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# A non-metrizable collectionwise Hausdorff tree with no uncountable chains and no Aronszajn subtrees

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*Abstract.* It is independent of the usual (ZFC) axioms of set theory whether every collectionwise Hausdorff tree is either metrizable or has an uncountable chain. We show that even if we add "or has an Aronszajn subtree," the statement remains ZFC-independent. This is done by constructing a tree as in the title, using the set-theoretic hypothesis $\diamondsuit^*$, which holds in Gödel's Constructible Universe.

*Keywords:* tree, collectionwise Hausdorff, metrizable, Aronszajn tree

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## 1. Introduction

The interval topology on trees is a rich source of examples of locally compact spaces, particularly under extra set-theoretic hypotheses, such as the axiom $\diamondsuit^*$ which will be used in this paper. The interplay between order and topology in trees also highlights the contrast between various set-theoretic hypotheses. For example, Theorem 3.3 of [1] has the following corollary:

**Theorem A.** *An Aronszajn tree is collectionwise Hausdorff iff it does not have a stationary antichain.*

(Terminology relating to trees is explained below.) A corollary is that every Souslin tree is collectionwise Hausdorff. On the other hand, MA($\aleph_1$) implies that all Aronszajn trees are special, and hence that there are no Souslin trees and no collectionwise Hausdorff Aronszajn trees. MA($\aleph_1$) also implies [4, Theorem 3.1] that every collectionwise Hausdorff tree is either metrizable or has an uncountable chain. In the first author's doctoral dissertation [2], the set-theoretic hypothesis was weakened to one involving Aronszajn trees:

**Theorem B.** *If every Aronszajn tree is special, then every collectionwise Hausdorff tree is either metrizable or has an uncountable chain.*

It is natural to inquire, in light of Theorem A, whether the set-theoretic hypothesis in Theorem B can be weakened to "every Aronszajn tree has a stationary antichain". In the main part of the first author's dissertation [2], it was shown that the answer is negative:

**Theorem C.** *It is consistent with ZFC and also with ZFC+CH that every Aronszajn tree has a stationary antichain, and that there is a collectionwise Hausdorff tree which is not metrizable and does not have an uncountable chain.*

Theorem C was established by an iterated forcing beginning with a ground model of $\diamondsuit^*$. A specific tree $T$ was constructed in the ground model to satisfy (01) through (04) below. In the intermediate forcing stages, all Aronszajn trees were given stationary antichains (and thus rendered non-collectionwise Hausdorff), while preserving all four key properties of $T$. These were:

(01) $T$ is collectionwise Hausdorff,

(02) $T$ is not metrizable,

(03) $T$ has no uncountable chains, and

(04) $T$ has no Aronszajn subtrees.

The main theorem of this paper is that such a tree exists if one assumes $\diamondsuit^*$ (Theorem 3.4). As far as we know, it is the only construction of such a tree under any hypotheses. From Theorem B it follows that, under MA($\aleph_1$), there does not exist a tree satisfying (01), (02) and (03). On the other hand, $\diamondsuit$ implies that there exists a Souslin tree, which satisfies (01), (02) and (03) (but obviously not (04)). To establish Theorem C, the tree $T$ we construct must satisfy (04) because if $T$ contains an Aronszajn subtree in the ground model, then $T$ would contain one in forcing extension where every Aronszajn tree has a stationary antichain, which implies that $T$ would not be collectionwise Hausdorff anymore in the forcing extension.

For a construction of a Souslin tree using $\diamondsuit$, see e.g. Theorem 7.8 in Chapter 2 of [3]. Our $\diamondsuit^*$ construction is a modification which employs two disjoint stationary subsets of $\omega_1$ in contrasting ways.

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**Definitions and notations 1.1.** A partial ordered set $\langle T, <_T \rangle$ is a *tree* iff for every $t \in T$, $\{s \in T : s <_T t\}$ is a well-ordered set. If there is no confusion, we simply write $<$ for $<_T$. Also for simplicity we write $T$ for $\langle T, <_T \rangle$.

For an element $t$ of a tree $T$, the *height* of $t$, denoted by $\mathrm{ht}_T(t)$, is the order type of $\{s \in T : s <_T t\}$. If it is clear that $T$ is referred to, then we simply write $\mathrm{ht}(t)$ for $\mathrm{ht}_T(t)$. The set $\{t \in T : \mathrm{ht}_T(t) = \alpha\}$ is called the $\alpha$-th *level* of $T$, and it is denoted by $T(\alpha)$. The *height of a tree* $T$, denoted by $\mathrm{ht}(T)$, is the least ordinal $\alpha$ such that $T(\alpha) = \emptyset$. We let $T{\restriction}\alpha = \bigcup_{\xi<\alpha} T(\xi)$.

We let $\hat{t} = \{x \in T : x \leq_T t\}$, and we say that a subtree $S$ of $T$ is *downward closed*, provided for all $t \in S$, $\hat{t} \subseteq S$.

A *chain* in a tree $T$ is a linearly ordered subset of $T$. An *antichain* in a tree is a set of pairwise incomparable elements.

$T$ is an $\omega_1$-tree iff $\mathrm{ht}(T) = \omega_1$ and for all $\alpha < \omega_1$, $|T(\alpha)| \leq \aleph_0$. $T$ is an *Aronszajn tree* iff $T$ is an $\omega_1$-tree and every chain is countable. $T$ is a *Souslin tree* iff $T$ is an Aronszajn tree and every antichain is countable.

We say $A$ is a *stationary antichain* of an $\omega_1$-tree $T$ if $\{\alpha \in \omega_1 : A \cap T(\alpha) \neq \emptyset\}$ is stationary.

The *tree topology* (also known as the *interval topology* on a tree $T$) is the topology such that if $t \in T$ is not a minimal element, then all sets of the form $(s, t] := \{x \in T : s <_T x \leq_T t\}$ constitute a local base at $t$, and if $t \in T$ is a minimal element, then $\{t\}$ is an open set. We only consider Hausdorff trees. Note that a tree $T$ is Hausdorff iff whenever $\{s \in T : s < t\} = \{s \in T : s < t'\}$, we have $t = t'$.

A topological space $X$ is *collectionwise Hausdorff* iff for every closed discrete subspace $D$, there exists a family of disjoint open sets $\{U_d : d \in D\}$ such that $U_d \cap D = \{d\}$ for each $d \in D$. Note that a subspace $D$ of a tree is closed discrete iff there is no sequence $\{d_n \in D : n \in \omega\}$ with $d_n < d_{n+1}$ for all $n \in \omega$ converging to some $t$.

A topological space is *$\omega$-fair*, provided every countable subspace has a countable closure.

For ordinals $\alpha$ and $\beta$, we let

$$^{<\alpha}\beta = \{x : x \text{ is a function}, \ \ \mathrm{dom}(x) < \alpha \ \text{ and } \ \mathrm{ran}(x) \subseteq \beta\},$$

where $\mathrm{dom}(x)$ is the domain of $x$ and $\mathrm{ran}(x)$ is the range of $x$. We consider $^{<\alpha}\beta$ as a tree ordered by inclusion. For the sake of simplicity, we assume that every function in $^{<\alpha}\beta$ has non-empty domain.

## 2. Downward closed subtrees of $^{<\omega_1}\omega_1$

In this section, we study a downward closed subtree $T$ of $^{<\omega_1}\omega_1$ such that $|T| \leq \aleph_1$ and $T$ is $\omega$-fair; the height of $T$ is $\leq \omega_1$, but $T$ can have a level which contains uncountable many nodes.

Here we introduce the notation $T_\alpha$, which is useful when one deals with a subtree of $^{<\omega_1}\omega_1$.

**Definition 2.1.** Suppose that $T$ is a subtree of $^{<\omega_1}\omega_1$. For each $\alpha \in \omega_1$, we define

$$T_\alpha = \{x \in T : \mathrm{dom}(x) < \alpha, \, \mathrm{ran}(x) \subseteq \alpha \ \text{ and } \ \mathrm{ran}(x) \text{ is not cofinal in } \alpha\}.$$

Note that some authors use $T_\alpha$ for the $\alpha$-th level of $T$, which is $T(\alpha)$ in our notation.

**Remark 2.2.** If $\alpha$ is a successor ordinal, say $\alpha = \beta + 1$, then

$$T_\alpha = \{x \in T : \mathrm{dom}(x) \leq \beta \;\text{ and }\; \mathrm{ran}(x) \subseteq \beta\}.$$

We investigate properties of $T_\alpha$.

**Lemma 2.3.** *Suppose that $T$ is a downward closed subtree of ${}^{<\omega_1}\omega_1$. The following statements are true:*

1. *$T_\alpha$ is downward closed for all $\alpha \in \omega_1$;*
2. *$T_\alpha$ is open for all $\alpha \in \omega_1$;*
3. *if $\alpha$ is a successor ordinal, then $T_\alpha$ is closed;*
4. *if $\alpha$ is a limit ordinal, then $T_\alpha = \bigcup_{\xi<\alpha} T_\xi$;*
5. *if $T$ is $\omega$-fair, then $T_\alpha$ is countable for all $\alpha \in \omega_1$.*

Proof: We will show (3) and (5). To prove (3), suppose that $\alpha = \beta + 1$, and let $x \in T \setminus T_\alpha$; then either $\mathrm{dom}(x) > \beta$ or $\mathrm{ran}(x) \not\subseteq \beta$ by Remark 2.2. If $\mathrm{dom}(x) > \beta$, then $(x{\upharpoonright}\beta, x]$ is a neighborhood of $x$ missing $T_\alpha$. If $\mathrm{ran}(x) \not\subseteq \beta$, then $x(\xi) \notin \beta$ for some $\xi \in \mathrm{dom}(x)$, and $(x{\upharpoonright}\xi, x]$ is a neighborhood of $x$ missing $T_\alpha$.

To prove (5), first observe that $T_\alpha = \bigcup_{\xi<\alpha}\{x \in T_\alpha : \mathrm{dom}(x) = \xi\}$. So it is enough to show that $E_\xi := \{x \in T_\alpha : \mathrm{dom}(x) = \xi\}$ is countable for all $\xi < \alpha$. We do this by induction. For $n < \omega$, $E_n$ is countable because for all $x \in E_n$ $\mathrm{dom}(x) = n - 1$ and $\mathrm{ran}(x) \subseteq \alpha$. Suppose that $E_\xi$ is countable for all $\xi < \gamma (< \alpha)$.

**Case 1:** $\gamma$ is a successor ordinal.
Suppose $\gamma = \xi + 1$. Observe that

$$E_\gamma \subseteq \{x \cup \{\langle \xi, \eta\rangle\} : x \in E_\xi \;\text{ and }\; 0 \leq \eta < \alpha\}.$$

Since $E_\xi$ is countable, so is $E_\gamma$.

**Case 2:** $\gamma$ is a limit ordinal.
Observe that

$$E_\gamma \subseteq \overline{\{x \in T_\alpha : \mathrm{dom}(x) < \gamma\}} = \overline{\bigcup_{\xi<\gamma} E_\xi}.$$

Since $T$ is $\omega$-fair and $\bigcup_{\xi<\gamma} E_\xi$ is countable, we have that $E_\gamma$ is countable. □

Observe that if $\mathrm{ht}(T) \leq \omega_1$, then for every $t \in T$, $\{s \in T : s < t\}$ is isomorphic to a countable ordinal so $t$ has a countable compact nbhd and so $T$ is locally metrizable, which implies that every countable subspace of $T$ is metrizable. We will use the next lemma to show that the tree we shall construct is not metrizable. Recall that every metrizable space is paracompact.

**Lemma 2.4.** *Suppose $T \subseteq {}^{<\omega_1}\omega_1$ is a downward closed subtree such that $|T| \leq \aleph_1$ and $T$ is $\omega$-fair. Then the following are equivalent:*

(1) *$T$ is paracompact;*
(2) *$\{\alpha \in \omega_1 : \overline{T_\alpha} \setminus T_\alpha \neq \emptyset\}$ is not a stationary subset of $\omega_1$.*

Proof: (1)$\Longrightarrow$(2). Suppose that $S = \{\alpha \in \omega_1 : \overline{T_\alpha} \setminus T_\alpha \neq \emptyset\}$ is stationary. Let $\mathcal{U}$ be an open refinement of the open cover $\{T_\alpha : \alpha \in \omega_1\}$ of $T$. We will show that $\mathcal{U}$ is not locally finite. For each $\alpha \in S$, pick $x_\alpha \in \overline{T_\alpha} \setminus T_\alpha$, and choose $U_\alpha \in \mathcal{U}$ such that $x_\alpha \in U_\alpha$. Then we can pick $y_\alpha \in U_\alpha$ such that $y_\alpha < x_\alpha$, and we can choose $\alpha' < \alpha$ such that $y_\alpha \in T_{\alpha'}$. By Fodor's Theorem, there exists a $\beta \in \omega_1$ such that $\{\alpha : \alpha' = \beta\}$ is stationary. Since $T_\beta$ is countable (Lemma 2.3(5)), there exists an $x \in T_\beta$ such that $\{\alpha : x \in U_\alpha\}$ is stationary.

(2)$\Longrightarrow$(1). If $\{\alpha \in \omega_1 : \overline{T_\alpha} \setminus T_\alpha = \emptyset\}$ contains a club subset of $\omega_1$, say $C = \{\alpha_\xi : \xi < \omega_1\}$ (enumerated in increasing order), then $T = \bigcup_{\xi<\omega_1}(T_{\alpha_{\xi+1}} \setminus \overline{T_{\alpha_\xi}})$, and for every $\xi \in \omega_1$, $T_{\alpha_{\xi+1}} \setminus \overline{T_{\alpha_\xi}}$ is a clopen metrizable subspace of $T$. If a space is the union of clopen paracompact subspaces, then it is paracompact. □

We will use the following lemma to show that the tree we shall construct is collectionwise Hausdorff.

**Lemma 2.5.** *Suppose $T \subseteq {}^{<\omega_1}\omega_1$ is a downward closed subtree such that $|T| \leq \aleph_1$ and $T$ is $\omega$-fair. Then the following are equivalent:*

(1) *$T$ is collectionwise Hausdorff;*
(2) *for every antichain $A$ of $T$, $\{\alpha \in \omega_1 : A \cap (\overline{T_\alpha} \setminus T_\alpha) \neq \emptyset\}$ is not a stationary subset of $\omega_1$.*

Proof: (1)$\Longrightarrow$(2). A proof is similar to that of Lemma 2.4.

(2)$\Longrightarrow$(1). Let $D$ be a closed discrete subspace of $T$. Let $A_0 = \{d \in D : (\forall\, t < d)(t \notin D)\}$, and for $n > 0$ let $A_{n+1} = \{d \in D \setminus \bigcup_{k\leq n} A_k : (\forall\, t < d)(t \notin D \setminus \bigcup_{k\leq n} A_k)\}$; then each $A_n$ is an antichain of $T$ and $D = \bigcup_{n\in\omega} A_n$ (see the comment about discrete spaces of a tree in Definitions and notations 1.1). By the assumption, we can find a club subset $C_n$ of $\omega_1$ for each $n$ such that for each $\alpha \in C_n$, $A_n \cap (\overline{T_\alpha} \setminus T_\alpha) = \emptyset$. Let $C = \bigcap_{n\in\omega} C_n$, and use the same argument as in the proof of Lemma 2.4. □

**Lemma 2.6.** *Suppose that $T$ is a tree such that $\mathrm{ht}(T) = \omega_1$. If $U$ is an $\omega_1$-subtree of $T$, then $\bigcup\{\widehat{u} : u \in U\}$ is also an $\omega_1$-subtree of $T$.*

Proof: Omitted. □

The following lemma will be used to show the tree we will construct has neither uncountable chains nor Aronszajn subtrees.

**Lemma 2.7.** *Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a downward closed subtree such that $T$ is $\omega$-fair and $\{\alpha \in \omega_1 : \overline{T_\alpha} \setminus T_\alpha = \emptyset\}$ is a stationary subset of $\omega_1$. Then $T$ has no $\omega_1$-subtrees; equivalently, it has neither uncountable chains nor Aronszajn subtrees.*

Proof: Let $S = \{\alpha \in \omega_1 : \overline{T_\alpha} \setminus T_\alpha = \emptyset$ and $\alpha$ is a limit ordinal$\}$. Assume, on the contrary, that $T$ has an $\omega_1$-subtree $U$. By Lemma 2.6, we may assume that $U$ is downward closed.

**Claim.** $\{\alpha \in \omega_1 : U(\alpha) \subseteq \overline{T_\alpha} \setminus T_\alpha\}$ is a club subset of $\omega_1$.

Assuming the claim holds, we can find $\alpha \in S$ such that $U(\alpha) \subseteq \overline{T_\alpha} \setminus T_\alpha$. This implies that $U(\alpha) = \emptyset$, a contradiction. Now it remains to show the claim.

Proof of Claim: To show the set is unbounded, fix an arbitrary $\alpha_0 \in \omega_1$. Take $\alpha_1 > \alpha_0$ such that $U\restriction \alpha_0 \subseteq T_{\alpha_1}$; then choose $\alpha_2 > \alpha_1$ so that $T_{\alpha_1} \cap U \subseteq U\restriction \alpha_2$; this is possible because $T_{\alpha_1}$ is countable (Lemma 2.3(5)). Take $\alpha_3 > \alpha_2$ so that $U\restriction \alpha_2 \subseteq T_{\alpha_3}$. Continuing in the same way, let $\beta = \sup\{\alpha_n : n \in \omega\}$. We have $U\restriction \beta \subseteq T_\beta$ so $U(\beta) \subseteq \overline{T_\beta}$. For every $x \in U(\beta)$, $\mathrm{dom}(x) = \beta$ so $x \notin T_\beta$. We can show that the set is closed in a similar way. □

**Corollary 2.8.** *$\omega_1$-trees are not metrizable. In particular, Aronszajn trees are not metrizable.*

Proof: Suppose that $T$ is an $\omega_1$-tree, i.e. $\mathrm{ht}(T) = \omega_1$ and $|T(\alpha)| \le \aleph_0$ for all $\alpha < \omega_1$. By the claim in Lemma 2.7, $\{\alpha \in \omega_1 : T(\alpha) \subseteq \overline{T_\alpha} \setminus T_\alpha\}$ contains a club subset of $\omega_1$. Since $T(\alpha) \neq \emptyset$ for each $\alpha \in \omega_1$, $T$ is not paracompact by Lemma 2.4. □

## 3. Construction of the tree

In this section, we construct the tree mentioned in the beginning of this paper.

**Notation 3.1.**
- $\Lambda = \{\alpha \in \omega_1 : \alpha$ is a limit ordinal$\}$
- $\Lambda_{\mathrm{suc}} = \{\alpha \in \Lambda : \alpha = \beta + \omega$ for some $\beta \in \omega_1\}$
- $\Lambda_{\mathrm{lim}} = \Lambda \setminus \Lambda_{\mathrm{suc}}$

**Definition 3.2.** $\diamondsuit^*$ is the following statement: There exists a sequence $\langle \mathcal{A}_\alpha : \alpha \in \Lambda\rangle$ of subsets of $\mathcal{P}(\omega_1)$ such that:

(1) $\forall\, \alpha \in \Lambda\ (\mathcal{A}_\alpha \subseteq \mathcal{P}(\alpha))$,
(2) $\forall\, \alpha \in \Lambda\ (\mathcal{A}_\alpha$ is countable), and
(3) $\forall\, X \subseteq \omega_1\ \{\alpha \in \Lambda : X \cap \alpha \in \mathcal{A}_\alpha\}$ contains a club subset of $\omega_1$.

We use the following lemma to show the tree, which we will construct, has no antichain which meets stationary many $\overline{T_\alpha} \setminus T_\alpha$'s.

**Lemma 3.3.** *Suppose that $T \subseteq {}^{<\omega_1}\omega_1$ is a downward closed subtree tree such that $|T| \leq \aleph_1$ and $T$ is $\omega$-fair, and that $M$ is a maximal antichain of $T$; then*

$$\{\alpha \in \omega_1 : M \cap T_\alpha \text{ is a maximal antichain of } X_\alpha\}$$

*is a club subset of $\omega_1$.*

Proof: A proof is similar to that of Lemma 2.7.6(b) in [3]. □

Now, we construct the tree.

**Theorem 3.4.** *Assuming $\diamondsuit^*$, there exists a tree $T$ such that*

1. *$T$ is collectionwise Hausdorff,*
2. *$T$ is not metrizable,*
3. *$T$ has no uncountable chains, and*
4. *$T$ has no Aronszajn subtrees.*

Proof: Fix a $\diamondsuit^*$-sequence $\{\mathcal{A}_\alpha \subseteq \mathcal{P}(\omega_1) : \alpha \in \Lambda\}$. Along with the construction, we will enumerate $T = \{t_\xi : \xi < \omega_1\}$ so that if $t_\xi \in T_\alpha$ and $t_\eta \in T_\beta \setminus T_\alpha$, then $\xi < \eta$. Observe that $\{\alpha \in \omega_1 : \{\xi \in \omega_1 : t_\xi \in T_\alpha\} = \alpha\}$ is a club subset of $\omega_1$. For $\alpha \in \Lambda$, we define

$$\mathcal{F}_\alpha = \{\{t_\xi : \xi \in A\} : A \in \mathcal{A}_\alpha\}.$$

Observe that for every subset $U$ of $T$, $\{\alpha \in \omega_1 : U \cap T_\alpha \in \mathcal{F}_\alpha\}$ contains a club subset of $\omega_1$ by the property of a $\diamondsuit^*$-sequence.

Fix a stationary and co-stationary subset $S_1$ of $\Lambda_{\text{lim}}$, and let $S_2 = \Lambda_{\text{lim}} \setminus S_1$. We will construct a downward closed subtree $T$ of ${}^{<\omega_1}\omega_1$ by induction on limit ordinals $\alpha$ with $T_{\alpha+1}$ so that

(a) $T_{\alpha+1}$ is downward closed,
(b) $(\forall\, x \in T_{\alpha+1})(x$ is non-decreasing),
(c) $(\forall\, \alpha \in \omega_1)(T_{\alpha+1}$ is countable),
(d) $(\forall\, \alpha \in S_1)(\overline{T_\alpha} \setminus T_\alpha = \emptyset)$,
(e) $(\forall\, \alpha \in S_2)(\overline{T_\alpha} \setminus T_\alpha \neq \emptyset)$,
(f) $(\alpha \in S_2$ and $x \in \overline{T_\alpha} \setminus T_\alpha) \Longrightarrow (\text{dom}(x) = \alpha)$,
(g) $(\alpha \in \Lambda_{\text{suc}}$ and $x \in T_\alpha) \Longrightarrow (\exists\, y \in \overline{T_\alpha})(x < y$ and $\text{dom}(y) = \alpha)$, and
(h) $(\alpha \in S_2$ and $F \in \mathcal{F}_\alpha$ is a maximal antichain of $T_\alpha) \Longrightarrow (\forall\, y \in \overline{T_\alpha}$ with $\text{dom}(y) = \alpha)(\exists\, x \in F)(x < y)$.

Now we start the construction. Let $T_{\omega+1} = \{x \in {}^{\leq\omega}\omega : x \equiv 0\}$. Suppose that we have constructed $T_{\xi+1}$ for all $\xi \in \Lambda$ with $\xi < \alpha$.

**Case 1:** $\alpha \in \Lambda_{\text{suc}}$.
Suppose that $\alpha = \beta + \omega$, where $\beta \in \Lambda$. Let

$$T_{\alpha+1} = T_{\beta+1} \cup \{y : y = x \cup \{\langle \xi, \beta\rangle : \text{dom}(x) \leq \xi < \gamma\} \text{ for some } x \in T_{\beta+1} \text{ and for some } \gamma \leq \alpha\}.$$

We have to make sure that we did not add new functions to $T_{\beta+1}$; every new function contains $\beta$ in its range, so it does not belong to $T_{\beta+1}$.

To show (g) is satisfied, fix an arbitrary $x$ in $T_\alpha$. If $x \in T_{\beta+1}$, then let $y = x \cup \{\langle \xi, \beta \rangle : \mathrm{dom}(x) \leq \xi < \alpha\}$, and $y$ works. If $x \notin T_{\beta+1}$, then there exist an element $z \in T_{\beta+1}$ and an ordinal $\gamma < \alpha$ such that $x = z \cup \{\langle \xi, \beta \rangle : \mathrm{dom}(z) \leq \xi < \gamma\}$. Let $y = z \cup \{\langle \xi, \beta \rangle : \mathrm{dom}(z) \leq \xi < \alpha\}$; then $y$ is as required.

**Case 2:** $\alpha \in \Lambda_{\mathrm{lim}}$.
In this case, $T_\alpha$ is already defined because $T_\alpha = \bigcup\{T_\xi : \xi < \alpha \text{ and } \xi \in \Lambda\}$.

**Subcase 2.1:** $\alpha \in S_1$.
We simply let $T_{\alpha+1} = T_\alpha$. It is easy to see that $T_{\alpha+1}$ satisfies (a)–(h). (For (d), notice that $\overline{T_\alpha} \subseteq T_{\alpha+1}$ because $T_{\alpha+1}$ is closed (Lemma 2.3).)

**Subcase 2.2:** $\alpha \in S_2$.
Let $\{F_n : n \in \omega\}$ enumerate $\{F \in \mathcal{F}_\alpha : F \text{ is a maximal antichain of } T_\alpha\}$. Fix $x \in T_\alpha$ and an increasing sequence $\langle \beta_n : n \in \omega \rangle$ such that $\sup\{\beta_n : n \in \omega\} = \alpha$. We will define a chain $\{x_n : n \in \omega\}$ in $T_\alpha$ and an increasing sequence $\langle \alpha_n : n \in \omega \rangle$ such that $\alpha_n \in \Lambda_{\mathrm{suc}}$ for all $n \geq 0$ so that

- $\beta_n \leq \alpha_n < \alpha$ for all $n \geq 0$,
- $x_n \in \overline{T_{\alpha_n}}$,
- $\mathrm{dom}(x_n) = \alpha_n$, and
- $\widehat{x_n} \cap F_l \neq \emptyset$ for all $l < n$.

Take $\alpha_0 \in \Lambda_{\mathrm{suc}}$ such that $x \in T_{\alpha_0}$ and $\alpha_0 \geq \beta_0$. Using the item (g), take $x_0 \geq_{T_\alpha} x$ such that $x_0 \in \overline{T_{\alpha_0}}$ and $\mathrm{dom}(x_0) = \alpha_0$. Suppose that we have picked $x_n$ and $\alpha_n$ satisfying the above.

If $\widehat{x}_n \cap F_n = \emptyset$, then pick $y \in F_n$ so that $x_n <_{T_\alpha} y$. Take $\alpha_{n+1} > \alpha_n$ such that $y \in T_{\alpha_{n+1}}$. Using the item (g), pick $x_{n+1} \in \overline{T_{\alpha_{n+1}}}$ such that $y <_{T_\alpha} x_{n+1}$ and $\mathrm{dom}(x_{n+1}) = \alpha_{n+1}$.

If $\widehat{x}_n \cap F_n \neq \emptyset$, then pick $\alpha_{n+1} > \alpha_n$ such that $x_n \in T_{\alpha_{n+1}}$, and using the item (g) pick $x_{n+1} \in \overline{T_{\alpha_{n+1}}}$ such that $x_n <_{T_\alpha} x_{n+1}$ and $\mathrm{dom}(x_{n+1}) = \alpha_{n+1}$.

Now, we have obtained a chain $\{x_n : n \in \omega\}$ in $T_\alpha$. Let

$$y^* = \bigcup\{x_n : n \in \omega\}.$$

We have $\mathrm{dom}(y^*) = \bigcup_{n\in\omega} \mathrm{dom}(x_n) = \bigcup_{n\in\omega} \alpha_n = \alpha$ and for all $n \in \omega$, $\widehat{y^*} \cap F_n \neq \emptyset$. Finally, let

$$T_{\alpha+1} = T_\alpha \cup \{y^*\}.$$

It is easy to see that $T_{\alpha+1}$ satisfies (a)–(g). (For (e), observe that $y^* \in \overline{T_\alpha} \setminus T_\alpha$.) So let us check if $T_{\alpha+1}$ satisfies (h). Suppose $F \in \mathcal{F}_\alpha$ is a maximal antichain of $T_\alpha$. Pick an arbitrary $y$ from $\overline{T_\alpha}$ such that $\mathrm{dom}(y) = \alpha$; then $y = y^*$ and we know that $\widehat{y^*} \cap F \neq \emptyset$ so there is an $x' < y$ such that $x' \in F$. This finishes the construction.

Now, we have to verify that $T$ satisfies (1)–(4).

$T$ is not metrizable because of the item (e) and Lemma 2.4. For (3) and (4), first observe that $T$ is $\omega$-fair because of the item (c), and by the item (d) and Lemma 2.7 these hold.

So it remains to show that $T$ is collectionwise Hausdorff. For contradiction, assume that $T$ is not collectionwise Hausdorff. By Lemma 2.5, there exists an antichain $A$ in $T$ such that

$$E := \{\alpha \in \omega_1 : A \cap (\overline{T_\alpha} \setminus T_\alpha) \neq \emptyset\}$$

is a stationary subset of $\omega_1$. We may assume that $E \subseteq \Lambda_{\text{lim}}$. By the item (d), this implies that $E \subseteq S_2$. Take a maximal antichain $M$ of $T$ containing $A$ and let

$$C_1 = \{\alpha \in \omega_1 : M \cap T_\alpha \text{ is a maximal antichain in } T_\alpha\}.$$

By Lemma 3.3, $C_1$ is a club subset of $\omega_1$. Let

$$C_2 = \{\alpha \in \omega_1 : M \cap T_\alpha \in \mathcal{F}_\alpha\}.$$

$C_2$ contains a club subset of $\omega_1$ by the property of a $\diamondsuit^*$-sequence. Pick $\alpha$ from $E \cap C_1 \cap C_2$ and $x$ from $A \cap (\overline{T_\alpha} \setminus T_\alpha)$; then $\text{dom}(x) = \alpha$ by (f). By the item (h) and the fact that $M \cap T_\alpha$ is a maximal antichain in $T_\alpha$, there exists an $y$ in $M \cap T_\alpha$ such that $y <_T x$, but this is a contradiction because $x \in M$. This finishes the proof of the theorem. □

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