid M, there exists a least congruence $\asymp$ on M such that $M/{\asymp}$ is a $(\vee, 0)$-semilattice (see [6]). The quotient $M/{\asymp}$, denoted by $\nabla(M)$, is called the *maximal semilattice quotient of* M. The correspondence $M \to \nabla(M)$ naturally extends to a direct limits preserving functor from the category of all commutative monoids to the category of all $(\vee, 0)$-semilattices ([6]). Given an element a of M, we denote by $\boldsymbol{a}$ the corresponding element in $\nabla(M)$.

A commutative monoid M satisfies the *refinement property* provided that for every $a_0, a_1, b_0, b_1 \in M$, the equality $a_0 + a_1 = b_0 + b_1$ implies that there exist c_{ij}, $i, j = 0, 1$, in M satisfying $a_i = c_{i0} + c_{i1}$ for every $i = 0, 1$, and $b_j = c_{0j} + c_{1j}$ for every $j = 0, 1$. We say that a commutative monoid M is *a Riesz monoid* provided that for every $a, b, c \in M$ with $a \leq b + c$, there exist $b' \leq b$ and $c' \leq c$ in M with $a = b' + c'$. Every commutative monoid satisfying the refinement property is a Riesz monoid while the converse is not true in general. However, for join-semilattices, i.e., monoids in which every element is an idempotent, these two properties coincide. A $(\vee, 0)$-semilattice satisfying the refinement property is called *distributive* (see [7, Section II.5]).

A nonzero element x of a join-semilattice S is *join-irreducible* if $x = y \vee z$ implies that $x = y$ or $x = z$ for every y, $z \in S$. We denote by $\mathrm{J}(S)$ the partially ordered set of all join-irreducible elements of a join-semilattice S. A distributive join-semilattice in which every element is a finite join of join-irreducible elements is called *strongly distributive*.

A hereditary subset of a partially ordered set P is a subset H of P satisfying: $p \in H$ and $q \leq p$ implies that $q \in H$ as well. We denote by $H(P)$ the distributive lattice of all hereditary subsets of P. Notice that a $(\vee, 0)$-semilattice is strongly distributive if and only if it is isomorphic to $H_c(P)$, the $(\vee, 0)$-semilattice of compact elements of $H(P)$, for some partially ordered set P. A subset P of a $(\vee, 0)$-semilattice S is *dense*, if $0 \notin P$ and for every nonzero $a \in S$, there is $p \in P$ with $p \leq a$.

We denote by G^+ the *positive cone* of a partially ordered abelian group G, that is, $G^+ = \{a \in G \mid 0 \leq a\}$. A partially ordered abelian group G is *unperforated* if $na \geq 0$ implies $a \geq 0$ for all $a \in G$ and every positive integer n. It is *directed*, if each of its element is the difference of two elements from G^+. It is easy to see that a partially ordered abelian group is directed if and only if it is directed as a partially ordered set. A partially ordered abelian group G is an *interpolation group* if for every a_0, a_1, b_0, and $b_1 \in G$ with $a_i \leq b_j$, $i, j = 0, 1$, there exists $c \in G$ such that $a_i \leq c \leq b_j$, for every $i, j = 0, 1$. A partially ordered abelian group G is an interpolation group if and only if its positive cone is a refinement monoid ([4, Proposition 2.1]). A *dimension group* is an unperforated, directed, interpolation group.

An *ordered vector space* is a partially ordered vector space over the field of rational numbers such that the multiplication by positive scalars is order-preserving. A *dimension vector space* is an ordered vector space which is, as a partially ordered abelian group, a dimension group.

We denote the first infinite ordinal by ω, its successor cardinal by ω_1. Given a set X, we denote by $\mathcal{P}(X)$ the set of all subsets of X and by $[X]^{<\omega}$ the set of all finite subsets of X. Given a Boolean algebra B and an element $x \in B$, we denote by $B\restriction x$ the Boolean algebra $\{y \in B \mid y \leq x\}$. If x, y are elements of a partially ordered set P such that there is no element of P smaller both than x and y, we write $x \perp y$.

The construction

Let B be a Boolean algebra, let F be a filter of B, and let I be the dual ideal of the filter F. Given a distributive $(\vee, 0)$-semilattice S, we denote by $S \times_F B$ the subsemilattice

$$S \times_F B = ((S \setminus \{0\}) \times F) \cup (\{0\} \times I)$$

of $S \times B$ (see [8] and [12]). It could be proved similarly as [8, Lemma 3.3] that if S is a distributive $(\vee, 0)$-semilattice, then $S \times_F B$ is distributive. Here, we prove this fact alternatively, by presenting the $(\vee, 0)$-semilattice $S \times_F B$ as a union of a direct system of its distributive $(\vee, 0)$-subsemilattices.

Lemma 1. *Let B be a Boolean algebra, let F be a filter of B, and let I be the dual ideal of the filter F. Let S be a distributive $(\vee, 0)$-semilattice. Then the $(\vee, 0)$-semilattice $S \times_F B$ is distributive.*

Proof: Let X be a basis of the ideal I. For all $x \in X$, set

$$S_x = \{(0, u) \mid u \in B\restriction x\} \cup \{(a, u \vee (-x)) \mid a \in S \setminus \{0\} \text{ and } u \in B\restriction x\}.$$

It is easy to see that S_x is a $(\vee, 0)$-subsemilattice of $S \times_F B$ isomorphic to $S \times (B\restriction x)$.

We will prove that $S \times_F B$ is a directed union of the distributive $(\vee, 0)$-semilattices S_x. Trivially we have that $\{0\} \times I \subseteq \bigcup_{x \in X} S_x$. Let a be a nonzero element of S and let $u \in F$. Then for some $x \in X$, $-x \leq u$, whence $(u \wedge x) \vee (-x) = u$, and so $(a, u) \in S_x$. Therefore $(S \setminus \{0\}) \times F \subseteq \bigcup_{x \in X} S_x$, and we have proved that $S \times_F B = \bigcup_{x \in X} S_x$. It is obvious from the definition that $x \leq y$ implies $S_x \subseteq S_y$, which implies that the union is directed. This completes the proof. $\square$

Remark 2. Let $\mathfrak{F}$ denote the Fréchet filter on $\mathcal{P}(\omega)$. Then

$$S \times_{\mathfrak{F}} \mathcal{P}(\omega) = \varinjlim_{n \in \omega} \big(S \times \mathcal{P}(n+1)\big),$$

with the transition maps being the one-to-one $(\vee,0)$-embeddings defined by

$$f_{n,m}(a,F)=\begin{cases}(a,F\cup\{n+1,\dots,m\}) & : a>0,\\ (a,F) & : a=0,\end{cases}$$

where $n<m$ are natural numbers, $a\in S$, and $F\subseteq\{0,\dots,n\}$. In particular, if S is a Boolean join-semilattice or a strongly distributive $(\vee,0)$-semilattice, then $S\times_{\mathfrak{F}}\mathcal{P}(\omega)$ is a directed union of a countable chain of Boolean join-semilattices or strongly distributive $(\vee,0)$-semilattices, respectively. Moreover, if S has a greatest element, then the transition maps are $(\vee,0,1)$-homomorphisms.

We modify some notation from [8]. Let a, b be elements of a monoid M. Then

$$\mathrm{Q}(a/b)=\{n/m\mid n,m\in\mathbb{N}\text{ and }\exists\,k\in\mathbb{N}: knb\leq kma\}$$

is a lower interval in $\mathbb{Q}^+$. Indeed, if $n'/m'\leq n/m$ and $n/m\in Q(a/b)$, then $knb\leq kma$ for some natural number k, whence $(kn)n'b\leq kmn'a\leq(kn)m'a$. We define $(a/b)=\sup\mathrm{Q}(a/b)$.

Lemma 3. *Let a, b and c be elements of a monoid M. Then the following hold.*

(i) *$(na/b)=n(a/b)$ for every positive integer n.*
(ii) *$(a+b/c)\geq(a/c)+(b/c)$.*
(iii) *Suppose that M is a Riesz monoid and that $\boldsymbol{b}\wedge\boldsymbol{c}=\boldsymbol{0}$. Then $c\leq a+b$ implies $c\leq a$. In particular, we have that $(a+b/c)=(a/c)$ (compare to [8, Corollary 2.5]).*

Proof: (i) Observe that $n'/nm\in\mathrm{Q}(a/b)$ iff $n'/m\in\mathrm{Q}(na/b)$, for all $n',m\in\mathbb{N}$.

(ii) It is obvious that if $k/n\in\mathrm{Q}(a/c)$ and $l/n\in\mathrm{Q}(b/c)$, then $k/n+l/n\in\mathrm{Q}(a+b,c)$.

(iii) Let $c\leq a+b$. Since M is a Riesz monoid, there are $a'\leq a$, $b'\leq b$ with $c=a'+b'$. From $\boldsymbol{b}\wedge\boldsymbol{c}=\boldsymbol{0}$ it follows that $b'=0$, whence $c\leq a$. For the equality $(a+b/c)=(a/c)$, it suffices to check that $(a+b/c)\leq(a/c)$. But if $kmc\leq kn(a+b)=kna+knb$ for some k, m, $n\in\mathbb{N}$, then we have just proved that $kmc\leq kna$. $\square$

We denote by $(\mathbb{R}^+)^\omega$, resp. $(\mathbb{R}^+)^{(\omega)}$ the monoid of all maps from ω to $\mathbb{R}^+$, resp. the monoid of all maps from ω to $\mathbb{R}^+$ with finite support. We denote by $\boldsymbol{R}$ the quotient $(\mathbb{R}^+)^\omega/(\mathbb{R}^+)^{(\omega)}$, and for all $f\in(\mathbb{R}^+)^\omega$, we denote by $\overline{\boldsymbol{f}}$ the corresponding element of $\nabla(\boldsymbol{R})$.

Let S be a $(\vee,0)$-semilattice, M a monoid, and let $\iota: S\times_{\mathfrak{F}}\mathcal{P}(\omega)\to\nabla(M)$ be an isomorphism. Fix a set $E=\{e_i\mid i\in\omega\}$ of elements of M such that $\boldsymbol{e}_i=\iota(0,\{i\})$, for every $i\in\omega$. For all $a\in M$ and all $i\in\omega$, define $f_a(i)=(a/e_i)$.

Lemma 4. *Let M be a Riesz monoid and e_i, $i \in \omega$, anti-idempotent elements of M. Then $(a/e_i) < \infty$, for all $i \in \omega$ and $a \in M$, that is, f_a is a map from ω to $\mathbb{R}^+$, for every $a \in M$.*

Proof: Fix $i \in \omega$, $a \in M$. Let $(x, A) \in S \times_{\mathfrak{F}} \mathcal{P}(\omega)$ be such that $\boldsymbol{a} = \iota(x, A)$. Pick $b \in M$ satisfying $\boldsymbol{b} = \iota(x, A \smallsetminus \{i\})$. Then $\boldsymbol{a} \leq \boldsymbol{b} \vee \boldsymbol{e}_i$, hence $a \leq nb + ne_i$, for some positive integer n. Suppose that $2n < (a/e_i)$. Then $2nke_i \leq ka$, for some $k \in \mathbb{N}$. It follows that $2nke_i \leq knb + kne_i$. Since $\boldsymbol{b} \wedge \boldsymbol{e}_i = \boldsymbol{0}$, we have, by Lemma 3(iii), that $2nke_i \leq kne_i$, which contradicts the assumption that e_i is anti-idempotent. Therefore $(a/e_i) \leq 2n$. □

Lemma 5. *If $\boldsymbol{a} = \iota(x, A)$ and $\boldsymbol{b} = \iota(x, B)$, then $\overline{\boldsymbol{f}_a} = \overline{\boldsymbol{f}_b}$, for every $a, b \in M$.*

Proof: There exists a finite subset F of ω such that $A \cup F = B \cup F$. Pick $c \in M$ satisfying $\boldsymbol{c} = \iota(0, F)$. Then $\boldsymbol{a} \leq \boldsymbol{b} \vee \boldsymbol{c}$, which means that $a \leq n(b + c)$ for some $n \in \mathbb{N}$. For every $i \in \omega \smallsetminus F$, $\boldsymbol{c} \wedge \boldsymbol{e}_i = \boldsymbol{0}$, and so, by Lemma 3, $f_a(i) \leq f_{n(b+c)}(i) = (nb + nc/e_i) = n(b/e_i) = nf_b(i)$. It follows that $\overline{\boldsymbol{f}_a} \leq \overline{\boldsymbol{f}_b}$. Similarly we prove that $\overline{\boldsymbol{f}_b} \leq \overline{\boldsymbol{f}_a}$. □

Lemma 4 and Lemma 5 entitle us to define a monotone map $\mu_{\iota,E}\colon S \to \nabla(\boldsymbol{R})$ as follows: Given $x \in S$, we pick $A \in \mathcal{P}(\omega)$ such that $(x, A) \in S \times_{\mathfrak{F}} \mathcal{P}(\omega)$, we put $a = \iota(x, A)$, and we define $\mu_{\iota,E}(x) = \overline{\boldsymbol{f}_a}$.

Lemma 6. *Let M be a Riesz monoid, let S be a distributive $(\vee, 0)$-semilattice, and let $\iota : S \times_{\mathfrak{F}} \mathcal{P}(\omega) \to \nabla(M)$ be an isomorphism. Let $E = \{e_i \mid i \in \omega\}$ be a set of anti-idempotent elements of M satisfying $\boldsymbol{e}_i = \iota(0, \{i\})$ for all $i \in \omega$. Finally, let $x \in S \smallsetminus \{0\}$, and let $\{y_\alpha \mid \alpha \in \Omega\}$ be an uncountable set of elements of $S \smallsetminus \{0\}$ such that $x \geq y_\alpha$ for every $\alpha \in \Omega$ and $y_\alpha \wedge y_\beta = 0$ for every $\alpha \neq \beta$ in Ω (we will call such a set a decomposition under x). Then there exists $\alpha \in \Omega$ with $\mu_{\iota,E}(x) > \mu_{\iota,E}(y_\alpha)$.*

Proof: Let a, and b_α, $\alpha \in \Omega$ be elements of M satisfying $\boldsymbol{a} = \iota(x, \omega)$ and $\boldsymbol{b}_\alpha = \iota(y_\alpha, \omega)$. Since $\boldsymbol{a} \geq \boldsymbol{b}_\alpha$, for every $\alpha \in \Omega$, there are positive integers m_α such that $m_\alpha a \geq b_\alpha$, $\alpha \in \Omega$. Since the set Ω is uncountable, there are a positive integer m and an uncountable subset U of Ω such that $m_\alpha = m$, for every $\alpha \in U$. We can replace a with ma, and so we can without loss of generality suppose that $m = 1$.

The map $\mu_{\iota,E}$ is monotone, and so $\mu_{\iota,E}(x) \geq \mu_{\iota,E}(y_\alpha)$, for every $\alpha \in U$. Toward a contradiction, suppose that $\mu_{\iota,E}(x) = \mu_{\iota,E}(y_\alpha)$, for every $\alpha \in U$. Then there are positive integers n_α and finite subsets F_α of ω such that $n_\alpha f_{b_\alpha}(j) \geq f_a(j)$, for every $j \in \omega \smallsetminus F_\alpha$. Since U is uncountable, there are $n \in \mathbb{N}$ and an infinite subset V of U such that $n_\alpha = n$, for all $\alpha \in V$. Pick distinct elements $\alpha_0, \dots, \alpha_n$ from V. By [12, Lemma 2.3], there exist a finite subset F of ω and an element $e_F \in M$ with $\boldsymbol{e}_F = \iota(0, F)$ satisfying

$$\sum_{i=0}^{n} b_{\alpha_i} \leq a + e_F.$$

According to Lemma 3(ii), $\sum_{i=0}^{n}(b_{\alpha_i}/e_j) \leq \left(\sum_{i=0}^{n} b_{\alpha_i}/e_j\right)$, hence

$$\sum_{i=0}^{n} f_{b_{\alpha_i}}(j) \leq f_{a+e_F}(j),$$

for every $j \in \omega$. If $j \in \omega \setminus F$, the equality $(a + e_F/e_j) = (a/e_j)$ holds by Lemma 3(iii), whence

$$\sum_{i=0}^{n} f_{b_{\alpha_i}}(j) \leq f_a(j).$$

Pick a natural number $j \notin (\bigcup_{i=0}^{n} F_{\alpha_i}) \cup F$. Then

$$nf_a(j) \geq n \sum_{i=0}^{n} f_{b_{\alpha_i}}(j) = \sum_{i=0}^{n} nf_{b_{\alpha_i}}(j) \geq (n+1) f_a(j),$$

hence $f_a(j) = 0$, whence $(a/e_j) = 0$, a contradiction as $(0, \{j\}) \leq (x, \omega)$. □

Definition 1. Let $\varkappa$ be an infinite cardinal. We define the following properties of a partially ordered set P.

($\mathrm{A}_\varkappa$) Every decreasing sequence of elements of P of length at most $\varkappa$ has a nonzero lower bound.

(B) Under every $x \in P$, there exists an uncountable set $\{y_\alpha \mid \alpha \in \Omega\}$ of elements of P such that $y_\alpha \perp y_\beta$, for every $\alpha \neq \beta$ in Ω.

Lemma 7. *For every infinite cardinal $\varkappa$, there exists a Boolean algebra $B_\varkappa$ of size $2^\varkappa$ such that $B_\varkappa \setminus \{0\}$ satisfies both* ($\mathrm{A}_\varkappa$) *and* (B).

Proof: For an ordinal number α, denote by ω^α the set of all maps from α to ω, and set

$$P_\varkappa = \bigcup_{\varkappa \leq \alpha < \varkappa^+} \omega^\alpha.$$

Order the set $P_\varkappa$ by reverse inclusion, that is, $f \leq g$, if f is an extension of g, for every f, $g \in P_\varkappa$. Observe that $P_\varkappa$ is a tree of cardinality $2^\varkappa$ satisfying both ($\mathrm{A}_\varkappa$) and (B). Denote by $L_\varkappa$ the sublattice of $H(P_\varkappa)$ generated by $P_\varkappa$. Denote by $B_\varkappa$ the Boolean algebra R-generated by $L_\varkappa$ [7, II.4. Definition 2]. Observe that for every $a \ngeq b$ in $L_\varkappa$, there is $p \in P_\varkappa$ such that $p \leq b$ and $p \wedge a = 0$. By [7, II.4. Lemma 3], there are $a < b$ in $L_\varkappa$ such that $b - a \leq c$, for every nonzero element $c \in L_\varkappa$. Pick $p \in P_\varkappa$ with $p \leq b$ and $p \wedge a = 0$. Then $p \leq c$, and so $P_\varkappa$ is a dense subset of $B_\varkappa$. It follows that $B_\varkappa \setminus \{0\}$ satisfies both ($\mathrm{A}_\varkappa$) and (B). It is straightforward that the cardinality of $B_\varkappa$ is $2^\varkappa$. □

Proposition 8. *Let $\varkappa$ be an infinite cardinal. Let S be a distributive $(\vee, 0)$-semilattice such that the partially ordered set $S \setminus \{0\}$ satisfies both $(\mathrm{A}_\varkappa)$ and (B). Suppose that $S \times_{\mathfrak{F}} \mathcal{P}(\omega)$ is isomorphic, via an isomorphism ι, to $\nabla(M)$ for some Riesz monoid M and that there are anti-idempotent elements e_i, $i \in \omega$ with $\boldsymbol{e}_i = \iota(0, \{i\})$. Then $\nabla(\boldsymbol{R})$ contains a strictly decreasing sequence of length $\varkappa^+$.*

Proof: By transfinite induction up to $\varkappa^+$, we define a sequence $\{x_\alpha \mid \alpha < \varkappa^+\}$ of elements of $S \setminus \{0\}$ inducing a strictly decreasing sequence $\{\mu_{\iota,E}(x_\alpha) \mid \alpha < \varkappa^+\}$ of elements of $\nabla(\boldsymbol{R})$. Let x_0 be any nonzero element of S. Suppose that the sequence $\{x_\alpha \mid \alpha \leq \beta\}$ is defined for some $\beta \leq \varkappa^+$. Since $S \setminus \{0\}$ satisfies property (B), there is a decomposition $\{y_\gamma \mid \gamma < \Omega\}$ under x_β. By Lemma 6, $\mu_{\iota,E}(x_\beta) > \mu_{\iota,E}(y_\gamma)$, for some $\gamma \in \Omega$, and so we can define $x_{\beta+1} = y_\gamma$. Let $\beta < \varkappa^+$ be a limit ordinal and suppose that we have already defined the sequence $\{x_\alpha \mid \alpha < \beta\}$ such that the sequence $\{\mu_{\iota,E}(x_\alpha) \mid \alpha < \beta\}$ in $\nabla(\boldsymbol{R})$ is strictly decreasing. By $(\mathrm{A}_\varkappa)$, there is a lower bound x_β of $\{x_\alpha \mid \alpha < \beta\}$ in $S \setminus \{0\}$. Since the map $\mu_{\iota,E}$ is monotone, we obtain that $\mu_{\iota,E}(x_\alpha) > \mu_{\iota,E}(x_{\alpha+1}) \geq \mu_{\iota,E}(x_\beta)$, for every $\alpha < \beta$. □

Denote by $\mathfrak{e}$ the supremum of the lengths of all strictly decreasing sequences in $\nabla(\boldsymbol{R})$.

Theorem 9. *There is a directed union D of a countable chain of Boolean join-semilattices (with $(\vee, 0, 1)$-preserving inclusion maps) which is not isomorphic to $\nabla(M)$ for any Riesz monoid M in which every nonzero element is anti-idempotent. The cardinality of D is $2^{\mathfrak{e}}$.*

Proof: The $(\vee, 0, 1)$-semilattice $D = B_{\mathfrak{e}} \times_{\mathfrak{F}} \mathcal{P}(\omega)$ is a direct limit of a countable chain of Boolean lattices and one-to-one $(\vee, 0, 1)$-preserving transition maps (Remark 2). Since, by Lemma 7, $B_{\mathfrak{e}} \setminus \{0\}$ satisfies both $(\mathrm{A}_{\mathfrak{e}})$ and (B), and M is a Riesz monoid in which every nonzero element is anti-idempotent, the assertion follows from Proposition 8. The cardinality of $B_{\mathfrak{e}} \times_{\mathfrak{F}} \mathcal{P}(\omega)$ is clearly $2^{\mathfrak{e}}$. □

Remark 10. This result contrasts with the answer to the analogue of Problem 1 for semilattices of compact congruences of lattices: Every direct limit of a countable sequence of distributive lattices with zero and $(\vee, 0)$-homomorphisms is isomorphic to the semilattice $\mathrm{Con_c}\, L$ of compact congruences of some *relatively complemented* lattice L with zero ([13, Corollary 21.3]).

Theorem 11. *There is a union H of a countable chain of strongly distributive $(\vee, 0, 1)$-semilattices (with $(\vee, 0, 1)$-preserving inclusion maps) which is not isomorphic to the maximal semilattice quotient of any Riesz monoid in which every nonzero element is anti-idempotent.*

Proof: As in the proof of Theorem 9, $H = H_c(P_{\mathfrak{e}}) \times_{\mathfrak{F}} \mathcal{P}(\omega)$ is a direct limit of a countable chain of strongly distributive $(\vee, 0, 1)$-semilattices and one-to-one

$(\vee, 0, 1)$-preserving transition maps (Remark 2). Now argue as in the proof of Theorem 9. □

A commutative monoid M is *conical* if $a \leq 0$ implies that $a = 0$ for all $a \in M$. Since $2ne + x = ne$ implies $2(ne + x) = ne + x$, the conical monoids without nonzero idempotent elements are exactly conical monoids with all elements anti-idempotent. The positive cone of any dimension group forms a conical monoid without nonzero idempotent elements which satisfy the refinement property.

Corollary 12. *There is a union of a countable chain of Boolean algebras, resp. strongly distributive* $(\vee, 0, 1)$*-semilattices* (*with* $(\vee, 0, 1)$*-preserving inclusion maps*) *which is not isomorphic to* $\nabla(M)$ *for any conical Riesz monoid* M *without nonzero-idempotent elements. In particular, it is not isomorphic to* $\nabla(G^+)$ *for any dimension group* G.

Recall [12] that a commutative monoid M is *strongly separative* if $a + b = 2b$ implies $a = b$ for every $a, b \in M$. An element e of a commutative monoid M has *finite stable rank* if there is $k \in \mathbb{N}$ such that $ke + a \leq e + b$ implies $a \leq b$, for all $a, b \in M$. It is straightforward that every element of a strongly separative monoid has finite stable rank. In a conical monoid, every nonzero idempotent element has infinite stable rank. Therefore, we can replace the assumption that the monoid M has no nonzero idempotent elements by any of the following requirements: every element of M has finite stable rank, M is strongly separative (compare to [12, Corollary 5.3]). We could derive from Corollary 12 similar consequences to the ones obtained from [12, Corollary 5.3] in [12, Section 6]. In particular, neither the $(\vee, 0, 1)$-semilattice D nor the $(\vee, 0, 1)$-semilattice H are isomorphic to the join-semilattice of finitely generated ideals of any strongly separative von Neumann regular ring, resp. the join-semilattice $\mathrm{Con_c}\, L$ of all compact congruences of any modular lattice L of locally finite length.

Remark 13. Observe that every element $\boldsymbol{f} \in \nabla(\boldsymbol{R})$ is represented by a map with rational values. It follows that the cardinality of $\nabla(\boldsymbol{R})$ is $2^{\aleph_0}$, and so we have the estimate $\aleph_1 \leq \mathfrak{e} \leq 2^{\aleph_0}$. Of course, if $2^{\aleph_0} = \aleph_1$ and $2^{\aleph_1} = \aleph_2$, then $2^{\mathfrak{e}} = \aleph_2$. On the other hand, $\aleph_2 < 2^{\aleph_1}$ implies that $\aleph_2 < 2^{\mathfrak{e}}$.

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