On absolutely submetrizable spaces

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Abstract. We introduce a notion of absolute submetrizability (= "every Tychonoff subtopology is submetrizable") and investigate its behavior under basic topological operations. The main result is an example of an absolutely submetrizable space that contains an uncountable set of isolated points (hence the space is neither separable nor hereditarily Lindelöf). This example is used to show that absolute submetrizability is not preserved by some topological operations, in particular, by free sums.

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1. Introduction

In the theory of generalized metric spaces, submetrizability is one of the central and well-investigated properties. Recall that a topological space is called *submetrizable* if it admits a continuous bijection onto a metric space. It is known that submetrizability can be destroyed by continuous maps belonging to very nice classes. Motivated by this unfortunate fact we propose to study the following property.

We say that a Tychonoff space X is *absolutely submetrizable* if any Tychonoff subtopology of the topology of X is submetrizable. In other words, X is absolutely submetrizable if any one-to-one continuous Tychonoff image of X is submetrizable.

It is incorporated in the definition now that the property cannot be destroyed by continuous bijections, yet we do not know much about how sensitive this property is to other nice mappings. The main result of our study is an example of an absolutely submetrizable space that contains an uncountable set of isolated points (hence the space is neither separable nor hereditarily Lindelöf). This example is used to show that absolute submetrizability is not preserved by some topological operations, in particular, by free sums. In addition to our main result, we make a number of observations and pose motivated questions regarding the behavior of this property in continuous images, products, free sums, unions, subspaces, etc.

We will consider only Tychonoff spaces. In notation and terminology we will follow [ENG].

2. Study

Clearly, any (absolutely) submetrizable space has countable pseudocharacter. We will use this fact quite often throughout the paper. Let us start our study with two simple observations that will motivate our main result.

Theorem 1. *Any absolutely submetrizable space is Lindelöf.*

Proof: Suppose X is not Lindelöf. Then there exists a compactum $C \subset \beta X \setminus X$ such that any G_δ-set in βX containing C meets X. Fix any $x^* \in X$. Let Y be the quotient space defined by the partition on $C \cup X$ whose only non-trivial element is $\{x^*\} \cup C$. Clearly, Y is a one-to-one continuous Tychonoff image of X. Since C is not a G_δ-set in $C \cup X$, the image of x^* has uncountable pseudocharacter. Therefore Y is not submetrizable. $\square$

Of course, a submetrizable Lindelöf (even hereditarily Lindelöf) space need not be absolutely submetrizable. For example, Sorgenfrey Line is hereditarily Lindelöf and submetrizable, but it admits a continuous bijection onto the two arrows of Alexandroff (without one end-point). Since the latter is a non-metrizable compactum, the former is not absolutely submetrizable.

As was pointed out to the author, one class of absolutely submetrizable spaces is given by the following known corollary to the Sneider's theorem [SNE].

Fact 2. *Let X^2 be hereditarily Lindelöf. Then any continuous image of X is submetrizable. In particular, any space with countable network is absolutely submetrizable.*

Proof: If Y is a continuous image of X then Y^2 is hereditarily Lindelöf as well. Therefore, the diagonal Δ_Y is a G_δ-set in Y^2. Since Y is Tychonoff and Lindelöf, Y is submetrizable.

If Z has countable network, then Z^2 is hereditarily Lindelöf. Therefore, Z is absolutely submetrizable. $\square$

By Theorem 1, if every subspace of X is absolutely submetrizable then X is hereditarily Lindelöf. Moreover, every continuous Tychonoff image of X is submetrizable. Indeed, let f be a continuous map of X onto a Tychonoff Y. Choose $Z \subset X$ such that $f(Z) = Y$ and the restriction of f on Z is one-to-one. Since Z is absolutely submetrizable, Y is submetrizable as a continuous one-to-one Tychonoff image of Z. This observation and Fact 2 motivate the following question.

Question 3. *Let every subspace of X be absolutely submetrizable. Is X^2 hereditarily Lindelöf?*

In connection with Theorem 1 and Fact 2, it is interesting to have an example of an absolutely submetrizable space that is neither separable nor hereditarily Lindelöf. The next example serves this goal. In what follows, by R we denote the space of real numbers endowed with the Euclidean topology. Given a function $f : X \to Y$, by M_f we denote the set of all points of non-one-to-one-ness, that is, $M_f = \{x \in X : |f^{-1}f(x)| > 1\}$.

Example 4. *There exists an absolutely submetrizable space which is neither separable nor hereditarily Lindelöf.*

Construction. For each $A \subset R$, let F_A be the set of all continuous functions from A to R^ω such that $|M_f| = 2^\omega$. Clearly, $|F_A| \leq 2^\omega$. Let $F = \bigcup\{F_A : R \setminus A$ be countable$\}$. Enumerate F as $\{f_\alpha : \alpha < 2^\omega\}$. Inductively, we will choose points $a_\alpha, b_\alpha, c_\alpha$, and d_α. Points d_α's will be used in the definition of our space, while a_α's, b_α's, and c_α's will be used to prove absolute submetrizability. Let $C = \{C_\alpha : \alpha < 2^\omega\}$ be the set of all uncountable compact subsets of R. Assume for each $\beta < \alpha < 2^\omega$, points $a_\beta, b_\beta, c_\beta$, and d_β are defined.

Step $\alpha < 2^\omega$: Pick distinct $a_\alpha, b_\alpha \in M_{f_\alpha}$ such that $f_\alpha(a_\alpha) = f_\alpha(b_\alpha)$, $c_\alpha \in C_\alpha$, and $d_\alpha \in R$ that are distinct from each other and all $a_\beta, b_\beta, c_\beta, d_\beta$ picked before. This can be done because M_{f_α}, C_α, and R are of cardinality 2^ω.

Put $D = \{d_\alpha : \alpha < 2^\omega\}$ and $L = R \setminus D$. Let $R_L = (R, \mathcal{T})$ be the space with underlying set R and the topology $\mathcal{T}$ defined using Bing-Hanner construction (see [ENG, 5.1.22]). Namely, base neighborhoods at points of L are Euclidean, while points of D are declared isolated. Clearly, D is open in R_L. Since D is uncountable, R_L is neither separable nor hereditarily Lindelöf. Construction is complete. $\square$

To prove absolute submetrizability of R_L, we need the following three statements. Each of these statements is a corollary to some classical results and can be found in some form in literature but we will give them with proofs for completeness.

Proposition 5. *Suppose A has countable network and is a subspace of X. Then there exists a continuous map f of X to R^ω whose restriction on A is one-to-one.*

Proof: Let $\mathcal{N}$ be a countable network of A. For every $N, N' \in \mathcal{N}$, fix, if possible, a continuous map $f_{NN'} : X \to R$ that maps N into $[0, 1/3)$ and N' into $(2/3, 1]$. Let $\mathcal{S}$ consist of all pairs (N, N') for which $f_{NN'}$ is fixed. Clearly, $F = \Delta\{f_{NN'} : (N, N') \in \mathcal{S}\}$ is a continuous map of X to R^ω. Let us show that $F|_A$ is one-to-one. Fix distinct $x, y \in A$. Since X is Tychonoff, there exists a continuous function $f : X \to [0, 1]$ that maps x to 0 and y to 1. Since $\mathcal{N}$ is a network of A, there exist $N, N' \in \mathcal{N}$ such that $x \in N \subset f^{-1}([0, 1/3))$ and $y \in N' \subset f^{-1}((2/3, 1])$. Therefore, $(N, N') \in \mathcal{S}$ and $F(x) \neq F(y)$. $\square$

Lemma 6. *Let $f : R \to [0, 1]^\omega$ be a function which is continuous with respect to the topology of R_L. Let $B \subset R$ be the set of all points at which f is discontinuous with respect to the Euclidean topology on R. Then B is countable.*

Proof: Since the topology at points of L in R_L is Euclidean, $B \subset D$. Assume B is uncountable. For each $x \in B$, fix a sequence $\{s_x(k)\}_k$ of elements of R such that

1. $s_x(k) \to x$ in the Euclidean R;
2. $f(s_x(k)) \to p_x$ in $[0,1]^\omega$ and $p_x \neq f(x)$.

There exist an uncountable $B' \subset B$ and $\epsilon > 0$ such that the distance $\rho(p_x, f(x))$ in $[0,1]^\omega$ is greater than ϵ for every $x \in B'$.

Since all compact subsets of D are countable (due to c_α's in L) there exist $x \in L$ and a sequence $\{x_n\}_n$ of points of B' such that $x_n \to x$. Since f is continuous at x, $f(x_n) \to f(x)$. By 2 and the choice of B', we may assume that the following holds:

3. $\rho(s_{x_n}(k), f(x_n)) > \epsilon/2$ for all k.

For each n select an element x'_n of the sequence s_{x_n} so that $x'_n \to x$. By 3, $f(x'_n) \not\to f(x)$ contradicting continuity of f at x. □

Proposition 7. *Let X be a subspace of R such that $R \setminus X$ is countable. Let f be a continuous function from X to a Hausdorff space. Then $|M_f| \leq \omega$ or $|M_f| = 2^\omega$.*

Proof: For each $k \in \omega \setminus \{0\}$, the set $S_k = \{(x,y) : |x-y| \geq 1/k, f(x) = f(y)\}$ is, obviously, closed in $X \times X$. If M_f is uncountable then so is S_k for some $k \in \omega \setminus \{0\}$. Since S_k is closed in $X \times X$ and $R \setminus X$ is countable, by Cantor's theorem, $|S_k| = 2^\omega$. Since $x \neq y$ and $f(x) = f(y)$ for each $(x,y) \in S_k$, the projection of S_k to each coordinate axis is a subset of M_f. If the projection of S_k to the first coordinate axis is of cardinality 2^ω then we are done. Otherwise, there exists $x \in M_f$ such that $f^{-1}f(x)$ is uncountable. Since $f^{-1}f(x)$ is closed in X, by Cantor's theorem, it has cardinality 2^ω. □

Lemma 8. *R_L is absolutely submetrizable.*

Proof: Let $\mathcal{T}_t$ be a Tychonoff subtopology of the topology $\mathcal{T}$ of R_L. Since $\mathcal{T}|_L$ is separable and metrizable, $\mathcal{T}_t|_L$ has countable network. By Proposition 5, there exists a continuous map $f : (R, \mathcal{T}_t) \to [0,1]^\omega$ whose restriction on L is one-to-one. To show that $(R, \mathcal{T}_t)$ is submetrizable it suffices to prove that the set M_f of all points of non-one-to-one-ness of f is countable.

Assume M_f is uncountable. By Lemma 6, there exists $A \subset R$ such that $R \setminus A$ is countable and f is continuous at all points of A with respect to the Euclidean topology. Since M_f is uncountable, $M_{f|_A}$ is uncountable as well. By Proposition 7, $|M_{f|_A}| = 2^\omega$. Therefore, $f|_A = f_\alpha \in F$. By our construction, $a_\alpha, b_\alpha \in L$ and $f|_A(a_\alpha) = f|_A(b_\alpha)$, a contradiction with the fact that f is one-to-one on L. □

One of properties of the Bing-Hanner construction implies that $R_L \times D$, where $D = R \setminus L$ has the Euclidean topology, contains an uncountable closed discrete subset, namely, $\{(d,d) : d \in D\}$.

Corollary 9. *Absolute submetrizability is not finitely productive. Moreover, an absolutely submetrizable space times a separable metric space need not be absolutely submetrizable.*

Observe that the quotient space defined by the partition p on R_L whose only non-trivial element is L is not submetrizable. Indeed, the set D is not an F_σ-set in R_L because otherwise R_L would have contained an uncountable closed discrete subset, contradicting the Lindelöf property of R_L. Since D is not an F_σ-set, $p(L)$ is not a G_δ-set in $p(R_L)$. Since $p(L)$ is a one-point set, $p(R_L)$ has uncountable pseudocharacter. Hence, $p(R_L)$ is not submetrizable. Clearly, this quotient map is in addition closed. Thus, absolute submetrizability is not preserved by closed continuous maps.

Question 10. *Is absolute submetrizability preserved by continuous open maps, perfect maps?*

Now let us look how addition affects our property.

Example 11. *There exist two absolutely submetrizable spaces whose free sum is not absolutely submetrizable.*

Construction. Consider $R_L \oplus D$, where $D = R \setminus L$ has the Euclidean topology. By Fact 2, D is absolutely submetrizable. Let us show that this free sum is not absolutely submetrizable.

Let $X = R \cup R'$ be the Alexandroff double of R, where points of $R' = \{x' : x \in R\}$ are isolated. Let $Y = X \setminus \{x' : x \in L\}$. Clearly, $R_L \oplus D$ admits a continuous bijection onto Y. However, Y is not submetrizable because some non-isolated points of Y have neighborhoods with non-metrizable compact closures. $\square$

Question 12. *Let X be absolutely submetrizable. Is $X \times 2$ absolutely submetrizable?*

Next we study the behavior of our property in countable unions with some additional properties. For our further discussion a family $\mathcal{F}$ of subsets of X is called *Hausdorff separating* if for every distinct $x, y \in X$ there exist disjoint $F_x, F_y \in \mathcal{F}$ that contain x and y, respectively. If X admits a continuous bijection onto a separable metric space M then X has a countable Hausdorff separating family of closed subsets. To construct such a family take the inverse images of closures of elements of some countable base in M. It is also clear that if a normal (in particular, a regular Lindelöf) space X has a countable Hausdorff separating family of closed sets then X is submetrizable.

Proposition 13. *Let $X = \bigcup_n X_n$, where X_n is submetrizable and Lindelöf and $X_n \subset X_{n+1}$. Then X is submetrizable. Moreover, if every X_n is absolutely submetrizable then so is X.*

Proof: For each n, fix a countable Hausdorff separating family $\mathcal{S}_n$ of closed subsets of X_n.

For each n and each pair (P, S) of disjoint elements of $\mathcal{S}_n$ fix a countable family $\mathcal{S}_{PS}$ of closed sets in X such that $\bigcup \mathcal{S}_{PS}$ contains P and does not meet $\mathrm{Cl}_X(S)$. Let us show that such a family exists. Since P and S are closed and disjoint in X_n, the sets P and $\mathrm{Cl}_X(S)$ are disjoint as well. For each $x \in P$, fix $O_x \ni x$ open in X whose closure in X does not meet $\mathrm{Cl}_X(S)$. Since P is closed in X_n and the latter is Lindelöf, P is Lindelöf as well. Choose a countable $P' \subset P$ such that $\mathcal{S}_{PS} = \{\mathrm{Cl}_X(O_x) : x \in P'\}$ covers P.

Define $\mathcal{F}$ as follows: $F \in \mathcal{F}$ iff F is in some $\mathcal{S}_{PS}$ or is the closure in X of some element of $\mathcal{S}_n$. Clearly, $\mathcal{F}$ is a countable family of closed subsets of X. We only need to show that it is Hausdorff separating. Fix distinct $x, y \in X$. Then $x, y \in X_n$ for some n. Then there exist disjoint $P, S \in \mathcal{S}_n$ that contain x and y, respectively. Then there exists $F_x \in \mathcal{S}_{PS}$ that contains x and does not meet $F_y = \mathrm{Cl}_X(S) \ni y$. Since both F_x and F_y are in $\mathcal{F}$, we are done.

For "moreover" part observe that every continuous one-to-one image of X satisfies the hypothesis and, therefore, is submetrizable. $\square$

It is worth mentioning that the union of two Lindelöf submetrizable spaces need not be submetrizable. Such is the two arrows space. Recall that the two arrows space is not submetrizable because it is a non-metrizable compactum. Therefore, the requirement that $\{X_n\}_n$ is a chain is important. Also, the union of a countable chain of non-Lindelöf submetrizable spaces need not be submetrizable. Such is the Mrówka space [MRO] constructed from a maximal disjoint family on ω. Since the Mrówka space is the union of countably many closed discrete subspaces, it can be represented as a countable chain of even metrizable spaces. Since the Mrówka space is pseudocompact and non-compact it is not submetrizable.

Proposition 14. *Let $X = Y \cup S$, where Y is absolutely submetrizable and S is countable. Then X is absolutely submetrizable.*

Proof: Since any continuous one-to-one image of X satisfies the hypothesis of our proposition it suffices to show that X is submetrizable. By virtue of Proposition 13, we may assume that $S = \{p\}$. By Theorem 1, Y is Lindelöf. Therefore, p has countable pseudocharacter.

Now fix $\mathcal{S}$ a countable Hausdorff separating family of closed subsets of Y. Fix a countable nested family $\mathcal{B}$ of open neighborhoods of p in X such that $\bigcap \mathcal{B} = \{p\}$. Define $\mathcal{F}$ as follows: $F \in \mathcal{F}$ iff $F = S \setminus B$ for some $S \in \mathcal{S}$ and $B \in \mathcal{B}$ or $F = \{p\}$. Clearly, $\mathcal{F}$ is a countable Hausdorff separating family of closed subsets of X. $\square$

Note that Proposition 14 holds if we replace "absolute submetrizable" by "Lindelöf and submetrizable". In general, however, the union of a submetrizable space and a countable space need not be submetrizable. This is witnessed for example by a one-point Lindelöfication of an uncountable discrete space. Of course, in the above proposition, S can be replaced by a countable union of metrizable compact subspaces. We will show later that removing finitely many compact subsets from absolutely submetrizable spaces does not destroy absolute submetrizability either. However, it is not clear to the author what happens if we remove an infinite countable subset.

Question 15. *Let $X = Y \cup S$ be absolutely submetrizable, where S is countably infinite. Is Y absolutely submetrizable?*

Since R_L has an open discrete uncountable subspace, we conclude that absolute submetrizability is not inherited by open subspaces. It is easy to prove that the property is inherited by closed subspaces and even open F_σ-subspaces.

Theorem 16. *Absolute submetrizability is inherited by F_σ-subspaces.*

Proof: By virtue of Proposition 13, it suffices to prove the conclusion for closed subspaces. Let X be absolutely submetrizable and let A be a closed subspace of X. Let $h : A \to Y \subset [0,1]^\tau$ be a continuous one-to-one map of A onto Y. Since X is Lindelöf and A is closed, there exists a continuous $\tilde{h} : X \to [0,1]^\tau$ that coincides with h on A. Since X is Tychonoff and A is closed in X there exists a continuous map f from X to some cube $[0,1]^\kappa$ with the following properties:

1. A is mapped to the point with all coordinates 0;
2. $f(A) \cap f(X \setminus A) = \emptyset$;
3. f is one-to-one on $X \setminus A$.

The map $G = f \Delta \tilde{h} : X \to [0,1]^\kappa \times [0,1]^\tau$ is, clearly, a continuous injection. Hence, $G(X)$ is submetrizable. By 1, $G(A)$ is homeomorphic to Y. Therefore, Y is submetrizable. □

This theorem implies, in particular, that $X \setminus C$ is submetrizable if X is submetrizable and C is compact. Indeed, since X is submetrizable, $X \setminus C$ is an F_σ-subset of X, and therefore, absolutely submetrizable.

Question 17. *Suppose that every open subspace of X is absolutely submetrizable. Is every subspace of X absolutely submetrizable?*

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