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Weak orderability of some spaces which admit a weak selection

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Abstract. We show that if a Hausdorff topological space X satisfies one of the following properties:
a) X has a countable, discrete dense subset and X^2 is hereditarily collectionwise Hausdorff;
b) X has a discrete dense subset and admits a countable base;
then the existence of a (continuous) weak selection on X implies weak orderability. As a special case of either item a) or b), we obtain the result for every separable metrizable space with a discrete dense subset.

Keywords: weak (continuous) selection, weak orderability, Vietoris topology, dense countable subset, isolated point, countable base, collectionwise Hausdorff space

Classification: Primary 54C65, 54F05; Secondary 54D15, 54D70, 54E35

1. Introduction

A *weak selection* on a topological space X is a function f associating to every non-ordered pair of elements of X an element of the pair, in such a way that f is continuous with respect to the Vietoris topology (for a detailed definition, see the beginning of the next section).

The link between weak selections and orderability properties of a space traces back to [6, Lemma 7.2], which states that a connected Hausdorff space X admits a weak selection if and only if there exists a linear order on X such that its related order topology is coarser than (or equal to) the original topology of X (i.e., X is *weakly orderable*).

30 years later, van Mill and Wattel in [7] proved that the same result holds if we replace the assumption of connectedness on X with that of compactness (as a relevant consequence of this fact, they obtain that for a compact space X there exists a Vietoris-continuous selection on the space of all nonempty closed subsets of X if and only if the topology of X is generated by a linear order). In the same paper, the authors raise explicitly the question of whether the assumption of compactness may be dropped in their result.

In the subsequent years, the hypothesis of compactness on X to have the implication

$$\text{existence of weak selection} \Longrightarrow \text{weak orderability}$$

(which actually turns out to be an equivalence, as the reverse implication always holds) was relaxed, in the realm of Tychonoff spaces, first to countable compactness in [1], and then to pseudompactness in [4].

Moreover, in [3, Theorem 3.1] it is shown that the above implication holds for all Hausdorff countable spaces (actually, this could also have been proved using the original technique by van Mill and Wattel, which does not require any assumption of compactness as long as transfinite steps are not involved in the inductive argument).

In the present paper, we give a further contribution to this problem, showing that if a (Hausdorff) space X either has a countable, discrete dense subset and a hereditarily collectionwise Hausdorff square, or has a dense discrete subset and a countable base (in particular, if X is a separable metrizable space with a dense discrete subset), then weak orderability of X is still equivalent to the existence of a weak selection.

2. Basic facts and definitions

Let X be a T_2-topological space and $\tilde{\mathcal{F}}_2(X) = [X]^2$ be the collection of all subsets of X having exactly 2 elements. Let V be the Vietoris topology on $\tilde{\mathcal{F}}_2(X)$; for every $C = \{x, y\} \in \tilde{\mathcal{F}}_2(X)$, a fundamental system of (open) neighbourhoods for C with respect to V is then given by:

$$\big\{\{\{x', y'\} \,|\, x' \in V,\, y' \in W\} \,\big|\, V, W \ \text{ open in } \ X,\, x \in V,\, y \in W,\, V \cap W = \emptyset\big\}.$$

Whenever f is a selection from $\big(\tilde{\mathcal{F}}_2(X), \mathsf{V}\big)$ to X, following [5] we will denote by $\preccurlyeq_f$ the binary relation on X defined by:

$$x \preccurlyeq_f y \Longleftrightarrow \big(x = y \quad \vee \quad (x \neq y \quad \wedge \quad f(\{x, y\}) = x)\big);$$

(of course, $x \prec_f y$ will then mean $x \preccurlyeq_f y$ and $x \neq y$). Observe that $\preccurlyeq_f$ is not an order (nor a preorder) relation, in general, as it is not transitive. For every $x \in X$, we set:

$$(\leftarrow, x)_{\preccurlyeq_f} = \{y \in X \,|\, y \prec_f x\}, \quad (\leftarrow, x]_{\preccurlyeq_f} = \{y \in X \,|\, y \preccurlyeq_f x\}$$

and, symmetrically:

$$(x, \rightarrow)_{\preccurlyeq_f} = \{y \in X \,|\, x \prec_f y\}, \quad [x, \rightarrow)_{\preccurlyeq_f} = \{y \in X \,|\, x \preccurlyeq_f y\}.$$

Notice that if f is continuous, then for every $x \in X$ both $(\leftarrow, x)_{\preccurlyeq_f}$ and $(x, \rightarrow)_{\preccurlyeq_f}$ are open subsets of X. Indeed, let for example $\bar{x}, \bar{y} \in X$ with $\bar{y} \in (\bar{x}, \rightarrow)_{\preccurlyeq_f}$, i.e. $f(\{\bar{x}, \bar{y}\}) = \bar{x}$. Fix two disjoint neighbourhoods V and W of $\bar{x}$

and $\bar{y}$, respectively. Then by the Vietoris-continuity of f, since $f(\{\bar{x},\bar{y}\}) = \bar{x} \in V$, there exist V', W' open disjoint neighbourhoods of $\bar{x}$ and $\bar{y}$, respectively, such that

$$\forall x' \in V' : \forall y' \in W' : f(\{x', y'\}) \in V, \tag{2.1}$$

and of course we may further assume that $W' \subseteq W$. Then from (2.1) it follows in particular that $f\{\bar{x}, y'\} \in V$ for every $y' \in W'$: since $f(\{\bar{x}, y'\})$ is either $\bar{x}$ or y', and y' belongs to W' which is disjoint from V, we have the equality $f(\{\bar{x}, y'\}) = \bar{x}$ for every $y' \in W'$, i.e. W' is an open neighbourhood of y' included in $(\bar{x}, \rightarrow)_{\preccurlyeq_f}$.

In the following, we will often have to deal with a partial or linear ordering $\sqsubseteq$ defined on a topological space X. We will still adopt (for $x \in X$) the notations: $(\leftarrow, x)_{\sqsubseteq}$, $(\leftarrow, x]_{\sqsubseteq}$, $(x, \rightarrow)_{\sqsubseteq}$ and $[x, \rightarrow)_{\sqsubseteq}$ to denote, respectively, the sets: $\{y \in X \mid y \sqsubset x\}$, $\{y \in X \mid y \sqsubseteq x\}$, $\{y \in X \mid y \sqsupset x\}$ and $\{y \in X \mid y \sqsupseteq x\}$.

Proposition 2.1. *Let X be a topological space and, for every $n \in \omega$, let $\mathcal{A}_n = \{A_{n,i} \mid i \in I_n\}$ be a* (*cl*)*open partition of X, with $i \mapsto A_{n,i}$ one-to-one, and $\sqsubseteq_n$ be a linear order on the set I_n. Let also, for every $n \in \omega$, $j_n \colon X \to I_n$ be the* (*unique*) *function such that $x \in A_{n,j_n(x)}$ for every $x \in X$, and consider the relation $\sqsubseteq$ on X defined by:*

$$x \sqsubseteq y \Longleftrightarrow \Big(x = y \quad \vee \quad \exists\, n \in \omega : \big((\forall i < n \colon j_i(x) = j_i(y)) \quad \wedge \quad j_n(x) \sqsubset_n j_n(y)\big)\Big).$$

Then $\sqsubseteq$ is a (*not necessarily linear*) *ordering of X, and for every $x \in X$ the sets $(\leftarrow, \bar{x})_{\sqsubseteq}$ and $(\bar{x}, \rightarrow)_{\sqsubseteq}$ are open in X.*

Proof: The proof that $\sqsubseteq$ is an ordering of X is routine, as $\sqsubseteq$ is similar to a lexicographic order.

Now, let $\bar{x}$ be an arbitrary element of X. Observe that

$$(\leftarrow, \bar{x})_{\sqsubseteq} = \bigcup_{n \in \omega} \bigcup_{\substack{i \in I_n \\ i \sqsubset_n j_n(\bar{x})}} \left(A_{n,i} \cap \bigcap_{n' < n} A_{n', j_{n'}(\bar{x})}\right).$$

Since each $A_{n,i}$ with $n \in \omega$ and $i \in I_n$ is open, it follows that $(\leftarrow, \bar{x})_{\sqsubseteq}$ is open too.

A symmetric argument applies to $(\bar{x}, \rightarrow)_{\sqsubseteq}$. □

Remark. As is easy to see (keeping the same assumptions of the previous proposition), two distinct elements $x, y \in X$ are incomparable with respect to $\sqsubseteq$ if and only if $j_n(x) = j_n(y)$ for every $n \in \omega$. It follows that the binary relation on X, of being either equal or incomparable with respect to $\sqsubseteq$, is transitive. Indeed, if $x, y, z \in X$ and x is comparable with z but distinct from it, then for some $n \in \omega$ the points x and z belong to distinct (hence disjoint) elements of $\mathcal{A}_n$ — say A' and A''. Therefore, y cannot belong to both A' and A'', hence it is comparable with either x or z (or both).

Theorem 2.2. *Let X be a Hausdorff space having a countable dense subset D consisting of isolated points, and such that either $X \times X$ is hereditarily collectionwise Hausdorff or X has a countable base. Then, if X admits a weak selection, X is also weakly orderable.*

Proof: Put $D = \{a_n \mid n \in \omega\}$, and let $f: \tilde{\mathcal{F}}_2(X) \to X$ be a weak selection for X. Then, for every $n \in \omega$, the collection $\mathcal{A}_n = \{A_{n,0}, A_{n,1}, A_{n,2}\}$, where

$$A_{n,0} = (\leftarrow, a_n)_{\preccurlyeq_f}, \quad A_{n,1} = \{a_n\} \text{ and } A_{n,2} = (a_n, \to)_{\preccurlyeq_f},$$

is an open partition of X. Let $\sqsubseteq$ be the binary relation on X defined as:

$$x \sqsubseteq y \iff \Big(x = y \quad \vee \quad \exists\, n \in \omega : \big((\forall i < n \colon j_i(x) = j_i(y)) \quad \wedge \quad j_n(x) < j_n(y)\big)\Big)$$

where, for every $x \in X$ and $n \in \omega$, $j_n(x)$ is the unique $i \in \{0,1,2\}$ such that $x \in A_{n,i}$. Then it follows from Proposition 2.1 that $\sqsubseteq$ is an order on X such that, for every $x \in X$, the sets $(\leftarrow, x)_{\sqsubseteq}$ and $(x, \to)_{\sqsubseteq}$ are open. Let us prove more peculiar properties of the relation $\sqsubseteq$.

Fact 1. *Each element of D is comparable with every element of X.*

Proof: Let $a_{\bar n} \in D$ and $\bar x \in X$: of course, we may assume $\bar x \neq a_{\bar n}$. Then we have the equality $j_{\bar n}(a_{\bar n}) = 1$, while $j_{\bar n}(\bar x) \in \{0,2\}$; by the remark after Proposition 2.1, this implies that $a_{\bar n}$ and $\bar x$ are comparable. □

Fact 2. *If $x, y \in X$ are incomparable with respect to $\sqsubseteq$, then*

$$\forall\, z \in X \backslash \{x,y\} : x \prec_f z \iff y \prec_f z.$$

Proof: For x and y to be incomparable, it is easily seen by the above definition of $\sqsubseteq$ that we must have $x \prec_f a \iff y \prec_f a$ for every $a \in D\backslash\{x,y\}$. Since $D\backslash\{x,y\}$ is dense in $X\backslash\{x,y\}$ and f is continuous, the same property clearly holds if we replace a with an arbitrary $z \in X\backslash\{x,y\}$. □

Fact 3. *Let $x, y \in X$ be incomparable with respect to $\sqsubseteq$ and such that $f(\{x,y\}) = x$ (i.e., $x \prec_f y$). Then there exist neighbourhoods V, W of x, y, respectively, such that*

$$\forall\, z \in V\backslash\{x\} : z \prec_f x \quad \textit{and} \quad \forall\, w \in W\backslash\{y\} : y \prec_f w.$$

Proof: By the continuity of f, there exist V, W neighbourhoods of x, y, respectively (and that clearly we may also assume to be disjoint), such that $f(\{z,w\}) = z$ for every $z \in V$ and $w \in W$; in particular, we then have the relations $z \prec_f y$ for every $z \in V$ and $x \prec_f w$ for every $w \in W$. Then it follows from Fact 2 that $z \prec_f x$ for every $z \in V\backslash\{x\}$ and that $y \prec_f w$ for every $w \in W\backslash\{y\}$. □

Fact 4. *For every $x \in X$, there is at most one $y \in X$ such that x is incomparable with y, with respect to $\sqsubseteq$.*

PROOF: By contradiction, suppose for a given $\bar{x} \in X$ there exist two distinct $x', x'' \in X$ which are both incomparable with $\bar{x}$: then, by the remark to Proposition 2.1, it follows that x' is incomparable with x'', too. Now, an easy check of all possible combinations shows that we may always rename the elements of $\{\bar{x}, x', x''\}$ as y_1, y_2, y_3, in such a way that $y_1 \prec_f y_2$ and $y_2 \prec_f y_3$. Applying Fact 3 successively to y_1, y_2 and then to y_2, y_3, we obtain neighbourhoods V', V'' of y_2 such that

$$\forall\, x \in V' \backslash \{y_2\} : x \prec_f y_2 \quad \wedge \quad \forall\, x \in V'' \backslash \{y_2\} : y_2 \prec_f x.$$

Of course, this implies that y_2 is an isolated point of X, which is impossible as $y_2 \in X \backslash D$ (by Fact 1) and D is dense in X. □

Fact 5. *The set*

$$\mathcal{I} = \left\{ \{x, y\} \in \tilde{\mathcal{F}}_2(X) \,\middle|\, x \;\text{ and }\; y \;\text{ are incomparable with respect to } \;\sqsubseteq \right\}$$

is countable, and consists of pairwise disjoint elements.

PROOF: The fact that the elements of $\mathcal{I}$ are pairwise disjoint follows from Fact 4. To prove countability, let us first prove that the set

$$\Sigma = \left\{ (x, y) \in X^2 \,\middle|\, \{x, y\} \in \mathcal{I} \quad \wedge \quad x \prec_f y \right\} \tag{2.2}$$

is discrete in X^2. For every $(x, y) \in \Sigma$, let $U_{x,y}$, $V_{x,y}$ be disjoint neighbourhoods of x, y, respectively, such that:

$$\forall\, x' \in U_{x,y} : \forall\, y' \in V_{x,y} : x' \prec_f y'. \tag{2.3}$$

To prove our claim, it will suffice to show that each set of the form $U_{x,y} \times V_{x,y}$, with $(x, y) \in \Sigma$, contains no element of $\Sigma \backslash \{(x, y)\}$. Indeed, if $(x', y') \in (U_{x,y} \times V_{x,y}) \cap (\Sigma \backslash \{(x, y)\})$, then $x' \prec_f y$ and $x \prec_f y'$ by (2.3). However, since $\{x, y\} \in \mathcal{I}$ and $y' \notin \{x, y\}$ (by Fact 4), the latter relation implies by Fact 2 that $y \prec_f y'$, and this implies in turn, still by Fact 2 (as $(x', y') \in \mathcal{I}$, and $y \notin \{x', y'\}$ by Fact 4), that $y \prec_f x'$. But, of course, $x' \prec_f y$ and $y \prec_f x'$ are incompatible relations.

Now, if X^2 is hereditarily collectionwise Hausdorff, then we may associate to every $(x, y) \in \Sigma$ a neighbourhood $Z_{x,y}$ of (x, y) in X^2, in such a way that $Z_{x',y'} \cap Z_{x'',y''} = \emptyset$ for distinct elements $(x', y'), (x'', y'')$ of Σ; thus, the separability of X clearly implies that Σ (hence $\mathcal{I}$, too) must be countable. On the other hand, if X has a countable base, then the same holds for X^2; then we may associate to every $(x, y) \in \Sigma$ an element $B_{x,y}$ of a previously fixed countable base $\mathcal{B}$ for X^2, in such a way that $B_{x,y} \cap \Sigma = \{(x, y)\}$. Again, this association is clearly one-to-one, hence Σ must be countable. □

Fact 6. *Let $x, y \in X \setminus D$ be incomparable with respect to $\sqsubseteq$, with $x \preccurlyeq_f y$. Then the set $(\leftarrow, x]_{\preccurlyeq_f}$ is clopen (symmetrically, the same holds for $[y, \rightarrow)_{\preccurlyeq_f}$).*

PROOF: We know that $(\leftarrow, x]_{\preccurlyeq_f}$ is closed, and since $(\leftarrow, x)_{\preccurlyeq_f}$ is open, we only have to show that the point x has a neighbourhood which is entirely contained in $(\leftarrow, x]_{\preccurlyeq_f}$. This clearly follows from Fact 3. $\square$

Now we achieve the proof of the theorem. Due to Fact 5, we may index the set Σ defined in (2.2) as $\{(x_n, y_n) \mid n \in \omega\}$; thus $x_n \prec_f y_n$ for every $n \in \omega$, and $\{\{x_n, y_n\} \mid n \in \omega\} = \mathcal{I}$ is the set of all pairs of incomparable elements of X. Of course, to get such an indexing we must assume that $\Sigma \neq \emptyset$; but we do not have to worry about the case $\Sigma = \emptyset$, as then the order $\sqsubseteq$ would automatically be linear, thus making X a weakly orderable space. Let, for $n \in \omega$ and $i \in \{0, 1, 2\}$, $A'_{2n,i} = A_{n,i}$; let also, still for $n \in \omega$:

$$A'_{2n+1,0} = (\leftarrow, x_n]_{\preccurlyeq_f} \quad \text{and} \quad A'_{2n+1,1} = (x_n, \rightarrow)_{\preccurlyeq_f}.$$

Now define

$$\mathcal{A}'_m = \begin{cases} \{A'_{m,0}, A'_{m,1}, A'_{m,2}\}, & \text{for } m \text{ even;} \\ \{A'_{m,0}, A'_{m,1}\}, & \text{for } m \text{ odd;} \end{cases}$$

and j'_m to be the function from X to either $\{0, 1, 2\}$ or $\{0, 1\}$ (according to whether m is even or odd), such that $x \in A'_{m, j'_m(x)}$ for every $x \in X$ and $m \in \omega$. Observe that, by Fact 6, $\mathcal{A}'_m$ is an open partition of X also for odd m's. Therefore, the binary relation $\sqsubseteq'$ on X, defined as:

$$x \sqsubseteq' y \Longleftrightarrow \Big(x = y \quad \vee \quad \exists\, m \in \omega : \big((\forall i < m : j'_i(x) = j'_i(y)) \quad \wedge \quad j'_m(x) < j'_m(y)\big)\Big)$$

is an order on X such that $(\leftarrow, x)_{\sqsubseteq'}$ and $(x, \rightarrow)_{\sqsubseteq'}$ are open for every $x \in X$. We claim that such an order is also linear, which will entail the weak orderability of X.

Indeed, let $x, y \in X$: if x and y are comparable with respect to $\sqsubseteq$, then by the remark after Proposition 2.1 it is easily seen that they are comparable as well with respect to $\sqsubseteq'$. Therefore, suppose x, y are $\sqsubseteq$-incomparable: we may assume that $x \prec_f y$. Then $(x, y) \in \Sigma$, so that $(x, y) = (x_{\bar{n}}, y_{\bar{n}})$ for some $\bar{n} \in \omega$; since $x = x_{\bar{n}} \in (\leftarrow, x_{\bar{n}}]_{\preccurlyeq_f} = A'_{2\bar{n}+1,0}$ and $y = y_{\bar{n}} \in (x_{\bar{n}}, \rightarrow)_{\preccurlyeq_f} = A'_{2\bar{n}+1,1}$, we have the inequality $j'_{2\bar{n}+1}(x) = 0 \neq 1 = j'_{2\bar{n}+1}(y)$. Hence x and y are $\sqsubseteq'$-comparable. $\square$

Corollary 2.3. *Let X be a (Hausdorff) space admitting a weak selection, and suppose that one of the following properties holds:*

(a) *X has a countable, discrete dense subset, and X^2 is hereditarily collectionwise Hausdorff;*
(b) *X has a discrete dense subset and admits a countable base.*

Then X is weakly orderable.

In particular, for every separable metrizable space with a dense and discrete subset the existence of a weak selection implies weak orderability.

PROOF: First of all, notice that each dense and discrete subset D of a T_1-topological space Y consists of isolated points. Indeed, if $x \in D$ and V is an open neighbourhood of x such that $V \cap D = \{x\}$, then $V \backslash \{x\}$ is an open subset of X missing D. Thus we must have the equality $V \backslash \{x\} = \emptyset$, i.e. $V = \{x\}$.

Now, if we are in case (a), then by the above considerations we may simply apply Theorem 2.2. And if we are in case (b), then first take a dense and discrete subset D of X (which actually will consist of isolated points), and then associate to every element B_n of a fixed countable base $\mathcal{B} = \{B_n \mid n \in \omega\}$ of X an element x_n of $B_n \cap D$. It turns out that $\{x_n \mid n \in \omega\}$ is a countable dense subset of X consisting of isolated points, so that we are again in condition to apply Theorem 2.2. $\square$

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