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Strict minimizers of order m in nonsmooth optimization problems

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Abstract. In the paper, some sufficient optimality conditions for strict minima of order m in constrained nonlinear mathematical programming problems involving (locally Lipschitz) (F, ρ)-convex functions of order m are presented. Furthermore, the concept of strict local minimizer of order m is also used to state various duality results in the sense of Mond-Weir and in the sense of Wolfe for such nondifferentiable optimization problems.

Keywords: nonsmooth programming, strict local minimizer of order m, Clarke's generalized gradient, (F, ρ)-convex function of order m with respect to θ

Classification: 90C29, 90C26, 90C46, 49J52

1. Introduction

The notion of a strict local minimizer of order m plays an important role in the convergence analysis of iterative numerical methods (see, for example, [6]) and in stability results (see, for example, [14], [20]). Some results and optimality conditions concerning characterizations of such minimizers for nonlinear constrained mathematical programming problems have been derived by Auslender [1], Studniarski [19] and [21], Ward [22]. These results, in general, suggest that these minimizers are often exactly those satisfying an "m-th derivative test".

In this paper, we present a different approach for identifying such minimizers. In the past few years, many methods have been proposed for solving the constrained mathematical programming problem, especially in the case where all functions involved are convex. Recently, some generalizations of convexity have been proposed in optimization theory. One such generalization is a class of differentiable functions introduced to optimization theory by Hanson [9] and later called invex by Craven [5]. In the recent years, the concept of invexity, previously introduced for differentiable functions, was generalized to the case of nonsmooth functions. Kaul et al. [13] proved sufficient optimality conditions and duality results in nonsmooth programming problems involving nonsmooth invex functions. Jeyakumar [12] defined the class of locally Lipschitz ρ-convex functions and proved saddle point and duality theorems for nonsmooth problems involving this type of functions. Hanson and Mond [10] introduced the concept of F-convexity (without naming it so) for differentiable functions. They proved optimality and duality results for mathematical programming problems involving such functions. The name "F-convex function" was given by Egudo and Mond [7]. Later, Preda [17] introduced generalized (F, ρ)-convexity, an extension of F-convexity and generalized ρ-convexity defined in [12].

In this paper, we use a class of nondifferentiable (locally Lipschitz) nonconvex functions, that is, a class of (F, ρ)-convex functions of order m with respect to the same function θ to characterize a strict minimizer of order m in standard mathematical programming problems. This class of nondifferentiable generalized convex functions generalizes the class of differentiable (F, ρ)-convex functions earlier introduced by Preda [17]. To describe a class of nonsmooth (F, ρ)-convex functions, we use Clarke's generalized gradient [4]. The purpose of this paper is to use the introduced notion of (F, ρ)-convex functions of order m with respect to the same function θ to establish sufficient optimality conditions for strict minimizer of order m in nonsmooth optimization problems involving this type of functions. Furthermore, duality results in the sense of Mond-Weir and in the sense of Wolfe for such class of nonsmooth programming problems are also obtained. The concept of strict minimizer of order m is also used to state various duality results.

2. Preliminaries

In this section, we provide some definitions and results that we shall use in the sequel.

Definition 1. Let X be an open subset of $\mathbb{R}^n$. The function $f : X \to \mathbb{R}$ is said to be *locally Lipschitz* (of rank K) at $x \in X$ if there exist a positive constant K and a neighborhood N of x such that, for any $y, z \in N$,

$$|f(y) - f(z)| \leq K \|y - z\| .$$

If the inequality above is satisfied for any $x \in X$ then f is said to be locally Lipschitz (of rank K) on X.

Definition 2 ([4])**.** If $f : X \to \mathbb{R}$ is locally Lipschitz at $x \in X$, the *generalized derivative* (in the sense of Clarke) of f at $x \in X$ in the direction $v \in \mathbb{R}^n$, denoted $f^{\,0}(x; v)$, is given by

$$f^{\,0}(x; v) = \limsup_{\substack{y \to x \\ \lambda \downarrow 0}} \frac{f(y + \lambda v) - f(y)}{\lambda} .$$

Definition 3 ([4])**.** The *generalized gradient* of f at $x \in X$, denoted $\partial f(x)$, is defined as follows:

$$\partial f(x) = \left\{ \xi \in \mathbb{R}^n : f^{\,0}(x; v) \geq \langle \xi; v \rangle \quad \text{for all} \quad v \in \mathbb{R}^n \right\} .$$

The following proposition collects some properties which can be found in [4].

Proposition 4.

(a) *$\partial f(x)$ is a nonempty compact subset of $\mathbb{R}^n$,*
(b) *$f^0(x; v) = \max\{\langle \xi; v\rangle : \xi \in \partial f(x)\}$,*
(c) *if x is a local minimum of f, then $0 \in \partial f(x)$,*
(d) *if $g : \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function, then $\partial(f+g)(x) \subset \partial f(x) + \partial g(x)$.*

Definition 5. A function $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ is *sublinear* (with respect to the third component) if for any $x, u \in X$, the following inequalities

$$F(x, u, q_1 + q_2) \leq F(x, u, q_1) + F(x, u, q_2), \tag{1}$$

$$F(x, u, \alpha q) \leq \alpha F(x, u, q) \tag{2}$$

hold for any $\alpha \geq 0$ and $q, q_1, q_2 \in \mathbb{R}^n$.

Remark 6. Note that from (1) and (2) it follows that $F(x, u, 0) = 0$.

On the basis of the definition of invexity for differentiable invex functions ([9]) and the notions of strong and weak invexity introduced by Jeyakumar [11], Jeyakumar [12] defined the concept of ρ-invexity for locally Lipschitz functions.

For the benefit of the reader, we recall the definition of ρ-invex functions introduced by Jeyakumar in [12].

Definition 7. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on a nonempty set $X \subset \mathbb{R}^n$. If there exist functions $\eta : X \times X \to \mathbb{R}^n$ and $\theta : X \times X \to \mathbb{R}^n$, $\theta(x, u) \neq 0$, whenever $x \neq u$, and a real number ρ such that the inequality

$$f(x) - f(u) \geq \langle \xi, \eta(x, u)\rangle + \rho \|\theta(x, u)\| \tag{3}$$

holds for any $\xi \in \partial f(u)$ and for all $x \in X$, then f is said to be (*locally Lipschitz*) *ρ-invex* with respect to η and θ at u on X.
If the relation (4) is satisfied at any point $u \in X$, then f is said to be *ρ-invex* with respect to η on X.
If $\rho > 0$, then f is said to be *strongly invex*. If $\rho = 0$, then f is said to be *invex*. If $\rho < 0$, then f is said to be *weakly invex*.
It is clear that strongly invex $\Longrightarrow$ weakly invex.

Now, we generalize the definition of ρ-invex functions with respect to η and θ introduced by Jeyakumar in [12]. We introduce the (locally Lipschitz) (F, ρ)-convexity of order m with respect to θ.

Definition 8. Let $f : X \to \mathbb{R}$ be a locally Lipschitz function on a nonempty set $X \subset \mathbb{R}^n$. If there exist some sublinear functional $F : X \times X \times \mathbb{R}^n \to \mathbb{R}$ with respect to the third component, a function $\theta : X \times X \to \mathbb{R}^n$, $\theta(x, u) \neq 0$ whenever $x \neq u$, a real number ρ and a positive integer number m such that the inequality

$$f(x) - f(u) \geq F(x, u, \xi) + \rho \|\theta(x, u)\|^m \tag{4}$$

holds for any $\xi \in \partial f(u)$ and for all $x \in X$, then f is said to be (*locally Lipschitz*) (F,ρ)-*convex* of order m with respect to θ at u on X.
If the relation (4) is satisfied at any point $u \in X$, then f is said to be (F,ρ)-*convex* of order m with respect to θ on X.
If $\rho > 0$, then f is said to be F-*strongly convex* of order m. If $\rho = 0$, then f is said to be F-*convex* of order m. If $\rho < 0$, then f is said to be *weakly* F-*convex* of order m.
It is clear that F-strongly convex of order $m \Longrightarrow F$-weakly convex of order m.

Remark 9. Note that if $F(x,u,\xi) = \langle \xi, \eta(x,u) \rangle$ and $m = 1$ then f is (locally Lipschitz) ρ-invex with respect to η and θ in the sense of the Jeyakumar's definition (see Definition 7).

Remark 10. In order to define an analogous class of (strictly) locally Lipschitz F-concave functions of order m with respect to θ, the direction of the inequalities in (4) should be reversed.

In the following example, we give an example of a (locally Lipschitz) (F,ρ)-convex function of order 2 with respect to some function θ.

Example 11. We consider the following Lipschitz function $f : (-1,1) \to \mathbb{R}$ defined as

$$f(x) = \begin{cases} x & \text{if } \; -1 < x \leq 0 \\ \frac{1}{2}x & \text{if } \; 0 \leq x < 1. \end{cases}$$

We show that f is (locally Lipschitz) (F,ρ)-convex of order 2 with respect to some function $\theta(x,u) = x - u$ at $u = 0$ on the set $X = (-1,1)$. We set

$$F(x,u,\xi) = \begin{cases} 2\left(x - x^2\right)\xi & \text{if } \; -1 < x \leq 0 \\ -x\xi & \text{if } \; 0 \leq x < 1 \end{cases}$$

for any $\xi \in \partial f(u) = \left[\frac{1}{2}, 1\right]$, and, moreover,

$$\theta(x,u) = x - u \;\text{ and }\; \rho = 1.$$

Thus, by Definition 8, f is (locally Lipschitz) (F,ρ)-convex of order 2 with respect to the function θ at $u = 0$ on X. Moreover, it is not difficult to prove that f is not ρ-invex in the sense of Jeyakumar [12] (see also Definition 7), where $\rho = 1$, with respect to θ and with respect to the function $\eta : X \times X \to \mathbb{R}$ satisfying $F(x,u,\xi) = \langle \xi, \eta(x,u) \rangle$ for any $\xi \in \partial f(u)$.

For the class of (F,ρ)-convex of order m functions with respect to θ, we generalize the results given by Ben-Israel and Mond in [3, Theorem 1]. To do this, we prove the following theorem:

Theorem 12. *Let f be a locally Lipschitz function defined on a nonempty open set $X \subset \mathbb{R}^n$. Then f is (F,ρ)-convex of order m with respect to θ on X if and only if $0 \in \partial f(u)$ implies $f(x) - f(u) \geq \rho \left\| \theta(x,u) \right\|^m$ for all $x \in X$ and some real number ρ.*

Proof: "$\Longrightarrow$" Assume that f is (F,ρ)-convex with respect to θ on X. Hence, by Definition 8, it follows that there exists a real number ρ, a function $\theta : X \times X \to \mathbb{R}^n$, and some sublinear functional $F(x,u,\cdot)$ such that, the inequality

$$f(x) - f(u) \geq F\left(x,u,\xi\right) + \rho \left\| \theta(x,u) \right\|^m$$

holds for all $x \in X$. From the assumption we have $0 \in \partial f(u)$. Then, by Remark 6, we get that the following inequality

$$f(x) \geq f(u) + \rho \left\| \theta(x,u) \right\|^m \tag{5}$$

holds for all $x \in X$ and some real number ρ.

"$\Longleftarrow$" We assume that $0 \in \partial f(u)$ implies (5). Then it is sufficient to take

$$F\left(x,u,\xi\right) = 0 \tag{6}$$

for all $x \in X$ and all $\xi \in \partial f(u)$. Now we suppose that $0 \notin \partial f(u)$. We set

$$F(x,u,\xi) = \begin{cases} \left(f(x) - f(u) - \rho\|\theta(x,u)\|^m\right) \frac{\|\xi\|}{\max_{\zeta \in \partial f(u)} \|\zeta\|} \\ \qquad\qquad\qquad \text{if } \; f(x) - f(u) - \rho\|\theta(x,u)\|^m \geq 0, \\ \left(f(x) - f(u) - \rho\|\theta(x,u)\|^m\right) \frac{\langle \xi_{\min}, \xi \rangle}{\|\xi_{\min}\|^2} \\ \qquad\qquad\qquad \text{if } \; f(x) - f(u) - \rho\|\theta(x,u)\|^m < 0, \end{cases} \tag{7}$$

where $\xi_{\min} = \min_{\zeta \in \partial f(u)} \|\zeta\|$. Taking into account (6) and (7), we conclude by Definition 8 that f is (F,ρ)-convex of order m with respect to θ on X. $\square$

3. Optimality

We consider the following mathematical programming problem:

$$\begin{gathered} f(x) \to \min, \\ \text{subject to } \; g_j(x) \leq 0, j \in J = \{1, \ldots, p\}, \end{gathered} \tag{P}$$

where $f : \mathbb{R}^n \to \mathbb{R}$, $g_j : \mathbb{R}^n \to \mathbb{R}$, $j \in J$, are locally Lipschitz functions defined on $\mathbb{R}^n$. The set of all feasible solutions D in (P) is the set

$$D := \left\{ x \in \mathbb{R}^n : g_j(x) \leq 0, \; \forall \, j \in J \right\}.$$

Further, the Lagrange function or the Lagrangian for problem (P) is defined as follows

$$L\left(x, \lambda, \mu\right) := \lambda f(x) + \mu g(x). \tag{8}$$

Definition 13. The point $\overline{x}$ is an *isolated local minimizer for* (P) if there is an open neighborhood U of $\overline{x}$ such that

$$f(x) \geq f(\overline{x}), \ \forall x \in D \cap U.$$

If this inequality is strictly satisfied for $x \neq \overline{x}$, then $\overline{x}$ is said to be a *strict local minimizer*.

Definition 14. We say that $\overline{x}$ is a *strict local minimizer of order m for problem* (P) if there exist an open neighborhood U of $\overline{x}$ and a positive real number β such that

$$f(x) \geq f(\overline{x}) + \beta \left\| x - \overline{x} \right\|^{m}, \ \forall x \in D \cap U. \tag{9}$$

If there is a neighborhood U such that $\overline{x}$ is the only local minimizer in U, then $\overline{x}$ is called an *isolated local minimizer*.

Remark 15. Observe that if $\overline{x}$ is a strict local minimizer of order m, it is also a strict local minimizer of order p for all $p > m$. If (9) holds for all $x \in D$ then $\overline{x}$ is a strict global minimizer of order m, or shortly, a strict minimizer of order m.

It is known that the following Karush-Kuhn-Tucker necessary optimality conditions are fulfilled for $\overline{x}$ to be a (strict) local minimizer in problem (P) (see, for example, [8] and also [21] for the case when $\overline{x}$ is a strict local minimizer of order one):

Theorem 16. *Let $\overline{x}$ be a (local) minimizer of order m in the considered optimization problem* (P) *and some suitable constraint qualification* [2] *be satisfied at $\overline{x}$. Then there exists numbers $\overline{\mu}_j$, $j = 1, \ldots, p$, such that*

$$0 \in \partial\Big(f(\overline{x}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x})\Big), \tag{10}$$

$$\overline{\mu}_j g_j(\overline{x}) = 0, \ j = 1, \ldots, p, \tag{11}$$

$$\overline{\mu}_j \geq 0, \ j = 1, \ldots, p. \tag{12}$$

Now, we give the sufficient optimality conditions for $\overline{x}$ to be a strict minimizer of order m in the considered optimization problem (P).

Theorem 17. *Let $\overline{x}$ be a feasible solution in problem* (P) *and let the Karush-Kuhn-Tucker optimality conditions be fulfilled at $\overline{x}$. Moreover, we assume that f is strongly (F, ρ)-convex of order m with respect to θ at $\overline{x}$ on D and g_j, $j = 1, \ldots, p$, are strongly $\left(F, \rho_j\right)$-convex with respect to θ at $\overline{x}$ on D. If $\theta\,(x, \overline{x}) = x - \overline{x}$ then $\overline{x}$ is a strict minimizer of order m in* (P).

Proof: Let $\overline{x}$ be a feasible solution in problem (P) and the Karush-Kuhn-Tucker necessary optimality conditions (10)–(12) be satisfied at $\overline{x}$. By assumption, f is strongly (F,ρ)-convex of order m with respect to θ at $\overline{x}$ on D and g_j, $j=1,\dots,p$, are strongly $\left(F,\rho_j\right)$-convex of order m with respect to θ at $\overline{x}$ on D. Then by Definition 8 we have that, for each $x\in D$, the following inequalities

$$\begin{aligned} f(x)-f\left(\overline{x}\right) &\geq F\left(x,\overline{x},\xi\right)+\rho\left\|\theta\left(x,\overline{x}\right)\right\|^{m}, \\ g_j(x)-g_j(\overline{x}) &\geq F\left(x,\overline{x},\zeta_j\right)+\rho_j\left\|\theta\left(x,\overline{x}\right)\right\|^{m},\ j=1,\dots,p, \end{aligned} \tag{13}$$

are satisfied for any $\xi\in\partial f(\overline{x})$, $\zeta_j\in\partial g_j(\overline{x})$, respectively. Since F is a sublinear function with respect to the third component, using the Karush-Kuhn-Tucker condition optimality (12), we get

$$\overline{\mu}_j g_j(x)-\overline{\mu}_j g_j(\overline{x})\geq F\left(x,\overline{x},\overline{\mu}_j\zeta_j\right)+\overline{\mu}_j\rho_j\left\|\theta\left(x,\overline{x}\right)\right\|^{m},\ j=1,\dots,p.$$

From $x\in D$ and by the Karush-Kuhn-Tucker optimality conditions (11) and (12),

$$0\geq F\left(x,\overline{x},\overline{\mu}_j\zeta_j\right)+\overline{\mu}_j\rho_j\left\|\theta\left(x,\overline{x}\right)\right\|^{m},\ j=1,\dots,p, \tag{14}$$

and, moreover, since F is sublinear with respect to the third component, (14) gives

$$0\geq F\Big(x,\overline{x},\sum_{j=1}^{p}\overline{\mu}_j\zeta_j\Big)+\Big(\sum_{j=1}^{p}\overline{\mu}_j\rho_j\Big)\left\|\theta\left(x,\overline{x}\right)\right\|^{m}. \tag{15}$$

Adding both sides of (13) and (14) we get

$$f(x)-f\left(\overline{x}\right)\geq F\left(x,\overline{x},\xi\right)+F\Big(x,\overline{x},\sum_{j=1}^{p}\overline{\mu}_j\zeta_j\Big)+\Big(\rho+\sum_{j=1}^{p}\overline{\mu}_j\rho_j\Big)\left\|\theta\left(x,\overline{x}\right)\right\|^{m},$$

thus

$$f(x)-f\left(\overline{x}\right)\geq F\Big(x,\overline{x},\xi+\sum_{j=1}^{p}\overline{\mu}_j\zeta_j\Big)+\Big(\rho+\sum_{j=1}^{p}\overline{\mu}_j\rho_j\Big)\left\|\theta\left(x,\overline{x}\right)\right\|^{m}. \tag{16}$$

By assumption, the Karush-Kuhn-Tucker optimality condition (10) is fulfilled at $\overline{x}$. Then there exist $\xi\in\partial f\left(\overline{x}\right)$ and $\zeta_j\in\partial g_j\left(\overline{x}\right)$, $j=1,\dots,p$, such that $0=\xi+\sum_{j=1}^{p}\overline{\mu}_j\zeta_j$. Thus, (16) implies

$$f(x)\geq f\left(\overline{x}\right)+\Big(\rho+\sum_{j=1}^{p}\overline{\mu}_j\rho_j\Big)\left\|\theta\left(x,\overline{x}\right)\right\|^{m}.$$

By assumption, $\theta\left(x,\overline{x}\right)=x-\overline{x}$. Hence, the inequality

$$f(x)\geq f\left(\overline{x}\right)+\overline{\rho}\left\|x-\overline{x}\right\|^{m} \tag{17}$$

holds for all $x\in D$, where

$$\overline{\rho}=\rho+\sum_{j=1}^{p}\overline{\mu}_{j}\rho_{j}>0.$$

Then, using Definition 14 together with (17), it follows that $\overline{x}$ is a strict global minimizer of order m. □

Remark 18. As follows from the proof of Theorem 17, we do not need to assume that all functions involving in problem (P) are strongly (F,ρ)-convex with respect to θ at $\overline{x}$ on D. It is sufficient to assume one of the following hypotheses:

(i) f is strongly (F,ρ)-convex of order m with respect to η and θ at $\overline{x}$ on D and g_j, $j\in J(\overline{x})$, are (F,ρ_j)-convex of order m with respect to η and θ at $\overline{x}$ on D with $\rho_j\geq 0$,

(ii) f is (F,ρ)-convex $(\rho\geq 0)$ of order m with respect to η and θ at $\overline{x}$ on D and g_j, $j\in J(\overline{x})$, are (F,ρ_j)-convex of order m with respect to θ at $\overline{x}$ on D with $\rho_j\geq 0$, but at least one of the constraint function g_s, $s\in J(\overline{x})$, is strongly (F,ρ_s)-convex of order m with respect to θ at $\overline{x}$ on D.

Example 19. To illustrate the results proved in Theorem 17, we consider the following nonsmooth optimization problem

$$f(x)=\begin{cases}\left(x_1+x_2\right)^2+3\left(x_1^2+x_2^2\right)+2x_2 & \text{if } \ x_2>0\\ 4x_1^2+x_2 & \text{if } \ x_2\leq 0\end{cases}\rightarrow\min$$

$$g(x)=-x_2\leq 0.$$

Note that by Definition 14 $\overline{x}=(0,0)$ is a strict local minimizer of order 2 in the considered nonsmooth optimization problem. To prove this results we use the sufficient optimality conditions from Theorem 17.
We have $D=\left\{(x_1,x_2)\in\mathbb{R}^2:-x_2\leq 0\right\}$, and by Definition 3,
$\partial f(\overline{x})=\left\{(\xi_1,\xi_2)\in\mathbb{R}^2:\xi_1=0,1\leq\xi_2\leq 2\right\}$, $\partial g(\overline{x})=\nabla g(\overline{x})=[0,-1]$. Moreover, it is not difficult to show by Definition 8 that both the objective function f and the constraint function g are (locally Lipschitz) strongly (F,ρ)-convex of order 2 with respect to θ at $\overline{x}$ on D, where

$$F(x,u,\xi)=\xi_2\left(x_2+x_1^2+x_2^2\right),$$
$$\theta\left(x,\overline{x}\right)=x-\overline{x},$$

and ρ is some real number such that $0 < \rho \leq 1$. Since $\overline{x} = (0,0)$ satisfies the Karush-Kuhn-Tucker necessary optimality conditions (10)–(12) and, moreover, all functions involved are (locally Lipschitz) strongly (F,ρ)-convex of order 2 with respect to θ at $\overline{x}$ on D, then by Theorem 17, $\overline{x} = (0,0)$ is a strict local minimizer of order 2 in the considered nonsmooth optimization problem.

Now, we give a sufficient condition for $\overline{x}$ to be a strict global minimizer of order m under Lagrangian type assumption.

Theorem 20. *Let $\overline{x}$ be a feasible solution in problem* (P) *and let the Karush-Kuhn-Tucker optimality conditions be fulfilled at $\overline{x}$. Moreover, we assume that the Lagrangian is strongly (F,ρ)-convex of order m with respect to θ at $\overline{x}$ on D. Then $\overline{x}$ is a strict global minimizer of order m in problem* (P)*.*

PROOF: Let $\overline{x}$ be a feasible solution in problem (P) and let the Karush-Kuhn-Tucker necessary optimality conditions be fulfilled at $\overline{x}$. Since the Lagrangian is strongly (F,ρ)-convex of order m with respect to θ at $\overline{x}$ on D, we have by Definition 8 that, for any feasible solution x of (P),

$$f(x) + \sum_{j=1}^{p} \overline{\mu}_j g_j\,(x) \geq f\,(\overline{x}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x}) + F\,(x,\overline{x},\zeta) + \rho\,\|\theta\,(x,\overline{x})\|^m$$

for any $\zeta \in \partial \left(f(\overline{x}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x}) \right)$, where ρ is a positive real number. Using the Karush-Kuhn-Tucker necessary optimality conditions together with the feasibility of x in (P) we obtain

$$f(x) - f\,(\overline{x}) \geq \rho\,\|\theta\,(x,\overline{x})\|^m .$$

By assumption, $\theta\,(x,\overline{x}) = x - \overline{x}$. Thus, the inequality

$$f(x) \geq f\,(\overline{x}) + \overline{\rho}\,\|x - \overline{x}\|^m$$

holds for all $x \in D$. This means, by Definition 14, that $\overline{x}$ is a strict global minimizer of order m in problem (P). □

4. Mond-Weir duality

We consider the following Mond-Weir type dual problem (MWD) [16] for the optimization problem (P):

$$\text{(MWD)} \qquad \begin{aligned} & f(y) \to \max \\ & \text{such that } \ 0 \in \lambda \partial f\,(y) + \sum_{j=1}^{p} \mu_j \partial g_j(y), \\ & \sum_{j=1}^{p} \mu_j g_j(y) \geq 0 \\ & \lambda > 0, \mu \geq 0. \end{aligned}$$

Let W denote the set of all feasible solutions to the dual problem (MWD). Further, we denote by Y the set $Y = \{y \in X : (y, \lambda, \mu) \in W\}$.

By the help of the concept of strict local minimizer of order m, we establish weak, strong, converse, and strict converse duality theorems in the sense of Mond-Weir between problems (MWD) and (P) under assumption that the functions constituting these problems satisfy some suitable (F, ρ)-convex condition. Before we prove various duality theorems, we give a useful lemma whose simple proof is omitted.

Lemma 21. *Let* (y, λ, μ) *be a feasible solution for* (MWD). *Assume that* g_j, $j \in J(y)$, *are* $\left(F, \rho_j\right)$*-convex of order* m *with respect to the same function* ϑ *at* y *on* $D \cup Y$, *where* $\sum_{j \in J(y)} \rho_j \mu_j \geq 0$. *Then, the following inequality*

$$\sum_{j=1}^{p} F\left(x, y, \mu_j \zeta_j\right) \leq 0 \tag{18}$$

holds for each $\zeta_j \in \partial g_j(y)$ *and for all* $x \in D$.

Theorem 22 (Weak duality)**.** *Let* x *and* (y, λ, μ) *be feasible solutions for* (P) *and* (MWD), *respectively. Moreover, we assume that* f *is* (F, ρ)*-convex of order* m *with respect to* θ *at* y *on* $D \cup Y$ *with* $\rho \geq 0$, *and* g_j, $j \in J$, *are* $\left(F, \rho_j\right)$*-convex of order* m *at* y *on* $D \cup Y$ *with respect to the same function* ϑ, *not necessarily equal to* θ, *where* $\sum_{j \in J(y)} \rho_j \mu_j \geq 0$. *Then* $f(x) \geq f(y)$.

PROOF: Let x and (y, λ, μ) be feasible solutions for (P) and (MWD), respectively. By assumption, f is (F, ρ)-convex of order m with respect to θ at y on $D \cup Y$, and g_j, $j \in J$, are $\left(F, \rho_j\right)$-convex of order m with respect to the same function θ at y on $D \cup Y$. Then, by Definition 8, we have

$$f(x) - f(y) \geq F(x, y, \xi) + \rho \|\theta(x, y)\|^m.$$

Since g_j, $j \in J$, are $\left(F, \rho_j\right)$-convex of order m with respect to ϑ at y on $D \cup Y$, then by Lemma 21, the inequality

$$\sum_{j=1}^{p} F\left(x, y, \mu_j \zeta_j\right) \leq 0$$

holds for each $\zeta_j \in \partial g_j(y)$. Thus, by $\lambda > 0$

$$\lambda(f(x) - f(y)) \geq F(x, y, \lambda \xi) + \sum_{j=1}^{p} F\left(x, y, \mu_j \zeta_j\right) + \lambda \rho \|\theta(x, y)\|^m.$$

From the first constraint of (MWD) it follows that

$$f(x) - f(y) \geq \rho \|\theta(x, y)\|^m,$$

and, so by $\rho \geq 0$, we get the conclusion of the theorem. □

Now, we give an example of a nondifferentiable optimization problem to illustrate the proved weak duality theorem.

Example 23. We consider the following nondifferentiable optimization problem

$$\text{(P)} \qquad \begin{aligned} f(x) &= e^{-|x|} - e^{\ln|x|} \to \min \\ g(x) &= x^2 - |x| \leq 0. \end{aligned}$$

We construct for the considered optimization problem (P) its dual problem in the sense of Mond-Weir

$$\text{(MWD)} \qquad \begin{gathered} e^{-|y|} - e^{\ln|y|} \to \max \\ \text{such that} \;\; 0 \in \lambda\partial f(y) + \mu\partial g(y) \\ \mu(y^2 - |y|) \geq 0 \\ \lambda > 0, \;\; \mu \geq 0. \end{gathered}$$

Note that $D = \{x \in \mathbb{R} : -1 \leq x < 0 \wedge 0 < x \leq 1\}$ and $Y = \{y \in \mathbb{R} : y \leq -1 \wedge y \geq 1\}$. Further, it is not difficult to show that f and g are (F, ρ)-convex of order 2 with respect to some function θ at any $y \in Y$ on $D \cup Y$ and g is (F, ρ)-convex of order 2 with respect to some function ϑ at any $y \in Y$ on $D \cup Y$. Indeed, if we set

$$F(x, y, \xi) = \begin{cases} (-y - |x|)\xi & \text{if } \; y \leq -1 \\ (-y + |x|)\xi & \text{if } \; y \geq 1 \end{cases} \quad \forall \xi \in \partial f(y) \;\text{ or }\; \forall \xi \in \partial g(y), \text{ respectively,}$$

$$\theta(u, y) = ||u| - y| \quad \text{for all} \;\; u \in D \cup Y,$$

$$\vartheta(u, y) = \sqrt{e^{-|u|} + e^{-|y|}(|u| - |y| - 1)} \;\; \text{for all} \;\; u \in D \cup Y,$$

$$\rho \text{ is any an arbitrary real number such that } \rho \in [0, 1],$$

then, by Definition 8, f and g are (F, ρ)-convex of order 2 with respect to θ and ϑ, respectively, at any $y \in Y$ on $D \cup Y$. Since all hypotheses of Theorem 22 are fulfilled, the weak duality in the sense of Mond-Weir holds between (P) and (MWD).

It turns out that Mond-Weir weak duality is also valid under the (F, ρ)-convex Lagrangian type assumption.

Theorem 24 (Weak duality)**.** *Let x and (y, λ, μ) be feasible solutions for* (P) *and* (MWD), *respectively. If the Lagrangian is (F, ρ)-convex of order m with respect to θ at y on $D \cup Y$, where $\rho \geq 0$, then weak duality also holds between problems* (P) *and* (MWD).

Theorem 25 (Strong duality)**.** *Let $\overline{x}$ be a strict minimizer of order m in* (P) *and some suitable constraint qualification* ([2]) *be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in \mathbb{R}$, $\overline{\lambda} > 0$, $\overline{\mu} \in \mathbb{R}^p_+$, $\overline{\mu} \geq 0$, such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible in* (MWD). *If, also weak duality holds between problems* (P) *and* (MWD), *then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a strict* *maximizer of order m in* (MWD) *and the optimal values in both problems are the same.*

Proof: Let $\overline{x}$ be a strict local minimizer of order m in (P) and some suitable constraint qualification ([2]) be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in \mathbb{R}$, $\overline{\lambda} > 0$, $\overline{\mu} \in \mathbb{R}^p$, $\overline{\mu} \geq 0$, such that the Karush-Kuhn-Tucker optimality conditions (10)–(12) are fulfilled at $\overline{x}$. Thus, by the Karush-Kuhn-Tucker optimality conditions (10)–(12), we conclude that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is feasible in dual problem (MWD). Suppose that $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is not a strict maximizer of order m in (MWD). Then there exists $\widetilde{y} \in Y$ such that

$$f(\widetilde{y}) > f(\overline{x}) + \rho \left\|\widetilde{y} - \overline{x}\right\|^m .$$

By assumption, $\rho \geq 0$. Hence,

$$f(\widetilde{y}) > f(\overline{x}).$$

But the inequality above contradicts weak duality. Thus, $\left(\overline{x}, \overline{\lambda}, \overline{\mu}\right)$ is a strict maximizer of order m in problem (MWD), and hence the optimal values in both problems are the same. □

Theorem 26 (Converse duality)**.** *Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ be a strict maximizer of order m in* (MWD) *such that $\overline{y} \in D$. Moreover, we assume that f is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, and g_j, $j \in J$, are $\left(F, \rho_j\right)$-convex of order m at $\overline{y}$ on $D \cup Y$ with respect to the same function ϑ (not necessarily equal to θ), where $\lambda\rho + \sum_{j \in J(\overline{y})} \rho_j \mu_j \geq 0$. Then $\overline{y}$ is a strict minimizer of order m in* (P)*.*

Proof: We proceed by contradiction. Suppose that $\overline{y}$ is not a strict minimizer of order m in (P). Then by Definition 14 there exists $\widetilde{x} \in D$ such that the inequality

$$f\left(\widetilde{x}\right) < f\left(\overline{y}\right) + \beta \left\|\widetilde{x} - \overline{y}\right\|^m \tag{19}$$

holds for all $\beta > 0$. Since f is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, by Definition 8 the following inequality

$$f(x) - f\left(\overline{y}\right) \geq F\left(x, \overline{y}, \xi\right) + \rho \left\|\theta\left(x, \overline{y}\right)\right\|^m \tag{20}$$

holds for any $\xi \in \partial f(\overline{y})$ and for all $x \in D$, hence also for $x = \widetilde{x}$. Since $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ is feasible in (MWD) and $\theta\left(x, \overline{y}\right) = x - \overline{y}$, (20) gives

$$\overline{\lambda}\left(f\left(\widetilde{x}\right) - f\left(\overline{y}\right)\right) \geq \overline{\lambda} F\left(\widetilde{x}, \overline{y}, \xi\right) + \overline{\lambda}\rho \left\|\widetilde{x} - \overline{y}\right\|^m . \tag{21}$$

By assumption, g_j, $j \in J(\overline{y})$, are $\left(F, \rho_j\right)$-convex of order m with respect to the same function ϑ at $\overline{y}$ on $D \cup Y$. Then, by Lemma 21, it follows that the inequality

$$F\Big(x, \overline{y}, \sum_{j=1}^{p} \overline{\mu}_j \zeta_j\Big) \leq 0 \tag{22}$$

holds for each $\zeta_j \in \partial g_j(y)$ and for all $x \in D$. Hence, also for $x = \widetilde{x}$. Adding both sides of (21) and (22) and using the sublinearity of F with respect to the third component, we obtain that the inequality

$$\overline{\lambda}\left(f\left(\widetilde{x}\right) - f\left(\overline{y}\right)\right) \geq F\Big(\widetilde{x}, \overline{y}, \overline{\lambda}\xi + \sum_{j=1}^{p} \overline{\mu}_j \zeta_j\Big) + \rho \left\|\widetilde{x} - \overline{y}\right\|^m$$

holds for all $\xi \in \partial f(\overline{y})$ and $\zeta_j \in \partial g_j(\overline{y})$. Thus, using the first constraint of problem (MWD) we get

$$\overline{\lambda}\left(f\left(\widetilde{x}\right) - f\left(\overline{y}\right)\right) \geq \overline{\lambda}\rho \left\|\widetilde{x} - \overline{y}\right\|^m ,$$

and, so the inequality

$$f\left(\widetilde{x}\right) \geq f\left(\overline{y}\right) + \rho \left\|\widetilde{x} - \overline{y}\right\|^m$$

which contradicts (19). □

The Mond-Weir converse duality theorem can also be proved when the Lagrange function is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, where $\rho \geq 0$.

Theorem 27 (Converse duality). *Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ be a strict maximizer of order m in* (MWD) *such that $\overline{y} \in D$. Moreover, we assume that the Lagrange function is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, where $\rho \geq 0$. Then $\overline{y}$ is a strict maximizer of order m in* (P).

A restricted version of converse duality for (P) and (MWD) is the following:

Theorem 28 (Restricted converse duality). *Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ be feasible for problem* (MWD). *Further, we assume that there exists $\overline{x} \in D$ such that $f(\overline{x}) = f(\overline{y})$. If f is (F, ρ)-convex of order m with respect to θ at $\overline{x}$ on $D \cup Y$, and g_j, $j \in J$, are $\left(F, \rho_j\right)$-convex of order m at $\overline{x}$ on $D \cup Y$ with respect to the same function ϑ (not necessarily equal to θ), where $\lambda\rho + \sum_{j \in J(\overline{x})} \rho_j \mu_j \geq 0$, then $\overline{x}$ is a strict minimizer of order m in problem* (P).

PROOF: By assumption, f is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, and g_j, $j \in J$, are $\left(F, \rho_j\right)$-convex of order m with respect to the same function ϑ at $\overline{y}$ on $D \cup Y$. Then, by Definition 8, we have that the inequality

$$f(x) - f\left(\overline{y}\right) \geq F\left(x, \overline{y}, \xi\right) + \rho \left\|\theta\left(x, \overline{y}\right)\right\|^m \tag{23}$$

holds for all $\xi \in \partial f(\overline{y})$. Since g_j, $j \in J$, are $\left(F, \rho_j\right)$-convex with respect to the same function ϑ at y on $D \cup Y$, we have by Lemma 21 that the inequality

$$\sum_{j=1}^{p} F\left(x, \overline{y}, \mu_j \zeta_j\right) \leq 0$$

holds for each $\zeta_j \in \partial g_j(y)$ and for all $x \in D$. Thus,

$$f(x) - f(\overline{y}) \geq F(x, y, \xi) + \sum_{j=1}^{p} F\left(x, \overline{y}, \mu_j \zeta_j\right) + \rho \left\|\theta(x, \overline{y})\right\|^m .$$

From the first constraint of (MWD) it follows that the inequality

$$f(x) - f(\overline{y}) \geq \rho \left\|x - \overline{y}\right\|^m \tag{24}$$

holds for all $x \in D$.

Now we show that $\overline{x}$ is a strict minimizer of order m in (MWD). We proceed by contradiction. Suppose that $\overline{x}$ is not a strict minimizer of order m in (MWD). Then by Definition 14 there exists $\widetilde{x} \in D$ such that the inequality

$$f(\widetilde{x}) < f(\overline{x}) + \beta \left\|\widetilde{x} - \overline{x}\right\|^m$$

holds for any $\beta \geq 0$. By assumption, $f(\overline{x}) = f(\overline{y})$. Therefore,

$$f(\widetilde{x}) - f(\overline{y}) < \beta \left\|\widetilde{x} - \overline{x}\right\|^m . \tag{25}$$

Since (24) is satisfied for all $x \in D$, it holds also for $x = \widetilde{x}$. Hence,

$$-f(\widetilde{x}) + f(\overline{y}) \leq \rho \left\|\widetilde{x} - \overline{y}\right\|^m . \tag{26}$$

Adding both sides of (25) and (26) we get a contradiction to the definition of β. $\square$

Now, we establish a Mangasarian-type strict converse duality theorem for (P) and (MWD).

Theorem 29 (Strict converse duality)**.** *Let $\overline{x}$ and $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be feasible for* (P) *and* (MWD), *respectively, such that*

$$\overline{\lambda} f(\overline{x}) < \overline{\lambda} f(\overline{y}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{y}). \tag{27}$$

Moreover, we assume that the Lagrangian is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, where $\rho \geq 0$ and $\theta(x, y) = x - y$. Then $\overline{x} = \overline{y}$, and also $\overline{y}$ is a strict minimizer of order m in (P).

Proof: We proceed by contradiction. Suppose that $\overline{x} \neq \overline{y}$. Since $\overline{x}$ is feasible in (P) and $\overline{\mu} \geq 0$, we have $\sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x}) \leq 0$. Hence, by (27),

$$\overline{\lambda} f(\overline{x}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x}) < \overline{\lambda} f(\overline{y}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{y}). \tag{28}$$

By assumption, the Lagrangian is (F,ρ)-convex of order m with respect to θ at $\overline{y}$ on $D\cup Y$. Therefore, by Definition 8 there exists $\theta:(D\cup Y)\times(D\cup Y)\to\mathbb{R}^n$, where $\theta(x,y)\neq 0$ whenever $x\neq y$, a nonnegative real number ρ and a positive integer m such that the inequality

$$\overline{\lambda}f(\overline{x})+\sum_{j=1}^{p}\overline{\mu}_j g_j(\overline{x})-\overline{\lambda}f(\overline{y})-\sum_{j=1}^{p}\overline{\mu}_j g_j(\overline{y})\geq F\Big(\overline{x},\overline{y},\overline{\lambda}\xi+\sum_{j=1}^{p}\overline{\mu}_j\zeta_j\Big)+\rho\left\|\theta(\overline{x},\overline{y})\right\|^m$$

holds for each $\xi\in\partial f(\overline{y})$ and $\zeta_j\in\partial g_j(\overline{y})$. Then by (28), we get that the inequality

$$F\Big(\overline{x},\overline{y},\overline{\lambda}\xi+\sum_{j=1}^{p}\overline{\mu}_j\zeta_j\Big)+\rho\left\|\theta(\overline{x},\overline{y})\right\|^m<0$$

holds for each $\xi\in\partial f(\overline{y})$ and $\zeta_j\in\partial g_j(\overline{y})$. From the first constraint of (MWD), we have that there exist $\xi\in\partial f(\overline{y})$ and $\zeta_j\in\partial g_j(\overline{y})$ such that $\overline{\lambda}\xi+\sum_{j=1}^{p}\overline{\mu}_j\zeta_j=0$. Thus, by Remark 6, it follows that $\rho\left\|\theta(\overline{x},\overline{y})\right\|^m<0$. But this inequality is a contradiction to the assumption $\rho\geq 0$. This completes the proof of the theorem. □

5. Wolfe duality

Now, we consider the following dual problem (WD) in the sense of Wolfe [23] for the primal optimization problem (P):

$$\text{(WD)}\qquad \begin{aligned} &f(y)+\mu g(y)\to\max\\ &\text{such that}\ \ 0\in\lambda\partial f(y)+\sum_{j=1}^{p}\mu_j\partial g_j(y),\ \ j=1,\dots,p\\ &\lambda>0,\ \ \mu\geq 0.\end{aligned}$$

Let $\widetilde{W}$ denote the set of all feasible solutions to the dual problem (WD). Further, we denote by $\widetilde{Y}$ the set $\widetilde{Y}=\Big\{y\in X:(y,\lambda,\mu)\in\widetilde{W}\Big\}$.

With the help of the concept of strict local minimizer of order m and using the (F,ρ)-convexity Lagrangian type assumption, we establish weak, strong, converse, and strict converse duality theorems in the sense of Wolfe [23] between problems (WD) and (P).

Theorem 30 (Weak duality)**.** *Let x and (y,λ,μ) be feasible solutions for* (P) *and* (WD)*, respectively. Moreover, assume that the Lagrangian is (F,ρ)-convex of order m with respect to θ at y on $D\cup Y$ with $\rho\geq 0$. Then $f(x)\geq f(y)+\mu g(y)$.*

Proof: Let x and (y,λ,μ) be feasible solutions for (P) and (WD), respectively. By assumption, the Lagrange function is (F,ρ)-convex of order m with respect to

θ at y on $D \cup \widetilde{Y}$ with $\rho \geq 0$. Then, using Definition 8 together with the definition of the Lagrange function (8), we have

$$\lambda f(x) + \mu g(x) - (\lambda f(y) + \mu g(y)) \geq F(x, y, \xi) + \rho \|\theta(x, y)\|^m$$

for each $\xi \in \lambda \partial f(y) + \sum_{j=1}^{p} \mu_j \partial g_j(y)$. Since $x \in D$, the inequality above gives

$$\lambda f(x) - (\lambda f(y) + \mu g(y)) \geq F(x, y, \xi) + \rho \|\theta(x, y)\|^m .$$

Thus, by the constraint of the Wolfe dual problem we get

$$f(x) - (\lambda f(y) + \mu g(y)) \geq \rho \|\theta(x, y)\|^m ,$$

and, so by $\rho \geq 0$, we get the conclusion of the theorem. □

Theorem 31 (Strong duality). *Let $\overline{x}$ be a strict minimizer of order m in* (P) *and let some suitable constraint qualification* [2] *be satisfied at $\overline{x}$. Then there exist $\overline{\lambda} \in \mathbb{R}$, $\overline{\lambda} > 0$, $\overline{\mu} \in \mathbb{R}^p$, $\overline{\mu} \geq 0$ such that $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is feasible in* (WD)*. If, also weak duality holds between problems* (P) *and* (WD)*, then $(\overline{x}, \overline{\lambda}, \overline{\mu})$ is a strict maximizer of order m in* (WD) *and the optimal values in both problems are the same.*

Proof: Along the lines of the proof of Theorem 25. □

Theorem 32 (Converse duality). *Let $(\overline{y}, \overline{\lambda}, \overline{\mu})$ be a strict maximizer of order m in* (WD) *such that $g(\overline{y}) = 0$. Moreover, assume that the Lagrange function is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup \widetilde{Y}$ with $\rho \geq 0$. Then $\overline{y}$ is a strict minimizer of order m in* (P).

Proof: Since $(\overline{y}, \overline{\lambda}, \overline{\mu})$ is a strict minimizer of order m in (WD), it is feasible in (WD). By assumption, $g(\overline{y}) = 0$. Then, by $\overline{\mu} \geq 0$,

$$\sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{y}) = 0. \tag{29}$$

We proceed by contradiction. Suppose that $\overline{y}$ is not a strict minimizer of order m in (P). Then by Definition 14 there exists $\widetilde{x} \in D$ such that the inequality

$$f(\widetilde{x}) < f(\overline{y}) + \beta \|\widetilde{x} - \overline{y}\|^m \tag{30}$$

holds for all $\beta > 0$. By assumption, the Lagrange function is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup \widetilde{Y}$. Hence, using Definition 8 together with the definition of the Lagrange function (8), we obtain

$$f(\widetilde{x}) + \sum_{j=1}^{p} \mu_j g_j(\widetilde{x}) - \Big(f(\overline{y}) + \sum_{j=1}^{p} \mu_j g_j(\overline{y}) \Big) \geq F(\widetilde{x}, \overline{y}, \xi) + \rho \|\theta(\widetilde{x}, \overline{y})\|^m \tag{31}$$

for any $\xi \in \left(\overline{\lambda}\partial f(\overline{y}) + \overline{\mu}\partial g(\overline{y})\right)$. Since $\widetilde{x} \in D$ and $\overline{\mu} \geq 0$, we have

$$\sum_{j=1}^{p} \overline{\mu}_j g_j(\widetilde{x}) \leq 0. \tag{32}$$

By assumption, $\rho \geq 0$. Thus, by (29), (31) and (32), the inequality

$$F\left(\widetilde{x}, \overline{y}, \xi\right) < 0$$

holds for any $\xi \in \left(\overline{\lambda}\partial f(\overline{y}) + \overline{\mu}\partial g(\overline{y})\right)$. This is a contradiction to the feasibility of $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ in (WD). □

A restricted version of converse duality for (P) and (WD) is the following:

Theorem 33 (Restricted converse duality)**.** *Let* $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ *be feasible for problem* (WD)*. Further, assume that there exists* $\overline{x} \in D$ *such that* $f(\overline{x}) = f(\overline{y}) + \overline{\mu}g(\overline{y})$*. If the Lagrange function is* (F, ρ)*-convex of order* m *with respect to* θ *at* $\overline{y}$ *on* $D \cup Y$ *with* $\rho \geq 0$ *then* $\overline{x}$ *is a strict minimizer of order* m *in problem* (P)*.*

Proof: Let $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ be a feasible solution for problem (WD). We also assume that there exists $\overline{x} \in D$ such that $f(\overline{x}) = f(\overline{y}) + \overline{\mu}g(\overline{y})$. We proceed by contradiction. Suppose that $\overline{x}$ is not a strict minimizer of order m in (WD). Then by Definition 14 there exists $\widetilde{x} \in D$ such that the inequality

$$f\left(\widetilde{x}\right) < f\left(\overline{x}\right) + \beta \left\|\widetilde{x} - \overline{x}\right\|^m$$

holds for any $\beta > 0$. By assumption, $f\left(\overline{x}\right) = f(\overline{y}) + \overline{\mu}g(\overline{y})$. Therefore,

$$f\left(\widetilde{x}\right) < f(\overline{y}) + \overline{\mu}g(\overline{y}) + \beta \left\|\widetilde{x} - \overline{x}\right\|^m$$

and by $\widetilde{x} \in D$,

$$f\left(\widetilde{x}\right) + \overline{\mu}g(\widetilde{x}) < f(\overline{y}) + \overline{\mu}g(\overline{y}) + \beta \left\|\widetilde{x} - \overline{x}\right\|^m . \tag{33}$$

By assumption, the Lagrange function is (F, ρ)-convex of order m with respect to θ at y on $D \cup Y$ with $\rho \geq 0$. Then, using Definition 8 together with the Lagrange definition (8) we have that the inequality

$$f(x) + \sum_{j=1}^{p} \mu_j g_j(x) - \left(f\left(\overline{y}\right) + \sum_{j=1}^{p} \mu_j g_j(\overline{y})\right) \geq F\left(x, \overline{y}, \xi\right) + \rho \left\|\theta\left(x, \overline{y}\right)\right\|^m \tag{34}$$

holds for any $\xi \in (\lambda\partial f(\overline{y}) + \mu\partial g(\overline{y}))$ and for all $x \in D$. Thus, also for $x = \overline{x}$. By assumption, $f\left(\overline{x}\right) = f(\overline{y}) + \overline{\mu}g(\overline{y})$. Therefore, using $\widetilde{x} \in D$, we get

$$F\left(\overline{x}, \overline{y}, \xi\right) + \rho \left\|\theta\left(\overline{x}, \overline{y}\right)\right\|^m \leq 0.$$

From the constraint of (WD) it follows that

$$\rho \left\|\theta\left(\overline{x}, \overline{y}\right)\right\|^{m} \leq 0. \tag{35}$$

Using (33) and (34) we get

$$F\left(\widetilde{x}, \overline{y}, \xi\right)+\beta\|\widetilde{x}-\overline{x}\|^{m}<0 \tag{36}$$

and so by the constraint of (WD),

$$\beta\|\widetilde{x}-\overline{x}\|^{m}<0.$$

By (35) and (36) we get the inequality

$$\rho\left\|\theta\left(\overline{x}, \overline{y}\right)\right\|^{m}+\beta\|\widetilde{x}-\overline{x}\|^{m}<0,$$

which contradicts $\rho \geq 0$ and $\beta>0$. □

Now, we establish a Mangasarian-type strict converse duality theorem for (P) and (WD).

Theorem 34 (Strict converse duality)**.** *Let $\overline{x}$ and $\left(\overline{y}, \overline{\lambda}, \overline{\mu}\right)$ be feasible for* (P) *and* (WD), *respectively, such that*

$$\overline{\lambda} f(\overline{x})<\overline{\lambda} f(\overline{y})+\sum_{j=1}^{p} \overline{\mu}_{j} g_{j}(\overline{y}). \tag{37}$$

Moreover, we assume that the Lagrangian is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$, where $\rho \geq 0$. Then $\overline{x}=\overline{y}$, and $\overline{y}$ is a strict minimizer of order m in (P).

PROOF: We proceed by contradiction. Suppose that $\overline{x} \neq \overline{y}$. Since $\overline{x}$ is feasible in (P) and $\overline{\mu} \geq 0$, $\sum_{j=1}^{p} \overline{\mu}_{j} g_{j}(\overline{x}) \leq 0$. Hence, by (37),

$$\overline{\lambda} f(\overline{x})+\sum_{j=1}^{p} \overline{\mu}_{j} g_{j}(\overline{x})<\overline{\lambda} f(\overline{y})+\sum_{j=1}^{p} \overline{\mu}_{j} g_{j}(\overline{y}). \tag{38}$$

By assumption, the Lagrangian is (F, ρ)-convex of order m with respect to θ at $\overline{y}$ on $D \cup Y$. Then, by Definition 8, the inequality

$$\overline{\lambda} f(\overline{x})+\sum_{j=1}^{p} \overline{\mu}_{j} g_{j}(\overline{x})-\overline{\lambda} f(\overline{y})-\sum_{j=1}^{p} \overline{\mu}_{j} g_{j}(\overline{y}) \geq F\Big(\overline{x}, \overline{y}, \overline{\lambda} \xi+\sum_{j=1}^{p} \overline{\mu}_{j} \zeta_{j}\Big)+\rho\|\theta(\overline{x}, \overline{y})\|^{m}$$

holds for any $\xi \in \left(\lambda\partial f(\overline{y}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x})\right)$. Then by (38), we get that the inequality

$$F\Big(\overline{x}, \overline{y}, \overline{\lambda}\xi + \sum_{j=1}^{p} \overline{\mu}_j \zeta_j\Big) + \rho \left\|\theta(\overline{x}, \overline{y})\right\|^m < 0$$

holds for any $\xi \in \left(\lambda\partial f(\overline{y}) + \sum_{j=1}^{p} \overline{\mu}_j g_j(\overline{x})\right)$. From the first constraint of (WD), we have that $\overline{\lambda}\xi + \sum_{j=1}^{p} \overline{\mu}_j \zeta_j = 0$. Thus, by Remark 6,

$$\rho \left\|\theta(\overline{x}, \overline{y})\right\|^m < 0.$$

But this inequality is a contradiction to the assumption $\rho \geq 0$. This completes the proof of the theorem. □

Remark 35. Note that to prove duality results in the sense of Wolfe, in opposition to the duality results in the sense of Mond-Weir, we do not need that the functional F is sublinear with respect to the third component. Also, we do not assume that each function constituting the considered mathematical programming problem is (F, ρ)-convex of order m with respect to θ at some suitable feasible point on $D \cup Y$. To prove duality results in the sense of Wolfe it is sufficient to assume only that the Lagrange function is (F, ρ)-convex of order m with respect to θ at some suitable feasible point on $D \cup Y$, but the functional F is not required to be sublinear with the third component in this case.

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