On the number of Russell's socks or $2+2+2+\ \ldots\ =\ ?$

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Abstract. The following question is analyzed under the assumption that the Axiom of Choice fails badly: Given a countable number of pairs of socks, then how many socks are there? Surprisingly this number is not uniquely determined by the above information, thus giving rise to the concept of Russell-cardinals. It will be shown that:

- some Russell-cardinals are even, but others fail to be so;
- no Russell-cardinal is odd;
- no Russell-cardinal is comparable with any cardinal of the form $\aleph_\alpha$ or $2^{\aleph_\alpha}$;
- finite sums of Russell-cardinals are Russell-cardinals, but finite products — even squares — of Russell-cardinals may fail to be so;
- some countable unions of pairwise disjoint Russell-sets are Russell-sets, but others fail to be so;
- for each Russell-cardinal a there exists a chain consisting of $2^{\aleph_0}$ Russell-cardinals between a and 2^a;
- there exist antichains consisting of $2^{\aleph_0}$ Russell-cardinals;
- there are neither minimal nor maximal Russell-cardinals;
- no Russell-graph has a chromatic number.

Keywords: Bertrand Russell, Axiom of Choice, Generalized Continuum Hypothesis, Dedekind-finite sets, Dedekind-cardinals, Russell-cardinals, odd and (almost) even cardinals, cardinal arithmetic, coloring of graphs, chromatic number, socks

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Background

In [6, pp. 47–48] Bertrand Russell illustrated his doubts about the validity of the Axiom of Choice by means of the following picture:

> *"A simple illustration may serve to show the nature of the difficulty as regards this axiom, and to introduce the analogous "multiplicative axiom". Given $\aleph_0$ pairs of boots, let it be required to prove that the number of boots is even. This will be the case if all the boots can be divided into two classes which are mutually similar. If now each pair*

has the right and the left boots different, we need only put all the right boots in one class, and all the left boots in another: the class of right boots is similar to the class of left boots, and our problem is solved. But, if the right and left boots in each pair are indistinguishable[1]*, we cannot discover any property belonging to exactly half the boots. Hence we cannot divide the boots into two equal parts, and we cannot prove that the number of them is even. If the number of pairs were finite, we would simply choose one out of each pair; but we cannot choose one out of each of an infinite number of pairs unless we have a* **rule** *of choice, and in the present case no rule can be found.*

The problem involved in the above illustration raises grave difficulties in regard to many elementary theorems about multiplication of cardinals."

Since normally left and right shoes are distinguishable, but left and right socks are not, mathematical folklore slightly transformed Russell's illustration into one concerning shoes and socks, as expressed, e.g., in [8, p. 140]:

"To select one sock from each of infinitely many pairs of socks requires the Axiom of Choice; but for shoes the Axiom is not needed."

We will adopt this latter formulation and investigate in the following the number of these socks.

Definition 1. A *Russell-sequence* is a sequence $(X_n)_{n\in\mathbb{N}}$ of pairwise disjoint 2-element sets such that for each infinite subset M of $\mathbb{N}$ the product $\prod_{m\in M} X_m$ is empty.

A *Russell-set* is the union $X = \bigcup_{n\in\mathbb{N}} X_n$ of some Russell-sequence $(X_n)_{n\in\mathbb{N}}$.

A *Russell-cardinal* is the cardinal number $|X|$ of some Russell-set X.

Let us recall the following concepts:

A set X is called *Dedekind-finite* provided that it satisfies the following equivalent conditions:

- $\aleph_0 \not\leq |X|$,
- $|X| \neq |X| + 1$,
- $|A| < |X|$ for every proper subset A of X.

A set is called a *Dedekind-set* provided it is infinite and Dedekind-finite.

A cardinal a is called a *Dedekind-cardinal* provided it satisfies the following equivalent conditions:

- $a = |X|$ for some Dedekind-set X,
- a and $\aleph_0$ are incomparable.

[1] A few years later [7], Russell attributes such peculiar "bottines semblables, de sorte qu'il n'y avait pas une bottine droite et une bottine gauche dans chaque paire" to some "millionaire excentrique."

Observations and results

Do there exist Russell-cardinals?

The answer to this question is NO in **ZFC** (i.e., Zermelo-Fraenkel set theory including the Axiom of Choice). However, there exist models of **ZF** (i.e., Zermelo-Fraenkel set theory without the Axiom of Choice) in which the answer is YES (see [1]).

From now on, we assume to work in a **ZF**-model with at least one Russell-cardinal.

Are Russell-cardinals unique?

As the following results demonstrate the answer is an emphatic NO. If there is one Russell-cardinal, then there are many (even though $2\cdot\aleph_0 = \aleph_0\cdot 2 = \aleph_0$ in every **ZF**-model). This implies that if we know that Russell has countably many pairs of socks, we do **not** automatically know how many socks he has. To determine the latter number we need some additional information about the nature of his socks — a phenomenon that appears to be rather strange and unfamiliar to all those used to live in the **ZFC**-world.

Proposition 1. *Every Russell-set* (*resp. cardinal*) *is a Dedekind-set* (*resp. cardinal*)*.*

Though we are used to the idea that a person has at least as many socks as he has **pairs** of socks, Proposition 1 shows that in Russell's case that is not so. The number of his socks and the number of his pairs of socks ($=\aleph_0$) are incomparable!

Proposition 2. *Every Russell-cardinal a has a direct predecessor $a-1$ and a direct successor $a+1$.*

Later (Proposition 15) we will show that, for Russell-cardinals a, the cardinals $a-1$ and $a+1$ fail to be Russell-cardinals. However, among Russell-cardinals only, a has a direct predecessor $a-2$ and a direct successor $a+2$ (see Proposition 4).

Proposition 3. *If (X_n) is a Russell-sequence and M is an infinite proper subset of $\mathbb{N}$, then $\bigcup_{m\in M} X_m$ is a Russell-set and $|\bigcup_{m\in M} X_m| < |\bigcup_{n\in\mathbb{N}} X_n|$.*

Proposition 4. *If a is a Russell-cardinal, then so are $a+2$ and $a-2$ and $(a-2) < a < (a+2)$.*

This result implies that whenever Russell obtains an additional pair of socks (resp. if he gets rid of one of his pairs of socks), then the number of his socks properly increases (resp. properly decreases), though the number of his **pairs** of socks remains unaltered, namely $\aleph_0$. Contrast this phenomenon with the equally amazing one concerning *Hilbert's infinite hotel*. Observe further that there are neither maximal nor minimal Russell-cardinals.

Proposition 5. *If a and b are Russell-cardinals*[2] *then so is $a+b$ and there exists a family $(a_{(r,n)})_{(r,n)\in\mathbb{R}\times\mathbb{Z}}$ of Russell-cardinals such that*

$$a < a_{(r,n)} < a+b \quad \text{for each} \quad (r,n) \in \mathbb{R}\times\mathbb{Z}$$

and

$$a_{(r,n)} < a_{(s,m)} \quad \text{iff} \quad r < s \quad \text{or} \quad (r = s \quad \text{and} \quad n < m).$$

Proof: Let (X_n) be a Russell-sequence with $b = |\bigcup_{n\in\mathbb{N}} X_n|$, let $\rho\colon \mathbb{N} \to \mathbb{Q}$ be a bijection, and define $m(r) = \min\{n \in \mathbb{N} \mid \ |r-\rho(n)| < \frac{1}{2}\}$ for $r \in \mathbb{R}$. Consider

$$a_{(r,n)} = a + \begin{cases} |\bigcup_{\rho(k)<r} X_k \cup \bigcup_{k=1}^{n} X_{m(r+k)}| & \text{if } n \geq 0 \\ |\bigcup_{\rho(k)<r} X_k \setminus \bigcup_{k=1}^{n} X_{m(r-k)}| & \text{if } n < 0 \end{cases}.$$

□

Compare this result with [10] and [5, p. 161, Problem 6].

Proposition 6. *If a is a Russell-cardinal, then so is $2a$ and $a < 2a$.*

This result implies that there exist even Russell-cardinals. However, we will show (Proposition 19) below that Russell-cardinals may fail to be even. Thus Russell might not have been able to give precisely half the number of his socks to a friend (though he could give him half the number of his **pairs** of socks).

Proposition 7. *If a is a Russell-cardinal, then so are all cardinals $n\cdot a$ with $n \in \mathbb{N}^+$ and*

$$\begin{gathered} a < (a+1) < (a+2) < (a+2) < \cdots < (a+n) < \cdots \\ \cdots < 2a < 3a < \cdots < na < \cdots < \aleph_0 \cdot a \leq 2^a. \end{gathered}$$

Proof of $\aleph_0 \cdot a \leq 2^a$: Let (X_n) be a Russell-sequence, $X = \bigcup_{n\in\mathbb{N}} X_n$, $\mathcal{P}X$ the set of all subsets of X, and $a = |X|$. For each $n \in \mathbb{N}$, define $A_n = \bigcup_{m<n} X_m$. Then the map $f\colon \mathbb{N}\times X \to \mathcal{P}X$, defined by $f(n,x) = \begin{cases} A_n \cup \{x\}, & \text{if } x \notin A_n \\ A_{n+1}\setminus\{x\}, & \text{if } x \in A_n \end{cases}$ is injective. Thus $\aleph_0 \cdot a \leq 2^a$. □

Recall that **GCH**, the Generalized Continuum Hypothesis, implies the Axiom of Choice. So **GCH** must fail in any model under discussion here. Propositions 5 and 7 show that it fails **badly**.

[2] The conclusion also holds, if a is finite and even and b is a Russell-cardinal.

Countable sums and countable unions

By the last proposition, with each Russell-cardinal a all cardinals na with $n \in \mathbb{N}^+$ are Russell-cardinals as well. However, $\aleph_0 \cdot a$ fails to be a Russell-cardinal, since it is Dedekind-infinite. The related question, whether countable sums of Russell-cardinals are again Russell-cardinals, unfortunately makes no sense since in the **ZF**-models under discussion countable sums of cardinals cannot be defined properly. But, we can ask whether countable unions of pairwise disjoint Russell-sets are again Russell-sets. The surprising answer: sometimes they are, sometimes they are not:

Proposition 8. *Every Russell-set is expressible as a countable union of pairwise disjoint Russell-sets.*

Proof: Let (X_n) be a Russell-sequence with union X. For each $k \in \mathbb{N}$ define $Y_k = \bigcup_{m \in \mathbb{N}} 2^k \cdot (2m+1)$. Then the Y_k's are pairwise disjoint Russell-sets with union X. □

Proposition 9. *Let (X_n) be a Russell-sequence with union X. Then:*

1. *the sets $Y_n = \bigcup_{k \in \mathbb{N}} (X_k \times \{n\})$ form a sequence of pairwise disjoint Russell-sets whose union Y fails to be a Russell-set;*
2. *the sets $Z_n = \bigcup_{k \geq n} (X_k \times \{n\})$ form a sequence of pairwise disjoint Russell-sets whose union Z is a Russell-set.*

Moreover, $n \cdot |X| < |Z| < |Y| = \aleph_0 \cdot |X|$ for each $n \in \mathbb{N}$.

How nice can Russell-cardinals be?

As the following results show, Russell-cardinals and *nice* cardinals are rather unrelated.

Proposition 10. *If X is infinite and linearly orderable, then no Russell-cardinal is comparable with $|X|$.*

Proposition 11. *No Russell-cardinal is comparable with any $\aleph$.*

Observe that — though the noun *socks* is countable — Russell's socks are **uncountable**, not only in the sense that their number fails to be at most $\aleph_0$, but even in the stronger sense that they cannot be *counted* (= well-ordered) even if we would have an unlimited amount of time at our disposal.

Proposition 12. *No Russell-cardinal is comparable with any cardinal of the form $2^{\aleph}$.*

Proof: If $(X, \leq)$ is a well-ordered set, then the powerset $\mathcal{P}X$ is linearly ordered by

$$A < B \ \text{ iff } \ (A \neq B \ \text{ and } \ \min(A \triangle B) \in A).$$

□

Are Russell-cardinals even or odd?

In order to show that no Russell-cardinal is odd, observe first that every Russell-cardinal is *almost even* in the following sense:

Definition 2. A cardinal $a = |A|$ is called *almost even* provided it satisfies the following equivalent conditions:

- there exists a fixpoint-free map $\sigma\colon A \to A$ with $\sigma^2 = \mathrm{id}_A$;
- A can be expressed as the disjoint union of a family of 2-element sets.

Proposition 13. *If a and $a+1$ are almost even, then a is Dedekind-infinite.*

Proof: Consider $a = |A|$ and let $\sigma\colon A \to A$ be a fixpoint-free map with $\sigma^2 = \mathrm{id}_A$. Consider further $B = A \uplus \{0\}$ and let $\tau\colon B \to B$ be a fixpoint-free map with $\tau^2 = \mathrm{id}_B$. Define, via recursion, a sequence (a_n) in A as follows:

$$\begin{aligned} a_o &= \tau(0) \\ a_{n+1} &= \tau(\sigma(a_n)). \end{aligned}$$

Then (a_n) is injective. Thus A is Dedekind-infinite. □

Proposition 14. *No Russell-cardinal is odd.*

Proof: Immediate from Proposition 13, since Russell-cardinals and even cardinals are almost even. □

Proposition 15. *If a is a Russell-cardinal, then $a+1$ fails to be a Russell cardinal.*

For Bertrand Russell this result has undesirable consequences: Whenever one of his socks gets defective, he will not be able to rearrange the remaining socks into pairs (even if he does not care if the pairs match in color, structure, form or size), and so he may get rid not only of the defective sock but of its matching partner as well.

Proposition 16. *If a is a Russell-cardinal and b is an infinite cardinal with $b \leq a$, then exactly one of the two cardinals b and $b+1$ is a Russell-cardinal.*

Proof: Let (X_n) be a Russell-sequence and Y be an infinite subset of $\bigcup_{n\in\mathbb{N}} X_n$. Then the set $M = \{n \in \mathbb{N} \mid \ |X_n \cap Y| = 1\}$ is finite. If $|M|$ is even, then Y is a Russell-set. Otherwise $Y \uplus \{0\}$ is a Russell-set. By Proposition 15, both Y and $Y \uplus \{0\}$ cannot be Russell-sets. □

Proposition 17. *If a is a Russell-cardinal, then in the integer-indexed family*

$$\cdots < (a-2) < (a-1) < a < (a+1) < (a+2) < \cdots$$

Russell-cardinals alternate with non-Russell-cardinals.

As we have seen above (Proposition 6) in all models under discussion some Russell-cardinals are even. However, in some of these models not all Russell-cardinals are even:

Definition 3. A **ZF**-model is called a *Russell-model* provided that there exists a Russell-sequence (X_n) which satisfies the following equivalent conditions:

- no infinite subset of $\bigcup_{n\in\mathbb{N}} X_n$ has even cardinality;
- for each infinite subset M of $\mathbb{N}$ the cardinal $|\bigcup_{m\in M} X_m|$ fails to be even;
- $|\bigcup_{m\in M} X_m| \neq |\bigcup_{k\in K} X_k|$ for disjoint infinite subsets M and K of $\mathbb{N}$;
- for any two subsets M and K of $\mathbb{N}$ with infinite differences $M\backslash K$ and $K\backslash M$ the cardinals $|\bigcup_{m\in M} X_m|$ and $|\bigcup_{k\in K} X_k|$ are incomparable.

Proposition 18. *There exist Russell-models.*

Proof: The Second Cohen Model (M7 in [4, p. 152]) and the Second Fraenkel Model (N2 in [4, p. 178]) are Russell-models. We first recall the description of the Second Cohen Model. Let $(\mathcal{M}, \in)$ be a countable transitive model of $\mathbf{ZF} + V = L$ and $\mathbb{P} = \mathrm{Fin}(\omega\times 2\times\omega\times\omega, 2)$ be the set of all finite functions p with $\mathrm{dom}(p) \subset \omega\times 2\times\omega\times\omega$ and $\mathrm{ran}(p) \subset 2$. Partially order $\mathbb{P}$ by reverse inclusion, i.e. $p \le q$ iff $p \supseteq q$. Let G be a $\mathbb{P}$-generic set over $\mathcal{M}$ and $\mathcal{M}[G]$ the corresponding generic extension of $\mathcal{M}$. In $\mathcal{M}[G]$ define the following sets:

$$\begin{aligned} x_{nqi} &= \{j\in\mathbb{N}\colon \exists\, p\in G\ (p(n,q,i,j)=1)\}\\ X_{nq} &= \{x_{nqi}\colon i\in\mathbb{N}\}\\ X_n &= \{X_{n0}, X_{n1}\}. \end{aligned}$$

Let $\mathbb{G}$ be the group of all permutations on $\omega\times 2\times\omega$. Each $\pi\in\mathbb{G}$ induces an order automorphism on $\mathbb{P}$ which is defined as follows: $\pi p(\pi(n,q,i),j) = p(n,q,i,j)$. Consider the collection $\mathcal{E} = \{\mathrm{fix}(E)\colon E\in[\omega\times 2\times\omega]^{<\omega}\}$, where $\mathrm{fix}(E) = \{\pi\in\mathbb{G}\colon \forall\, e\in E\ \pi(e)=e\}$, of subgroups of $\mathbb{G}$. Clearly $\mathcal{E}$ is a filterbase for some normal filter $\mathcal{F}$. Let $\mathcal{N}$ be the corresponding symmetric model of **ZF**.

Now all the sets of $\mathcal{M}[G]$ defined above belong to $\mathcal{N}$ since they have hereditarily symmetric names. Furthermore, it is known that the sequence (X_n) does not have a partial choice function in $\mathcal{N}$ (i.e., for each infinite subset M of $\mathbb{N}$ the product $\prod_{m\in M} X_m$ is empty). Thus $\bigcup_{n\in\mathbb{N}} X_n$ is a Russell-set. We show next that for any two infinite subsets M and K of $\mathbb{N}$ such that $M\backslash K$ and $K\backslash M$ are infinite, the Russell-cardinals $a = |\bigcup\{X_m\colon m\in M\}|$ and $b = |\bigcup\{X_k\colon k\in K\}|$ are incomparable. Assume on the contrary that there exists an injective function $f: a\to b$ in $\mathcal{N}$. Let F be a hereditarily symmetric name for f with support E, i.e., $\mathrm{fix}(E)\subset \mathrm{sym}(F) = \{\pi\in\mathbb{G} : \pi(F)=F\}$. Since E is finite and f is injective, it follows that there exist $m\in M\setminus K$, $k\in K$, $m,k\notin \mathrm{dom}(\mathrm{dom}(E))$, and $q,q'\in 2$ such that $f(X_{mq}) = X_{kq'}$. Let $p\in G$ be such that:

$$p \Vdash F(\underline{X_{mq}}) = \underline{X_{kq'}},$$

where $\underline{X_{mq}}, \underline{X_{kq'}}$ are the canonical names of X_{mq} and $X_{kq'}$ respectively. Let $r\in\mathbb{N}$ be such that for all $i\ge r$ and all $q\in 2$, $(k,q,i)\notin\mathrm{dom}(\mathrm{dom}(p))$. We define the following permutation $\pi \in \mathbb{G}$:

$$\pi(k,s,i) = \begin{cases} (k, 1-s, i+r), & \text{if } \ s \in \{q', 1-q'\}, i < r, \\ (k, 1-s, i-r), & \text{if } \ s \in \{q', 1-q'\}, r \leq i < 2r, \\ (k, 1-s, i), & \text{if } \ s \in \{q', 1-q'\}, 2r < i \end{cases}$$

and

$$\pi(u,s,i) = (u,s,i) \ \text{ if } \ u \neq k.$$

Clearly, $\pi \in \text{fix}(E)$, hence $\pi(F) = F$. Furthermore, $\pi(\underline{X_{mq}}) = \underline{X_{mq}}$, $\pi(\underline{X_{kq'}}) = \underline{X_{k(1-q')}}$, and p, πp are compatible. Thus, $g = p \cup \pi p$ is a well-defined extension of p such that

$$g \Vdash (F(\underline{X_{mq}}) = \underline{X_{kq'}}) \wedge (F(\underline{X_{mq}}) = \underline{X_{k(1-q')}}).$$

This is a contradiction. Thus, a and b are incomparable. □

Proposition 19. *In every Russell-model there exist Russell-cardinals that fail to be even.*

Are Russell-cardinals comparable with each other?

All the Russell-cardinals we have constructed so far (Propositions 3, 4, 5, 6, 7 and 9) are comparable with each other. In [5, p. 162, Problem 7] Jech asks whether there exist two incomparable Dedekind-cardinals; and in [10] Tarski proves that, in case two incomparable Dedekind-cardinals exist, there exists a collection of $\aleph_0$ pairwise incomparable Dedekind-cardinals. Here we have:

Proposition 20. *In every Russell-model there exists a collection of $2^{\aleph_0}$ pairwise incomparable Russell-cardinals.*

Proof: Let (X_n) be a Russell-sequence as specified in the definition of Russell-models, let $\varphi \colon \mathbb{N} \to \mathbb{Q}$ be a bijection, and let $\mathbb{R}^+$ be the set of positive reals. For $r \in \mathbb{R}^+$ define

$$M_r = \{n \in \mathbb{N} \mid \ \varphi(n) < -r \ \text{ or } \ 0 < \varphi(n) < r\}.$$

Then the collection $\{|\bigcup_{n \in M_r} X_n| \; | r \in \mathbb{R}^+\}$ consists of $2^{\aleph_0}$ pairwise incomparable Russell-cardinals.

□

So, if Russell and one of his friends each have a collection of $\aleph_0$ pairs of socks, it may well happen that none of the two has at least as many socks as the other. Observe further that none of the $2^{\aleph_0}$ Russell-cardinals constructed above is even.

Squares of Russell-cardinals

As we have seen above (Proposition 5) sums $a+b$ of Russell-cardinals are Russell-cardinals. How about products $a \cdot b$, in particular squares a^2?

Consider a Russell-sequence (X_n) with $X_n = \{x_n, y_n\}$, $X = \bigcup_{n\in\mathbb{N}} X_n$, and $a = |X|$. Then the Dedekind-cardinal $a^2 = |X^2| = |\bigcup_{(n,m)\in\mathbb{N}^2}(X_n \times X_m)|$ is the cardinal-number of the union of a sequence of pairwise disjoint 4-element sets $X_n \times X_m$. Moreover, the latter can be expressed effectively as a union of two 2-element sets each; e.g.:

$$\begin{aligned} X_n \times X_m &= (X_n \times \{x_m\}) \cup (X_n \times \{y_m\}) \\ &= (\{x_n\} \times X_m) \cup (\{y_n\} \times X_m) \\ &= \{(x_n, x_m),\ (y_n, y_m)\} \cup \{(x_n, y_m),\ (y_n, x_m)\}. \end{aligned}$$

However, it may be impossible to express X^2 as a countable union of pairwise disjoint 2-element sets:

Proposition 21. *Squares of Russell-cardinals may fail to be Russell-cardinals.*

Proof: First let us recall the description of the Second Fraenkel Model $\mathcal{N}$ (N2 in [4, p. 178]). The set of atoms X is countable and partitioned into countably many disjoint 2-element sets $X_n = \{a_n, b_n\}$, $n \in \mathbb{N}$. Let G be the group of all permutations on X which stabilize each X_n, i.e., for all $\pi \in G$ and for all $n \in \mathbb{N}$, $\pi[X_n] = X_n$. Let I be the normal ideal of all finite subsets of X. Then $\mathcal{N}$ is the permutation model which is determined by G and I. It is known (see [4]) that the family $\{X_n : n \in \mathbb{N}\}$ is countable in $\mathcal{N}$ and has no partial choice function in $\mathcal{N}$. Thus, X is a Russell-set.

However, X^2 fails to be a Russell-set in $\mathcal{N}$. In fact, we show next that in $\mathcal{N}$ there does not even exist a disjoint family $Y = \{Y_n : n \in \mathbb{N}\}$ consisting of 2-element sets such that $X^2 = \bigcup Y$. Assume the contrary and let $Y = \{Y_n : n \in \mathbb{N}\} \in \mathcal{N}$ be such a family. Since Y is countable in $\mathcal{N}$, there is a $k \in \mathbb{N}$ such that the set $E = X_1 \cup X_2 \cup \ldots \cup X_k$ is a support for Y_n for all $n \in \mathbb{N}$, i.e., for each $n \in \mathbb{N}$, $\mathrm{fix}(E) = \{\pi \in G : \forall\, e \in E\ \pi(e) = e\} \subset \mathrm{sym}(Y_n) = \{\pi \in G : \pi[Y_n] = Y_n\}$. It can be easily verified that

$$(\exists\, n \in \mathbb{N})(\forall\, i > k)(\mathrm{dom}(Y_n) \cup \mathrm{ran}(Y_n) \not\subset E \cup X_i). \tag{1}$$

(Otherwise, for any $r, s \in \mathbb{N}$ such that $r > s > k$, the pair $(a_r, a_s) \in X^2 \setminus \bigcup Y$.) Fix a set Y_n satisfying (1). By (1) it follows that there exists an $i > k$ such that $[\mathrm{dom}(Y_n) \cup \mathrm{ran}(Y_n)] \cap X_i \neq \emptyset$. We shall construct a permutation $\pi \in \mathrm{fix}(E)$ such that $\pi[Y_n] \neq Y_n$. We consider the following cases.

(a) $|[\mathrm{dom}(Y_n) \cup \mathrm{ran}(Y_n)] \cap X_i| = 1$. Consider the transposition $\pi = (a_i, b_i)$. Clearly, $\pi \in \mathrm{fix}(E) \setminus \mathrm{sym}(Y_n)$.

(b) $X_i \subset \mathrm{dom}(Y_n) \cup \mathrm{ran}(Y_n)$ and $(a_i, b_i) \in Y_n$ or $(b_i, a_i) \in Y_n$ (and necessarily not both due to (1)). Then again we may let $\pi = (a_i, b_i)$.

(c) $X_i \subset \mathrm{dom}(Y_n) \cup \mathrm{ran}(Y_n)$ but neither $(a_i, b_i) \in Y_n$ nor $(b_i, a_i) \in Y_n$. Due to (1), we have that $[\mathrm{dom}(Y_n) \cup \mathrm{ran}(Y_n)] \cap X_m \neq \emptyset$ for some $m \neq 1, 2, \ldots, k, i$. Let $\pi = (a_m, b_m)$. It is easy to verify that $\pi \in \mathrm{fix}(E) \setminus \mathrm{sym}(Y_n)$.

Thus, $Y \notin \mathcal{N}$ and the proof of the proposition is complete. □

Russell's socks in the laundromat or Permutations of Russell-sets

If Russell's socks are washed in a laundromat and taken out in pairs, is it likely that some pairs remain unseparated? Surprisingly, this will happen with certainty for almost all pairs:

Definition 4. If (X_n) is a Russell-sequence with $X = \bigcup_{n \in \mathbb{N}} X_n$, then a map $f\colon X \to X$ is said to *separate* some X_n iff it maps the two elements of X_n into two different X_m's.

Proposition 22. *If (X_n) is a Russell-sequence, then each permutation of $\bigcup_{n \in \mathbb{N}} X_n$ separates only finitely many X_n's.*

Do Russell-graphs have chromatic numbers?

Russell-sets quite naturally can be considered as graphs. In view of the fact that all graphs have chromatic numbers if and only if the Axiom of Choice holds [2], the question arises whether at least the rather simple graphs, induced by Russell-sets, have chromatic numbers. The shocking truth: no such graph has a chromatic number!

Definition 5. A graph (G, ρ) is called a *Russell-graph* provided there exists some Russell-sequence (X_n) with $G = \bigcup_{n \in \mathbb{N}} X_n$ and

$$x \rho y \Leftrightarrow \exists\, n \ \ \{x, y\} = X_n.$$

Proposition 23. *No Russell-graph has a chromatic number.*

Proof: Let (X_n) be a Russell-sequence and let (G, ρ) be the associated Russell-graph. Let $f\colon G \to C$ be a C-coloration of this graph, i.e., a map such that $x \rho y$ implies $f(x) \neq f(y)$. Then for each $c \in C$ the set $f^{-1}(c)$ must be finite, and the set $f[G]$ must be a Dedekind-set. Select some point $c_0 \in f[G]$, then $M = \{n \in \mathbb{N} \mid f^{-1}(c_0) \cap X_n \neq \emptyset\}$ is finite, and thus there exists some $c_1 \in (f[G] \setminus \bigcup_{m \in M} f[X_n])$. Define a coloring $g\colon G \to (f[G] \setminus \{c_0\})$ by $g(x) = \begin{cases} f(x), & f(x) \neq c_0 \\ c_1, & f(x) = c_0 \end{cases}$. Since $|f[G] \setminus \{c_0\}| < |f[G]| \leq |C|$, the graph (G, ρ) has no chromatic number.

□

Problems

Problem 1. Does the inequality $a^2 < 2^a$ hold for Russell-cardinals?

Compare [5, p. 163, Problem 17]. Observe that $a^2 < 3^a$ for every cardinal a.

Problem 2. Is the class of Russell-cardinals bounded?

Observe that the only **lower** bounds are the finite cardinals.

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