On direct sums of $\mathrm{B}^{(1)}$-groups — II

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Abstract. $\mathrm{B}^{(1)}$-groups are a class of torsionfree Abelian groups of finite rank, part of the main class of Butler groups. In the paper C. Metelli, *On direct sums of* $\mathrm{B}^{(1)}$*-groups*, Comment. Math. Univ. Carolinae **34** (1993), 587–591, the problem of direct sums of $\mathrm{B}^{(1)}$-groups was discussed, and a necessary and sufficient condition was given for the direct sum of two $\mathrm{B}^{(1)}$-groups to be a $\mathrm{B}^{(1)}$-group. While sufficiency holds, necessity was wrongly claimed; we solve here the problem, and in the process study a curious hierarchy among indecomposable direct summands of $\mathrm{B}^{(1)}$-groups.

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Introduction

In the paper [M] the author claimed that a certain condition (called here *hooking*) was necessary and sufficient for the direct sum of two $\mathrm{B}^{(1)}$-groups — which is, in general, a $\mathrm{B}^{(2)}$-group — to be itself a $\mathrm{B}^{(1)}$-group. In fact, necessity was not true (the example is in Section 4); fortunately, no later result depended on that claim. In this paper we correct that old mistake.

We start by studying the hooking condition, that ties two complementary direct summands of G: there are situations in which the condition is also necessary for G to be $\mathrm{B}^{(1)}$, in particular when the two summands are strongly indecomposable. The Krull-Schmidt property then yields an interesting hierarchy among indecomposable direct summands of a $\mathrm{B}^{(1)}$-group (Sections 1, 2): we show that a $\mathrm{B}^{(1)}$-group always has a hooking decomposition, and decomposing again via hooking summands we can reach every indecomposable summand of G in a finite number of steps, giving it a hooking index. In Section 3 we solve the open problem, by giving a necessary and sufficient condition (called *generalized hooking*) for the general case. The proof consists in identifying a particular $\mathrm{B}^{(1)}$-group inside the direct sums of our two $\mathrm{B}^{(1)}$-groups, and showing that generalized hooking forces it to coincide with the whole group. Necessity is derived from a result in a previous paper [CDVM]. Section 4 is dedicated to some significant examples.

In our ongoing investigation of $\mathrm{B}^{(1)}$-groups, we always made a point in developing algorithms that would be manageable by hand in the small digit ranks: this made our work much more enjoyable, allowing us to meet so to say "in person" various interesting objects of our studies. Unfortunately, already for the hooking condition examples become quickly intractable (see [DVM8]); for generalized hooking summands one of which is indecomposable, Clorinda De Vivo made up a yard-long scheme that allows (only) her to assess concrete situations with some ease. The general case is hopeless.

All groups in the following are torsionfree Abelian of finite rank ([FII]). A $\mathrm{B}^{(i)}$-group is a group of rank $m-i$ which is the sum of m rank 1 subgroups; $\mathrm{B}^{(1)}$-groups have been vastly studied, for references see [A], [AV]. Throughout we deal with $\mathrm{B}^{(1)}$-groups up to *quasi isomorphism* ([FII]: G is quasi isomorphic to H if it is isomorphic to a finite index subgroup of H), and will write *isomorphic* for quasi-isomorphic, *indecomposable* instead of strongly indecomposable, *direct summand* instead of quasi-direct summand, etc.

0. Notation and definitions

Besides [M], most of the results we rely upon come from [DVM4]; we summarize them here.

A $\mathrm{B}^{(1)}$-group of rank $m-1$ is a torsionfree Abelian group that is the sum of m rank one subgroups

$$G = \langle g_1 \rangle_* + \cdots + \langle g_m \rangle_*$$

(where $\langle g \rangle_*$ denotes the pure subgroup generated by $g \in G$), subject to the only relation

$$g_1 + \cdots + g_m = 0.$$

This last condition used to characterize "regular" $\mathrm{B}^{(1)}$-groups, but we choose to consider non-regular $\mathrm{B}^{(1)}$-groups directly as $\mathrm{B}^{(2)}$-groups (see the introduction of [DVM8] for a discussion on the matter).

The m-tuple $(g_1, \dots, g_m)$ is called a *base* of G. For $g \in G$, let $t_G(g)$ denote the type in G of g, and let $t_i = t_G(g_i)$ for $i \in I = \{1, \dots, m\}$. Then the m-tuple $(t_1, \dots, t_m)$ is called a *type base* of G. A $\mathrm{B}^{(1)}$-group may have more than one base (type-base); in this case we say it has *base changes* (permutations of the indices yield trivial base changes).

An element of a $\mathrm{B}^{(1)}$-group G is called a *base element* if it belongs to some base of G; a type is called a *base type* if it is the type of a base element of G. As has been proved in [DVM4], when dealing up to quasi-isomorphism there is no loss of generality in supposing that the types of the elements of G consist only of zeros and a finite number of infinities: thus a type base is described by an $m \times n$ table of zeros and infinities for a suitable n (Section 4).

To proceed, we need to recall some notation. If $E \subseteq I$, set

$$\begin{aligned} g_E &= \sum \{g_i \mid i \in E\}, \qquad \text{then } g_E = -g_{I\setminus E}; \\ \tau(E) &= \wedge \{t_i \mid i \in E\}, \\ t_E = t_{I\setminus E} &= \tau(E) \vee \tau(I \setminus E), \quad \text{thus } t_E = t_G(g_E) \text{ (see [DVM4]);} \\ G_E = \langle g_i \mid i \in E\rangle_* &= \sum \{\langle g_i\rangle_* \mid i \in E\} + \langle g_{I\setminus E}\rangle_* : \end{aligned}$$

thus G_E is a $\mathrm{B}^{(1)}$-group with index set $J = E \cup \{I \setminus E\}$. For $\mathcal{A} = \{A_1, \dots, A_k\}$ a partition of I, define

$$G(\mathcal{A}) = \sum \left\{ \langle g_{A_j}\rangle_* \,|\, j = 1, \dots, k \right\};$$

then $G(\mathcal{A})$ is a $\mathrm{B}^{(1)}$-group with base

$$\left(g_{A_j} \,|\, j = 1, \dots, k \right)$$

and type base

$$\left(t_{A_j} \,|\, j = 1, \dots, k \right).$$

For $E \subseteq I$ define

$$p_E = \{\{i\} \mid i \in E\} \cup \{E \setminus I\}$$

the *pointed partition* pointed on E; then $G(p_E) = G_E$. Define further $t(\mathcal{A})$, the *type of* $\mathcal{A}$, by

$$t(\mathcal{A}) = t_{A_1} \wedge \cdots \wedge t_{A_h} = \tau(I \setminus A_1) \vee \cdots \vee \tau(I \setminus A_k);$$

in view of the following Observation, we have in particular $t(p_E) = \tau(E)$.

Observation 0.1 ([DVM4, 0.b)]). *The type of $\mathcal{A}$ remains unchanged if the infimum is taken over all but one of the terms.* (This is not true for the sup). □

The thus defined application $t : \mathcal{A} \mapsto t(\mathcal{A})$, from the lattice of partitions of I (ordered by "greater = coarser") into the lattice $\mathbb{T}$ of all types (with a maximum ∞ added for the type of 0), is a morphism of $\wedge$-semilattices, and is called *tent*. Since, given a base of G, every element g of G determines a partition $\mathcal{A}$ of I into "equal coefficient blocks" such that $t_G(g) = t(\mathcal{A})$ (see the next example and [DVM4]), $\mathrm{Im}(t)$ is the *typeset* of G, the set of all types of elements of G.

Example 0.2. Let $G = \langle g_1 \rangle_* + \cdots + \langle g_5 \rangle_*$ have type base

$$\begin{array}{lcccc} t_1 = & \infty & 0 & 0 & \infty \\ t_2 = & 0 & \infty & 0 & 0 \\ t_3 = & 0 & 0 & \infty & 0 \\ t_4 = & \infty & 0 & 0 & \infty \\ t_5 = & 0 & \infty & 0 & \infty \end{array}$$

and set $g = 2g_1 + g_2 + g_3 + 2g_4$.

Then $g = 2g_{\{1,4\}} + g_{\{2,3\}}$, its partition is $\mathcal{A} = \{\{1,4\},\{2,3\},\{5\}\}$, its type is $t_G(g) = t(\mathcal{A}) = \tau(\{2,3,5\}) \vee \tau(\{1,4,5\}) \vee \tau(\{1,2,3,4\}) = 0 \quad 0 \quad 0 \quad \infty$. □

In the next two sections we investigate a condition which is necessary and sufficient for the direct sum of two indecomposable $\mathrm{B}^{(1)}$-groups to be a $\mathrm{B}^{(1)}$-group.

1. The hooking condition

While a direct summand of a $\mathrm{B}^{(1)}$-group is always a $\mathrm{B}^{(1)}$-group, this is not true for direct sums of $\mathrm{B}^{(1)}$-groups. Below, we will state from [M] a sufficient condition $(*)$, called here *hooking condition*, for a direct sum $H = G' \oplus G''$ of two $\mathrm{B}^{(1)}$-groups to be a $\mathrm{B}^{(1)}$-group. The condition is necessary if G' and G'' are indecomposable, but not in general (see Example a) in Section 4, Theorem 2.1 and Section 3). Note that the condition is independent from the representation (the base) of the groups.

In order to state condition $(*)$, recall that the types of the elements of a $\mathrm{B}^{(1)}$-group form a finite lattice; in particular this implies that every $\mathrm{B}^{(1)}$-group has a minimum type. Let then G', G'' be $\mathrm{B}^{(1)}$-groups, with minimum types ρ' resp. ρ''.

Definition 1.1. For the $\mathrm{B}^{(1)}$-groups G' and G'' we say G' *hooks up to* G'' if there is a base type t' of G' such that $t' \leq \rho' \vee \rho''$; G' and G'' *hook up*, or satisfy the *hooking condition*, if each hooks up to the other, that is if

$(*)$ there is a base type t' of G' and a base type t'' of G'' such that $t' \vee t'' = \rho' \vee \rho''$ ($\geq$ always holds). □

We will say a direct summand G' of a group L is a *hooking summand* if it and its complement in L are $\mathrm{B}^{(1)}$-groups, and hook up (thus making L itself a $\mathrm{B}^{(1)}$-group). We will show (Lemma 1.1) that every decomposable $\mathrm{B}^{(1)}$-group G has at least one hooking decomposition. If $G = G' \oplus K'$ is such, the $\mathrm{B}^{(1)}$-group G' (if decomposable) has a hooking decomposition: $G' = G'' \oplus K'' \ldots$ Thus for certain direct summands S of a $\mathrm{B}^{(1)}$-group G there will be for some $n \leq \operatorname{rk} G$ a chain

$$S = G_n < G_{n-1} < \cdots < G_0 = G$$

such that each G_i is a hooking summand of G_{i-1}. If n is the least natural number for which this occurs, S will be called an *n-th-level-hooking* summand of G (first level = hooking). Note that, if a group H has an n-th-level-hooking summand S (for some $n \leq \operatorname{rk} H$), then every step of the chain is a $\mathrm{B}^{(1)}$-group; therefore a necessary and sufficient condition for a group H to be a $\mathrm{B}^{(1)}$-group is for it to have an n-th-level-hooking summand for some $n \leq \operatorname{rk} H$.

The hooking condition, as was proved in the sufficiency part of Theorem 1 in [M], is sufficient for $G = G' \oplus G''$ to be a $\mathrm{B}^{(1)}$-group: it yields a base for G by uniting the two bases of G' resp. G'' containing base element g' of type t', resp. g'' of type t'', and replacing g' and g'' with $g' + g''$ of type $t' \wedge t''$. This would always work if we dealt with vector spaces; but with $\mathrm{B}^{(1)}$-groups not satisfying the hooking condition such an operation will in general only produce a proper subgroup of the direct sum.

Given the type base $(t_1, \dots, t_m)$, a type σ of G always comes equipped with $\operatorname{part}_G(\sigma)$, the finest partition of I of type σ; part_G is a morphism of $\vee$-semilattices between the typeset of G and the lattice of partitions of I. (It would be more precise to call it part_t, as in [DVM4], but when the type base of G is fixed we prefer this notation). In particular for a base type t_i we have

$$\operatorname{part}_G(t_i) = \{\{i\}, A_{i1}, A_{i2}, \dots, A_{ik_i}\}$$

(see [DVM4] for the construction); thus we have the *partition base*

$$(\operatorname{part}_G(t_1), \dots, \operatorname{part}_G(t_m))$$

of G attached to the type base $(t_1, \dots, t_m)$.

We complete Theorem 1 from [DVM4] by

Lemma 1.2. *If $k_i > 1$, a splitting occurs: whenever $\{\{i\}, E, F\}$ is coarser than $\operatorname{part}_G(t_i)$ we get*

$$G = G_E \oplus G_F,$$

and this is always a hooking decomposition of G.

Proof: Let w.l.o.g. $i = 1$, $E = \{2, \dots, r\}$, $F = \{r+1, \dots, m\}$. The sum is direct, otherwise there would be a relation between the g_i's excluding g_1. To show that $H = G_E \oplus G_F$ equals G we only need to show that $g_1 = -(g_E + g_F)$ obtains its full type t_1 in H. In fact, we have $t_1 = t_G(g_1) \geq t_H(g_1) = t_E \wedge t_F$, but this, by Observation 1.1, equals $t(\{\{1\}, E, F\}) \geq t(\operatorname{part}_G(t_1)) = t_1$. To show that $(*)$ holds, observe that the minimum type of G_E is $\tau(E)$, the minimum of G_F is $\tau(F)$; and one checks that in this setting $t_E \wedge t_F = \tau(E) \vee \tau(F)$, as required. □

Since G is indecomposable if and only if $k_i = 1$ for each $i \in I$ (see [DVM4]), we have

Corollary 1.3. *Each decomposable* $\mathrm{B}^{(1)}$*-group has a proper hooking decomposition. Every indecomposable direct summand of a* $\mathrm{B}^{(1)}$*-group has a hooking level.*

Proof: In the above notation, set $F = \bigcup\{A_{ir} \mid r \neq j\}$, and note that the partition $\{\{i\}, A_{ij}, F\}$ is coarser than $\operatorname{part}_G(t_i)$; hence $G_{A_{ij}} = G(\mathfrak{p}_{A_{ij}})$ is always a hooking summand of $G = G_{A_{ij}} \oplus G_F$. Let now K be an indecomposable summand of G; K lies either in $G_{A_{ij}}$ or in G_F; finite induction will then prove the second statement. □

Indecomposable summands do not necessarily cover all levels, as can be seen in Example 3 a). In the next section we will prove that among the $G(\mathfrak{p}_{A_{ij}})$ there is an indecomposable summand.

2. Indecomposable hooking summands

Consider the partition base of G

$$\begin{aligned}\operatorname{part}_G(t_1) &= \{\{1\}, A_{11}, A_{12}, \dots, A_{1k_1}\}\\ \operatorname{part}_G(t_2) &= \{\{2\}, A_{21}, A_{22}, \dots, A_{2k_2}\}\\ &\dots\dots\dots\dots\dots\\ \operatorname{part}_G(t_m) &= \{\{m\}, A_{m1}, A_{m2}, \dots, A_{mk_m}\}.\end{aligned}$$

Theorem 2.1. *If* A_{ij} *is a block with minimum cardinality then the hooking summand* $G(\mathfrak{p}_{A_{ij}})$ *is indecomposable.*

Proof: Let w.l.o.g. $i = j = 1$, $A_{11} = \{2, 3, \dots, r\}$, $C = \bigcup\{A_{1j} \mid j \neq 1\} = I \setminus (A_{11} \cup \{1\})$, thus $G = G(\mathfrak{p}_{A_{11}}) \oplus G_C$. Set $G' = G(\mathfrak{p}_{A_{11}})$; its type base is $(t_2, t_3, \dots, t_r, t_{I \setminus A_{11}})$. Then by the decomposition algorithm in [DVM4] the partition base of G' is

$$\begin{aligned}\operatorname{part}_{G'}(t_2) &= \{\{2\}, B_{21}, B_{22}, \dots, B_{2h_2}\}\\ \operatorname{part}_{G'}(t_3) &= \{\{3\}, B_{31}, B_{32}, \dots, B_{3h_3}\}\\ &\dots\dots\dots\dots\dots\\ \operatorname{part}_{G'}(t_r) &= \{\{r\}, B_{r1}, B_{r2}, \dots, B_{rh_r}\}\\ \operatorname{part}_{G'}(t_{I \setminus A_{11}}) &= \{A_{11}, \{I \setminus A_{11}\}\} \quad \text{(a bipartition of } J = A_{11} \cup \{I \setminus A_{11}\}).\end{aligned}$$

If these are all bipartitions, $G(\mathfrak{p}_{A_{11}})$ is indecomposable, as wanted. Say then e.g. $h_2 > 1$. From the decomposition algorithm we know that all but one of the blocks of $\operatorname{part}_{G'}(t_2)$ are contained in A_{11}. But A_{11} contains $\{2\}$, and is of minimal cardinality, thus no other such block exists, and $\operatorname{part}_{G'}(t_2)$ is a bipartition, a contradiction. □

When both G' and G'' are indecomposable, the $(*)$ condition is necessary and sufficient for $G' \oplus G''$ to be a $\mathrm{B}^{(1)}$-group:

Theorem 2.2. *Let G', G'' be indecomposable* $\mathrm{B}^{(1)}$*-groups. Then $G = G' \oplus G''$ is a* $\mathrm{B}^{(1)}$*-group if and only if G', G'' are hooking summands of G.*

Proof: We only need to prove necessity. By the Krull-Schmidt property (enjoyed by $\mathrm{B}^{(1)}$-groups up to quasi-isomorphism) only one partition in the partition base of G is not a bipartition, and it is of the form $\{\{i\}, A_{i1}, A_{i2}\}$; then $G = G(\mathfrak{p}_{A_{i1}}) \oplus G(\mathfrak{p}_{A_{i2}})$ is a hooking decomposition, and the two summands are isomorphic to G' resp. G''. □

Theorem 2.1 implies that a necessary condition for any direct sum of indecomposable $\mathrm{B}^{(1)}$-groups to be a $\mathrm{B}^{(1)}$-group is that at least one of them must hook up with the direct sum of all the others. This seems promising enough, until we realize that, in order to check this hooking, the last sum must be a $\mathrm{B}^{(1)}$-group itself: the hooking condition requires the existence of base elements. Thus all we can say directly is

Corollary 2.3. *Let $G' = G'_1 \oplus \cdots \oplus G'_k$, $G'' = G''_1 \oplus \cdots \oplus G''_h$ be* $\mathrm{B}^{(1)}$*-groups, decomposed into indecomposable summands. Then $G = G' \oplus G''$ is a* $\mathrm{B}^{(1)}$*-group if and only if $h + k - 1$ summands sum up to a* $\mathrm{B}^{(1)}$*-group, and this hooks up to the remaining summand.* □

This yields no reasonable algorithm; we will thus have to investigate a more general setting, which will bring us to the solution of the direct sum problem.

3. Generalized hooking

In this section we generalize the hooking condition, to one that is necessary and sufficient for the direct sum of two $\mathrm{B}^{(1)}$-groups, one of which is indecomposable, to be a $\mathrm{B}^{(1)}$-group. This takes us to the final solution.

We start by giving necessary and sufficient conditions for a direct sum of $\mathrm{B}^{(1)}$-groups $G' \oplus G''$ to equal a particular subgroup G, which is a $\mathrm{B}^{(1)}$-group.

Integrate the previous notation for G, G', G'' with the following:

$G = \langle g_1 \rangle_* + \cdots + \langle g_r \rangle_* + \langle g_{r+1} \rangle_* + \cdots + \langle g_{r+k} \rangle_* + \langle g_{r+k+1} \rangle_* + \cdots + \langle g_m \rangle_*$;

$E = \{1, \dots, r\}, F = \{r+1, \dots, m\},$ thus $I = E \cup F$;

$G' = \langle g'_0 \rangle_* + \langle g'_1 \rangle_* + \cdots + \langle g'_r \rangle_*$;

$(t'_0, t'_1, \dots, t'_r)$ its type base;

$\rho' = \min(\text{typeset } G')$;

for $J \subseteq \{0\} \cup E$ (the index set of G'), $\tau'(J) = \wedge\{t'_i \mid i \in J\}$;

in particular, due to Observation 0.1, $\tau'(E) = \rho'$;

$t'_J = \tau'(J) \vee \tau'((\{0\} \cup E) \setminus J),$ is the type in G' of $g'_J,$ for $J \subseteq E$;

$\mathcal{X} = \{\{0\}, X_1, \dots, X_k\}$ is a partition of $\{0\} \cup E$,

$t'(\mathcal{X}) = t'_0 \wedge t'_{X_1} \wedge \cdots \wedge t'_{X_k} (= t'_{X_1} \wedge \cdots \wedge t'_{X_k},$ by Observation 0.1), hence
$t'(\mathcal{X}) \le t'(\{\{0\}, E\}) = t'_0$.

$G'' = \langle g''_{r+1}\rangle_* + \cdots + \langle g''_{r+k}\rangle_* + \langle g''_{r+k+1}\rangle_* + \cdots + \langle g''_m\rangle_*$;
$(t''_{r+1}, \ldots, t''_{r+k}, t''_{r+k+1}, \ldots, t''_m)$ its type base;
$\rho'' = \min(\text{typeset } G'')$;
for $J \subseteq F$ (F the index set of G''), $\tau''(J) = \wedge\{t''_i \mid i \in J\}$;
in particular, $\tau''(F) = \rho''$

Start now considering, in the group $L = G' \oplus G''$, the subgroup G defined, with the above notation (types are computed in L), by setting

$$\begin{array}{lll} g_i = g'_i & \text{for } i = 1, \ldots, r, & \text{with type } t_i = t'_i \\ g_{r+j} = g''_{r+j} - g'_{X_j} & \text{for } j = 1, \ldots, k, & \text{with type } t_{r+j} = t''_{r+j} \wedge t'_{X_j} \\ g_{r+s} = g''_{r+s} & \text{for } s = k+1, \ldots, m-r, & \text{with type } t_{r+s} = t''_{r+s}. \end{array}$$

The subgroup G of $G' \oplus G''$ is a $\mathrm{B}^{(1)}$-group containing all the generators g'_i, g''_j of G' resp. G''. For the sake of clarity, when we consider them as elements of G we redenominate them h'_i resp. h''_j, so that – say – $\langle h'_i\rangle_*$ will denote the pure subgroup *of* G generated by $h'_i = g'_i$. Their types in G: u'_i resp. u''_j, will in general be smaller than or equal to their original types t'_i resp. t''_j. In fact we have

– for $i \in E$, $h'_i = g_i$ and $u'_i = t_i = t'_i$;
  $h'_0 = -(g'_1 + \cdots + g'_r) = -g'_E = -g_E$, thus $u'_0 = t_E$;
– for $j = 1, \ldots, k$, $h'_{X_j} = g_{X_j}$, thus $h''_{r+j} = g_{r+j} + g_{X_j} = g_{\{r+j\} \cup X_j}$
  and $u''_{r+j} = t_{\{r+j\} \cup X_j}$;
– for $s = k+1, \ldots, m-r$, $h''_{r+s} = g''_{r+s}$,
  and $u''_{r+s} = t_{r+s} = t''_{r+s}$.

For the following, it is useful to compute in $G' \oplus G''$ the types $\tau(E)$, $\tau(F)$:

$$\begin{aligned} \tau(E) &= \tau'(E) = \rho'; \\ \tau(F) &= \wedge\{t''_{r+j} \wedge t'_{X_j} \mid j = 1, \ldots, r\} \wedge (\wedge\{t''_{r+s} \mid s = k+1, \ldots, m-r\} \\ &= \tau''(F) \wedge (\wedge\{t'_{X_j} \mid j = 1, \ldots, k\}) = \rho'' \wedge t'(\mathcal{X}). \end{aligned}$$

Proposition 3.1. *In the above notation, the conditions*

a) $t'_0 \leq \rho' \vee \rho''$,
b) $t'_0 = t'(\mathcal{X})$ (*that is,* $\mathrm{part}_{G'}(t'_0) \leq \mathcal{X} \leq \{\{0\}, E\}$),
c) $t''_{r+j} \leq \tau'(X_j) \vee \rho''$ *for all* $j = 1, \dots, k$,
d) $t''_{r+j} \leq \tau'(X_j) \vee \tau'(E \setminus X_j)$ *for all* $j = 1, \dots, k$

are necessary and sufficient for $G' \oplus G''$ *to be equal to* G.

Proof: Consider in G the subgroup $H' \oplus H'' = (\langle h'_0 \rangle_* + \cdots + \langle h'_r \rangle_*) \oplus (\langle h''_{r+1} \rangle_* + \cdots + \langle h''_{r+k} \rangle_* + \langle h_{r+k+1} \rangle_* + \cdots + \langle h''_m \rangle_*)$; we have $H' \leq G'$, $H'' \leq G''$.

Observe first that $G = G' \oplus G''$ if and only if $H' \oplus H'' = G' \oplus G''$: one way is obvious, the other is due to the fact that if $G = G' \oplus G''$ then the type u'_i of h'_i in G is equal to its type t'_i in G', thus $H' = G'$ (we are dealing up to quasi-isomorphism!); and the same for H''.

Clearly, to have $H' \oplus H'' = G' \oplus G''$ it is necessary and sufficient that $u'_i = t'_i$ for all $i = 0, 1, \dots, r$, and $u''_j = t''_j$ for all $j = r+1, \dots, m$.

For $i = 1, \dots, r$ and for $j = r+k+1, \dots, m$ this is already true.

For $i = 0$, we have $u'_0 = t_E = \tau(E) \vee \tau(F) = \rho' \vee (\rho'' \wedge t'(\mathcal{X}))$. The condition $u'_0 = t'_0$, i.e. $u'_0 \geq t'_0$, i.e. $t'_0 = t'_0 \wedge u'_0$, becomes

$$\begin{aligned} t'_0 &= t'_0 \wedge (\rho' \vee (\rho'' \wedge t'(\mathcal{X}))) = (t'_0 \wedge \rho') \vee (t'_0 \wedge \rho' \wedge t'(\mathcal{X})) \\ &= \rho' \vee (\rho'' \wedge t'(\mathcal{X})) = (\rho' \vee \rho'') \wedge t'(\mathcal{X}), \end{aligned}$$

hence a) follows; since we already have $t'_0 \leq t'(\mathcal{X})$ we get b).

For $i = r+j$, $j = 1, \dots, k$, we have

$$\begin{aligned} u''_{r+j} &= t_{\{r+j\} \cup X_j} = (t_{r+j} \wedge \tau(X_j)) \vee (\tau(F \setminus \{r+j\}) \wedge \tau(E \setminus X_j)) \\ &= (t''_{r+j} \wedge t'(X_j) \wedge \tau'(X_j)) \vee ((\wedge \{t''_{r+i} \mid i = 1, \dots, m-r; i \neq j\}) \\ &\quad \wedge t'(\{\{0\} \cup X_j, X_i \mid i = 1, \dots, k; i \neq j\}) \wedge \tau'(E \setminus X_j)). \end{aligned}$$

We have $t'(X_j) \geq \tau'(X_j)$; $\wedge \{t''_{r+i} \mid i = 1, \dots, m-r; i \neq j\} = \rho''$; and since $E \setminus X_j = \bigcup \{X_i \mid i = 1, \dots, k; i \neq j\}$ is a coblock of the partition $\{\{0\} \cup X_j, X_i \mid i = 1, \dots, k; i \neq j\}$, we have $\tau'(E \setminus X_j) \leq t'(\{\{0\} \cup X_j, X_i \mid i = 1, \dots, k, i \neq j\})$, therefore we may continue with

$$\begin{aligned} u''_{r+j} &= (t_{r+j} \wedge \tau'(X_j)) \vee (\rho'' \wedge \tau'(E \setminus X_j)) \\ &= t''_{r+j} \wedge (t''_{r+j} \wedge \tau'(E \setminus X_j)) \wedge (\tau'(X_j) \vee \rho'') \wedge (\tau'(X_j) \vee \tau'(E \setminus X_j)) \\ &= t''_{r+j} \wedge (\tau'(X_j) \vee \rho'') \wedge (\tau'(X_j) \vee \tau'(E \setminus X_j)) \leq t''_{r+j}. \end{aligned}$$

Finally, imposing $u''_{r+j} = t''_{r+j}$ is equivalent to conditions c): $t''_{r+j} \leq \tau'(X_j) \vee \rho''$ and d): $t''_{r+j} \leq \tau'(X_j) \vee \tau'(E \setminus X_j)$, as wanted. $\square$

Definition 3.2. Let G', G'' be $\mathrm{B}^{(1)}$-groups. If there is a type base $(t'_0, t'_1, \dots, t'_r)$ of G', a partition $\mathcal{X}$ of $\{0, 1, \dots, r\}$ and a type base $(t''_{r+1}, \dots, t''_m)$ of G'' such that conditions a), b), c), d) hold, we say G' and G'' satisfy the *generalized hooking condition*.

As an aside, observe that the condition starts with: "if there is a base … ". This is an indication that checking by hand is out of the question, except in trivial cases. Another difficulty comes from "if there is a partition $\mathcal{X}$ …". We saw that the condition on $\mathcal{X}$ is that $\mathrm{part}_{G'}(t'_0) \leq \mathcal{X} \leq \{\{0\}, E\}$. This condition cannot be made stricter in general, hence one would have to check all such partitions; the finer $\mathrm{part}_{G'}(t'_0)$, the longer the check.

The reason for the above definition is the following

Theorem 3.3. *Let G', G'' be $\mathrm{B}^{(1)}$-groups, with G'' indecomposable. Then $G' \oplus G''$ is a $\mathrm{B}^{(1)}$-group if and only if G' and G'' satisfy the generalized hooking condition.*

Proof: If G', G'' are $\mathrm{B}^{(1)}$-groups and a), … , d) hold for suitable bases, then Proposition 3.1 shows that $G' \oplus G''$ equals G, hence is a $\mathrm{B}^{(1)}$-group: the condition is sufficient.

Let then G',G'' and $H = G' \oplus G''$ be $\mathrm{B}^{(1)}$-groups with G'' indecomposable. From [CDVM, Theorem 2.2, Proposition 4.4] we have that there are partitions $\mathcal{D}$, $\mathcal{C}$ of the index set of H such that $H = H(\mathcal{D}) \oplus H(\mathcal{C})$ with $H(\mathcal{D}) \cong G'$ and $H(\mathcal{C}) \cong G''$; moreover, if $H(\mathcal{C})$ is indecomposable, then $\mathcal{D} = p_D$, a pointed partition. In [CDVM, Proposition 4.7] it is shown that the type bases $(t_D, t_1, \dots, t_r)$ of $H(p_D)$ and $(t_{C_1}, \dots, t_{C_k})$ of $H(\mathcal{C})$ satisfy conditions called there a), b), c), d). By isomorphism, G' will have a type base $t'_0 = t_D, t'_1 = t_1, \dots, t'_r = t_r$, and G'' will have a type base $t''_{r+1} = t_{C_1}, \dots, t''_{r+k} = t_{C_k}$, satisfying conditions a), … , d) which are now our conditions, renaming D by E. Therefore necessity holds as well. □

A help for computation is offered by the following

Corollary 3.4. *Let G', G'' be $\mathrm{B}^{(1)}$-groups, with G'' indecomposable. Then*

i) *a necessary condition for $G' \oplus G''$ to be a $\mathrm{B}^{(1)}$-group is that G' hooks on to G'';*
ii) *a necessary condition for (i) is that $G' \otimes R''$ be completely decomposable, where R'' is a subgroup of $\mathbb{Q}$ of type ρ''.*

Proof: i) is condition (0) a); ii) is in [DVM8]. □

Solution of the main problem. Let H be the direct sum of a finite number of $\mathrm{B}^{(1)}$-groups. After decomposing the summands, we will have $H = G_1 \oplus \cdots \oplus G_n$, a direct sum of indecomposable $\mathrm{B}^{(1)}$-groups. Start with $H_{n-1} = G_{n-1} \oplus G_n$, checking the hooking condition. If it does not work, H is not $\mathrm{B}^{(1)}$. If it does,

H_{n-1} is a $\mathrm{B}^{(1)}$-group; set $G' = H_{n-1}$, $G'' = G_{n-2}$, and check the generalized hooking condition. This is the first step in an obvious finite induction, which will lead to deciding whether H is, or is not, a $\mathrm{B}^{(1)}$-group, thus solving the general problem. □

4. Examples

a). The next is a decomposable $\mathrm{B}^{(1)}$-group where there are direct summands of various hooking levels, showing that $(*)$ is not a necessary condition.

Let G be the $\mathrm{B}^{(1)}$-group of rank 5 with type base

$$\begin{aligned} t_1 &= \infty \quad 0 \quad 0 \\ t_2 &= 0 \quad \infty \quad 0 \\ t_3 &= 0 \quad 0 \quad \infty \\ t_4 &= \infty \quad 0 \quad 0 \\ t_5 &= 0 \quad \infty \quad 0 \\ t_6 &= 0 \quad 0 \quad \infty. \end{aligned}$$

The decomposition algorithm shows that G is the direct sum of the following four indecomposable $\mathrm{B}^{(1)}$-groups: the rank 1 groups G_4, G_5, G_6 of types respectively t_4, t_5, t_6, and the rank 2 group $G' = G(\{\{1,4\},\{2,5\},\{3,6\}\})$ with type base

$$\begin{aligned} t_{\{1,4\}} &= \infty \quad 0 \quad 0 \\ t_{\{2,5\}} &= 0 \quad \infty \quad 0 \\ t_{\{3,6\}} &= 0 \quad 0 \quad \infty. \end{aligned}$$

G' has minimum type

$$\rho' = 0 \quad 0 \quad 0.$$

The rank 3 complement $G'' = G_4 \oplus G_5 \oplus G_6$ of G' has type base

$$\begin{aligned} t_4 &= \infty \quad 0 \quad 0 \\ t_5 &= 0 \quad \infty \quad 0 \\ t_6 &= 0 \quad 0 \quad \infty \\ t_4 \wedge t_5 \wedge t_6 &= 0 \quad 0 \quad 0 \end{aligned}$$

with minimum

$$\rho'' = 0 \quad 0 \quad 0.$$

Note that G' and G'' have no nontrivial base changes; therefore the only base types available to check the validity of $(*)$ are the ones shown above. It is then clear that $G = G' \oplus G''$ is not a hooking decomposition: there is no type in the base of G' that is $\leq \rho' \vee \rho''$. Instead, G_4, G_5 and G_6 are all hooking summands of G: this can be checked by computing a type base for their complements, or, directly, from the partition base of G:

$$\begin{aligned}
\mathrm{part}_G(t_1) &= \{\{1\},\{4\},\{2,3,5,6\}\}\\
\mathrm{part}_G(t_2) &= \{\{2\},\{5\},\{1,3,4,6\}\}\\
\mathrm{part}_G(t_3) &= \{\{3\},\{6\},\{1,2,4,5\}\}\\
\mathrm{part}_G(t_4) &= \{\{4\},\{1\},\{1,2,3,5,6\}\}\\
\mathrm{part}_G(t_5) &= \{\{5\},\{2\},\{1,2,3,4,6\}\}\\
\mathrm{part}_G(t_6) &= \{\{6\},\{3\},\{1,2,3,4,5\}\},
\end{aligned}$$

where the first three partitions show that $\{4\}$, $\{5\}$, $\{6\}$ are blocks of minimum cardinality.

To compute the hooking level of G', continue the decomposition via partitions: from the first we get $G = G_4 \oplus G_{\{2,3,5,6\}}$, where the type base of the hooking summand $H = G_{\{2,3,5,6\}}$ is $(t_2, t_3, t_5, t_6, t_{\{1,4\}})$ and its partition base starts with

$$\begin{aligned}
\mathrm{part}_H(t_2) &= \{\{2\},\{5\},\{3,6,\{1,4\}\}\},\\
\mathrm{part}_H(t_3) &= \{\{3\},\{6\},\{2,5,\{1,4\}\}\},\\
&\ldots\ldots\ldots\ldots\ldots
\end{aligned}$$

yielding $H = G_5 \oplus G(\{3,6,\{1,4\}\})$. Here G_5 is first level in H but was also first level in G; while $K = G(\{3,6,\{1,4\}\})$ is first level in H but second level in G: in fact its type base is $(t_3, t_6, t_{\{1,4\}}, t_{\{2,5\}})$, while the type base of its complement $G_4 \oplus G_5$ is $(t_4, t_5, t_4 \wedge t_5)$, and one checks that $(*)$ does not hold. Finally,

$$\mathrm{part}_K(t_3) = \{\{3\},\{6\},\{\{1,4\},\{2,5\}\}\},$$

yielding $K = G_6 \oplus G(\{\{1,4\},\{2,5\},\{3,6\}\}) = G_6 \oplus G''$, and a similar computation shows that G'' is indeed a level 3 summand of G.

b). We give the simplest example of $G = G_1 \oplus G_2 \oplus G_3$ where the G_i hook up pairwise, but G is not $\mathrm{B}^{(1)}$. Clearly, at least one of the G_i must be of rank ≥ 2. Note that $\mathrm{B}^{(1)}$-groups of rank 1 have a base consisting of two equal types; and that if H is rank 1, H hooks up to any $\mathrm{B}^{(1)}$-group.

G_1 is given by

$$\begin{array}{rcccc}
t_1 = & 0 & \infty & 0 & 0\\
t_1 = & 0 & \infty & 0 & 0,
\end{array}$$

G_2 is given by

$$\begin{array}{rcccc}
t_2 = & 0 & 0 & \infty & 0\\
t_2 = & 0 & 0 & \infty & 0,
\end{array}$$

G_3 is given by

$$\begin{array}{rcccc} t_3 = & \infty & \infty & 0 & 0 \\ t_4 = & \infty & 0 & \infty & 0 \\ t_5 = & \infty & 0 & 0 & \infty. \end{array}$$

To verify (for instance) that G_3 hooks up to G_1, we only need to note that

$$\rho_3 = \quad \infty \quad 0 \quad 0 \quad 0,$$

hence $t_3 = t_1 \vee \rho_3$. But $G_1 \oplus G_3$ has as a fourth base type $t_1 \wedge t_3 = t_1$, hence its minimum is

$$\rho = \quad 0 \quad 0 \quad 0 \quad 0;$$

thus $G_1 \oplus G_3$ (which has no base changes) does not hook up to G_2.

c). Theorem 2.1 states that summands determined by blocks of minimum cardinality are indecomposable: a very strong algebraic property is determined by a partition-theoretic property. We give an example where the block is minimum, but the cardinality of the summand is not, showing that this is a nontrivial characterization. Let G be the rank 11 $\mathrm{B}^{(1)}$-group given by

$$\begin{array}{rccccccccccccccc}
t_1 = & \infty & \infty & \infty & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_2 = & \infty & \infty & \infty & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_3 = & \infty & \infty & \infty & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_4 = & 0 & \infty & \infty & \infty & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_5 = & 0 & \infty & \infty & \infty & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
t_6 = & 0 & \infty & \infty & \infty & 0 & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 \\
t_7 = & \infty & 0 & \infty & 0 & \infty & 0 & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 \\
t_8 = & \infty & 0 & \infty & 0 & \infty & 0 & 0 & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 \\
t_9 = & \infty & 0 & \infty & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & \infty & 0 & 0 & 0 \\
t_{10} = & \infty & \infty & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & \infty & 0 & 0 \\
t_{11} = & \infty & \infty & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \infty & 0 \\
t_{12} = & \infty & \infty & 0 & 0 & 0 & \infty & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \infty.
\end{array}$$

G has no nontrivial base changes. We have

$$\begin{aligned}
\mathrm{part}(t_1) &= \{\{1\}\{4\ 5\ 6\}\{2\ 3\ 7\ 8\ 9\ 10\ 11\ 12\}\} \\
\mathrm{part}(t_2) &= \{\{2\}\{7\ 8\ 9\}\{1\ 3\ 4\ 5\ 6\ 10\ 11\ 12\}\} \\
\mathrm{part}(t_3) &= \{\{3\}\{10\ 11\ 12\}\{1\ 2\ 4\ 5\ 6\ 7\ 8\ 9\}\} \\
\mathrm{part}(t_i) &= p_i, \text{ for } i = 4, \dots, 12.
\end{aligned}$$

Decomposing, we get

$$\begin{aligned} G = G(\{\{1,4,5,6\}\{2,7,8,9\}\{3,10,11,12\}\}) \oplus \\ \oplus\, G(p\{4,5,6\}) \oplus G(p\{7,8,9\}) \oplus G(p\{10,11,12\}). \end{aligned}$$

The rank of indecomposable summands coming from minimum blocks is 3, while the rank 2 summand does not come from a minimum block in any representation of G (no base changes).

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